Same rules as Homework 1.

Note: In this assignment, R denotes a commutative ring.

Problem 19 Your signature below indicates you have:

(a.) I read Chapter 2 of Cook's lecture notes: ____________________.

Problem 20 Consider the system of equations given below:

$$\begin{aligned} &x - 2y + z = 3 \\ &-x + y + 2z = 12 \\ &3x + z = 31 \end{aligned}$$

Find the solution to this system via the technique of row reduction.

Problem 21 Consider the system of equations given below:

$$\begin{aligned} &x + z = -5 \\ &x + 2y = 1 \\ &x + 3z = -5 \\ &x + 2y + z + w = 10. \end{aligned}$$

Find the solution to this system (without technology, but, you are free to use whichever method seems wisest)

Problem 22 Consider the system of equations below:

$$\begin{aligned} &x_1 + 4x_2 - 7x_3 + x_4 = 2 \\ &2x_1 + 2x_2 + x_3 + x_4 = 3 \end{aligned}$$

Find the solution set of the equation via the techinque of row-reduction.

Problem 23 Consider the system of equations over $\mathbb{Z}/7\mathbb{Z}$ below:

$$\begin{aligned} &\overline{2}x - \overline{2}y + \overline{3}z = \overline{0} \\ &\overline{5}x + \overline{3}y - \overline{4}z = \overline{0} \end{aligned}$$

Find the solution set of the equation via the technique of row-reduction.

Problem 24 If $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ then calculate A^{-1} and use this to solve the following by matrix multiplication:

$$\begin{aligned} &x - 2y + z = a \\ &-x + y + 2z = b \\ &3x + z = c \end{aligned}$$

You should check this against Problem 20.

Problem 25 Is $v = (a,b,c,d) \in \text{span}\{(1,2,2,1),(3,3,1,1)\}$? What condition(s) must be made on a,b,c,d for v to be in the span? Write down the set of vectors which is not in the span.

Problem 26 Are $(1,2,3)$ and $(0,1,1)$ in $\text{span}\{(1,3,4),(-1,3,4)\}$? If so, explicitly give the linear combinations to prove both of your assertions.

Problem 27 For what value(s) of k is the set S of vectors below a linearly independent set ?

$$S = \{(1,2,3),(2,2,2),(3,4,k)\}.$$

Problem 28 You have two sets of vectors $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$ in $\mathbb{R}^5$. Let $[T|S]$ denote the matrix formed by listing the vectors in S and T as columns. Furthermore,

$$\text{rref}[S|T] = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Use the CCP to decide if S and T form LI sets. Also, determine which vectors in T fall inside span(S).

Problem 29 Prove the concatenation proposition for columns; that is prove Proposition 2.3.14 in my notes; If$A \in R^{m \times n}$ and $B \in R^{n \times p}$ then

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)]$$

Problem 30 Suppose N is nilpotent of degree 3. Show $I+N$ is an invertible matrix.
Hint: guess the inverse has the form $aI + bN + cN^2$ for some $a,b,c \in \mathbb{R}$ and work it out.

Problem 31 Let $Z = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ and find all real solutions of $Z^2 = I$.

Problem 32 Prove part (2.) of Proposition 2.4.12. In particular prove the following claim:
If $A \in R^{n \times n}$ then $(A^p)^q = A^{pq}$ for all $p,q \in \mathbb{N} \cup \{0\}$.

Problem 33 Let $S = \{D \in \mathbb{F}^{n \times n} \mid AD = DA \text{ for all } A \in \mathbb{F}^{n \times n}\}$. Prove that the matrices in S are diagonal matrices of a particularly simple type.

Problem 34 Let $v = (1,1)$ and $w = (1,-1)$. If $Av = (1,0)$ and $Aw = (0,1)$ then find A and A^{-1} explicitly.

Problem 35 Given that $A^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$ calculate $(A^TB)^{-1}$.

Problem 36 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$. Find 2×2 elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$. Also, while we're at it, what is $[\text{Col}_1(A)|\text{Col}_2(A)]^{-1}$? Can you read off the inverse without further calculation?

Problem 37 It is also possible to perform elementary column operations on a matrix A. Furthermore, if you perform a column operation on the identity matrix then the resulting matrix when multiplied on the **right** of A will perform that same column operation. For example, to swap columns 2 and 3 of $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ I first swap columns 2 and 3 of $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to obtain $C_{2\leftrightarrow 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and you can calculate

$$AC_{2\leftrightarrow 3} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

Find column operation matrices $C_1, C_2, \cdots, C_n$ **such that** $AC_1C_2\cdots C_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
I found either $n = 3$ or $n = 5$ depending on how you like to arrange the steps. This type of calculation has further significance if you study abstract algebra. In particular, see Dummit and Foote on the Smith Normal Form. For us, it's just a good exercise for matrix arithmetic.

