Same rules as Homework 1.

Note: In this assignment, R denotes a commutative ring.

Problem 19 Your signature below indicates you have:

(a.) I read Chapter 2 of Cook's lecture notes: ____________________.

Problem 20 Consider the system of equations given below:

$$\begin{aligned} &x - 2y + z = 3 \\ &-x + y + 2z = 12 \\ &3x + z = 31 \end{aligned}$$

Find the solution to this system via the technique of row reduction.

Problem 21 Consider the system of equations given below:

$$\begin{aligned} &x + z = -5 \\ &x + 2y = 1 \\ &x + 3z = -5 \\ &x + 2y + z + w = 10. \end{aligned}$$

Find the solution to this system (without technology, but, you are free to use whichever method seems wisest)

Problem 22 Consider the system of equations below:

$$\begin{aligned} &x_1 + 4x_2 - 7x_3 + x_4 = 2 \\ &2x_1 + 2x_2 + x_3 + x_4 = 3 \end{aligned}$$

Find the solution set of the equation via the techinque of row-reduction.

Problem 23 Consider the system of equations over $\mathbb{Z}/7\mathbb{Z}$ below:

$$\begin{aligned} &\overline{2}x - \overline{2}y + \overline{3}z = \overline{0} \\ &\overline{5}x + \overline{3}y - \overline{4}z = \overline{0} \end{aligned}$$

Find the solution set of the equation via the technique of row-reduction.

Problem 24 If $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ then calculate A^{-1} and use this to solve the following by matrix multiplication:

$$\begin{aligned} &x - 2y + z = a \\ &-x + y + 2z = b \\ &3x + z = c \end{aligned}$$

You should check this against Problem 20.

Problem 25 Is $v = (a,b,c,d) \in \text{span}\{(1,2,2,1),(3,3,1,1)\}$? What condition(s) must be made on a,b,c,d for v to be in the span? Write down the set of vectors which is not in the span.

Problem 26 Are $(1,2,3)$ and $(0,1,1)$ in $\text{span}\{(1,3,4),(-1,3,4)\}$? If so, explicitly give the linear combinations to prove both of your assertions.

Problem 27 For what value(s) of k is the set S of vectors below a linearly independent set ?

$$S = \{(1,2,3),(2,2,2),(3,4,k)\}.$$

Problem 28 You have two sets of vectors $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$ in $\mathbb{R}^5$. Let $[T|S]$ denote the matrix formed by listing the vectors in S and T as columns. Furthermore,

$$\text{rref}[S|T] = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Use the CCP to decide if S and T form LI sets. Also, determine which vectors in T fall inside $\text{span}(S)$.

Problem 29 Prove the concatenation proposition for columns; that is prove Proposition 2.3.14 in my notes; If$A \in R^{m\times n}$ and $B \in R^{n\times p}$ then

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)]$$

Problem 30 Suppose N is nilpotent of degree 3. Show $I + N$ is an invertible matrix. *Hint: guess the inverse has the form $aI + bN + cN^2$ for some $a,b,c \in \mathbb{R}$ and work it out.*

Problem 31 Let $Z = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ and find all real solutions of $Z^2 = I$.

Problem 32 Prove part (2.) of Proposition 2.4.12. In particular prove the following claim:
If $A \in R^{n\times n}$ then $(A^p)^q = A^{pq}$ for all $p,q \in \mathbb{N}\cup\{0\}$.

Problem 33 Let $S = \{D \in \mathbb{F}^{n\times n} \mid AD = DA \text{ for all } A \in \mathbb{F}^{n\times n}\}$. Prove that the matrices in S are diagonal matrices of a particularly simple type.

Problem 34 Let $v = (1,1)$ and $w = (1,-1)$. If $Av = (1,0)$ and $Aw = (0,1)$ then find A and A^{-1} explicitly.

Problem 35 Given that $A^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$ calculate $(A^TB)^{-1}$.

Problem 36 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$. Find 2×2 elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$. Also, while we're at it, what is $[\text{Col}_1(A)|\text{Col}_2(A)]^{-1}$? Can you read off the inverse without further calculation?

Problem 37 It is also possible to perform elementary column operations on a matrix A. Furthermore, if you perform a column operation on the identity matrix then the resulting matrix when multiplied on the **right** of A will perform that same column operation. For example, to swap columns 2 and 3 of $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ I first swap columns 2 and 3 of $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to obtain $C_{2\leftrightarrow 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and you can calculate

$$AC_{2\leftrightarrow 3} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

Find column operation matrices $C_1, C_2, \cdots, C_n$ **such that** $AC_1C_2 \cdots C_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
I found either $n = 3$ or $n = 5$ depending on how you like to arrange the steps. This type of calculation has further significance if you study abstract algebra. In particular, see Dummit and Foote on the Smith Normal Form. For us, it's just a good exercise for matrix arithmetic.

Problem 38 Suppose N is nilpotent of degree 2 and D is a 2×2 diagonal matrix with diagonal entries 2 and 3. If $M = \begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix}$ then is M invertible? Prove or disprove. Discuss.