Problem 38 Suppose N is nilpotent of degree 2 and D is a 2×2 diagonal matrix with diagonal entries 2 and 3. If $M = \begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix}$ then is M invertible? Prove or disprove. Discuss.

MATH 321 : HOMEWORK 2 SOLUTION

PROBLEM 20

$$\begin{aligned} x - 2y + z &= 3 \\ -x + y + 2z &= 12 \\ 3x + z &= 31 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -1 & 1 & 2 & 12 \\ 3 & 0 & 1 & 31 \end{array}\right] \xrightarrow[r_3-3r_1]{r_2+r_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & -1 & 3 & 15 \\ 0 & 6 & -2 & 22 \end{array}\right]$$

$$(*) \xrightarrow[r_3+6r_2]{r_1-2r_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -27 \\ 0 & -1 & 3 & 15 \\ 0 & 0 & 16 & 112 \end{array}\right] \xrightarrow[r_3/16]{r_2/-1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -27 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\xrightarrow[r_2+3r_3]{r_1+5r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right] \quad \therefore \quad \boxed{\begin{array}{l} x = 8,\ y = 6,\ z = 7 \\ \text{or } (8, 6, 7) \text{ is the sol}^{n} \end{array}}$$

Remark: I did not follow the Gauss-Jordan algorithm in the reduction above since I saw an opportunity at $(*)$ to resolve a few things at once.

PROBLEM 21

$$\begin{aligned} ①\ \ x + z &= -5 \\ ②\ \ x + 2y &= 1 \\ ③\ \ x + 3z &= -5 \\ ④\ \ x + 2y + z + w &= 10 \end{aligned} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -5 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & -5 \\ 1 & 2 & 1 & 1 & 10 \end{array}\right] \xrightarrow[r_4-r_1]{r_2-r_1,\ r_3-r_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -5 \\ 0 & 2 & -1 & 0 & 6 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 15 \end{array}\right]$$

$$\xrightarrow{r_4-r_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -5 \\ 0 & 2 & -1 & 0 & 6 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 9 \end{array}\right] \xrightarrow[r_4-r_3/2]{r_1-r_3/2,\ r_2+r_3/2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -5 \\ 0 & 2 & 0 & 0 & 6 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right] \xrightarrow[r_3/2]{r_2/2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

Thus $\boxed{(-5, 6, 0, 9) \text{ or } x = -5,\ y = 6,\ z = 0,\ w = 9 \text{ is sol}^{n}}$

Wait: ③ − ① : $2z = 0 \therefore \underline{z = 0} \Rightarrow x + 0 = -5 \therefore \underline{x = -5}$

But, $-5 + 2y = 1$ from ② thus $2y = 6 \therefore \underline{y = 3}$

and finally ④ $\longrightarrow -5 + 6 + 0 + w = 10 \therefore w = 15 - 6 = \underline{9 = w}$.

– (way easier to solve w/o row reduction) (in this case) –

PROBLEM 22

$$x_1 + 4x_2 - 7x_3 + x_4 = 2$$
$$2x_1 + 2x_2 + x_3 + x_4 = 3$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -7 & 1 & 2 \\ 2 & 2 & 1 & 1 & 3 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cccc|c} 1 & 4 & -7 & 1 & 2 \\ 0 & -6 & 15 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{r_2/-6} \left[\begin{array}{cccc|c} 1 & 4 & -7 & 1 & 2 \\ 0 & 1 & \frac{-15}{6} & \frac{1}{6} & \frac{1}{6} \end{array}\right] \xrightarrow{r_1 - 4r_2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 1/3 & 4/3 \\ 0 & 1 & -5/2 & 1/6 & 1/6 \end{array}\right]$$

Thus $x_1 + 3x_3 + \frac{1}{3}x_4 = \frac{4}{3}$ and $x_2 - \frac{5}{2}x_3 + \frac{1}{6}x_4 = \frac{1}{6}$

So, the sol^{n} set can be extracted by solving for pivot variables, $x_1 = -3x_3 - \frac{1}{3}x_4 + \frac{4}{3}$, $x_2 = \frac{5}{2}x_3 - \frac{1}{6}x_4 + \frac{1}{6}$

$$\boxed{\text{Sol}^{n}\text{ Set} = \left\{ \left(-3x_3 - \frac{1}{3}x_4 + \frac{4}{3}, \frac{5}{2}x_3 - \frac{1}{6}x_4 + \frac{1}{6}, x_3, x_4\right) \,\middle|\, x_3, x_4 \in \mathbb{R} \right\}}$$

Remark: a nice way to express the sol^{n} is that

$$x = (x_1, x_2, x_3, x_4) = x_3(-3, 5/2, 1, 0) + x_4\left(-\frac{1}{3}, -\frac{1}{6}, 0, 1\right) + \left(\frac{4}{3}, \frac{1}{6}, 0, 0\right)$$

Which indicates $\underline{x \in \left(\frac{4}{3}, \frac{1}{6}, 0, 0\right) + \text{span}\left\{(-3, 5/2, 1, 0), \left(\frac{-1}{3}, \frac{-1}{6}, 0, 1\right)\right\}}$

If not for $(4/3, 1/6, 0, 0)$ we could write the sol^{n} set as a span.

PROBLEM 23

Solve $\begin{array}{l} \bar{2}x - \bar{2}y + \bar{3}z = \bar{0} \\ \bar{5}x + \bar{3}y - \bar{4}z = \bar{0} \end{array}$ via row-reduction over $\mathbb{Z}/7\mathbb{Z}$

$$\begin{bmatrix} \bar{2} & -\bar{2} & \bar{3} \\ \bar{5} & \bar{3} & -\bar{4} \end{bmatrix} \xrightarrow{4r_1} \begin{bmatrix} \bar{1} & -\bar{8} & \bar{5} \\ \bar{5} & \bar{3} & -\bar{4} \end{bmatrix} \xrightarrow{r_2 - 5r_1} \begin{bmatrix} \bar{1} & -\bar{8} & \bar{5} \\ \bar{0} & \overline{43} & -\overline{29} \end{bmatrix} = \begin{bmatrix} \bar{1} & -\bar{1} & \bar{5} \\ \bar{0} & \bar{1} & -\bar{1} \end{bmatrix}$$

$$\xrightarrow{r_1 + r_2} \begin{bmatrix} \bar{1} & \bar{0} & \bar{4} \\ \bar{0} & \bar{1} & -\bar{1} \end{bmatrix} \therefore \begin{array}{l} x + \bar{4}z = 0 \longrightarrow x = -\bar{4}z = \bar{3}z \\ y - z = 0 \longrightarrow y = z \end{array}$$

Thus, $\boxed{\text{sol}^{n} \text{ set} = \{ (\bar{3}z, z, z) \mid z \in \mathbb{Z}/7\mathbb{Z} \}}$

$\boxed{\text{Sol}^{\underline{n}} \text{ set} = \{ (\bar{3}, \bar{0}, \bar{0}), (\bar{3}, \bar{1}, \bar{1}), (\bar{6}, \bar{2}, \bar{2}), (\bar{2}, \bar{3}, \bar{3}), (\bar{5}, \bar{4}, \bar{4}), (\bar{1}, \bar{5}, \bar{5}), (\bar{4}, \bar{6}, \bar{6}) \}}$

explicitly there are 7 distinct sol$^{\underline{n}}$'s.

Remark: as a check, $\bar{4}\begin{bmatrix} \bar{2} \\ \bar{5} \end{bmatrix} - \begin{bmatrix} -\bar{2} \\ \bar{3} \end{bmatrix} = \begin{bmatrix} \overline{10} \\ \overline{17} \end{bmatrix} = \begin{bmatrix} \bar{3} \\ \bar{3} \end{bmatrix} = \begin{bmatrix} \bar{3} \\ -\bar{4} \end{bmatrix}$. the CCP verifies our sol$^{\underline{n}}$.

PROBLEM 24

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 + r_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 6 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow[r_3 + 6r_2]{r_1 - 2r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -1 & -2 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 16 & 3 & 6 & 1 \end{array}\right]$$

$$\xrightarrow[r_3/16]{-r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -1 & -2 & 0 \\ 0 & 1 & -3 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3/16 & 3/8 & 1/16 \end{array}\right] \xrightarrow[r_2 + 3r_3]{r_1 + 5r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/16 & -1/8 & 5/16 \\ 0 & 1 & 0 & -7/16 & 1/8 & 3/16 \\ 0 & 0 & 1 & 3/16 & 3/8 & 1/16 \end{array}\right]$$

We find $\boxed{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{16}\begin{bmatrix} -1 & -2 & 5 \\ -7 & 2 & 3 \\ 3 & 6 & 1 \end{bmatrix}}$

Remark: factor out #'s when you can!

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} -1 & -2 & 5 \\ -7 & 2 & 3 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\boxed{Av = \vec{b} \Rightarrow A^{-1}Av = A^{-1}\vec{b}}$

Thus, $\boxed{x = \frac{1}{16}(-a - 2b + 5c),\ y = \frac{1}{16}(-7a + 2b + 3c),\ z = \frac{1}{16}(3a + 6b + c)}$

Set $a = 3$, $b = 12$, $c = 31$ to find $x = 8$, $y = 6$, $z = 7$.

$\boxed{\text{PROBLEM 25}}$

Consider,

$$\left[\begin{array}{cc|c} 1 & 3 & a \\ 2 & 3 & b \\ 2 & 1 & c \\ 1 & 1 & d \end{array}\right] \xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2-2r_1} \left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & -3 & b-2a \\ 0 & -5 & c-2a \\ 0 & -2 & d-a \end{array}\right] \xrightarrow{r_4/-2} \left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & -3 & b-2a \\ 0 & -5 & c-2a \\ 0 & 1 & (a-d)/2 \end{array}\right]$$

$$\xrightarrow[\substack{r_2+3r_4 \\ r_3+5r_4}]{r_1-3r_4} \left[\begin{array}{cc|c} 1 & 0 & a-\frac{3}{2}(a-d) \\ 0 & 0 & b-2a+\frac{3}{2}(a-d) \\ 0 & 0 & c-2a+\frac{5}{2}(a-d) \\ 0 & 1 & \frac{1}{2}(a-d) \end{array}\right] \begin{array}{l} \\ \longrightarrow \\ \longrightarrow \\ \\ \end{array} \underbrace{\begin{array}{l} b-a/2-(3/2)d=0 \\ c+a/2-(5/2)d=0 \end{array}}_{\substack{\text{necessary for this} \\ \text{system of eq}^{\text{ns}}\text{ to} \\ \text{be consistent.}}}$$

Thus,

$(a,b,c,d) \in \overbrace{\operatorname{span}\{(1,2,2,1),(3,3,1,1)\}}^{W}$ only if $2b-a-3d=0$ and $2c+a-5d=0$. On the other hand,

$$\{(a,b,c,d) \mid 2c+a-5d \neq 0 \text{ or } 2b-a-3d \neq 0\} \cap W = \emptyset.$$

$\boxed{\text{PROBLEM 26}}$

Are $(1,2,3), (0,1,1) \in \operatorname{span}\{(1,3,4),(-1,3,4)\}$?

Consider

$$\left[\begin{array}{cc|c|c} 1 & -1 & 1 & 0 \\ 3 & 3 & 2 & 1 \\ 4 & 4 & 3 & 1 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2-3r_1} \left[\begin{array}{cc|c|c} 1 & -1 & 1 & 0 \\ 0 & 6 & -1 & 1 \\ 0 & 8 & -1 & 1 \end{array}\right] \xrightarrow[\substack{8r_2 \\ 6r_3}]{48r_1} \left[\begin{array}{cc|cc} 48 & -48 & 48 & 0 \\ 0 & 48 & -8 & 8 \\ 0 & 48 & -6 & 6 \end{array}\right]$$

$$\xrightarrow[r_3-r_2]{r_1+r_2} \left[\begin{array}{cc|cc} 48 & 0 & 40 & 8 \\ 0 & 48 & -8 & 8 \\ 0 & 0 & 2 & -2 \end{array}\right] \xrightarrow[r_2+4r_3]{r_1-20r_3} \left[\begin{array}{cc|cc} 48 & 0 & 0 & 48 \\ 0 & 48 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right] \xrightarrow[\substack{r_2/48 \\ r_3/2}]{r_1/48} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus $c_1\begin{bmatrix}1\\3\\4\end{bmatrix} + c_2\begin{bmatrix}-1\\3\\4\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ or $\begin{bmatrix}0\\1\\1\end{bmatrix}$ have <u>no sol$^{\text{ns}}$</u>

$$\therefore \boxed{(1,2,3), (0,1,1) \notin \operatorname{span}\{(1,3,4),(-1,3,4)\}}$$

<u>Remark:</u> However, by CCP we do see $(1,2,3)+(0,1,1)=(1,3,4)$ (examine columns 3 and 4 and note their sum is column 1)

PROBLEM 27

$S = \{(1,2,3), (2,2,2), (3,4,k)\}$ is LI iff

$\underline{c_1 (1,2,3) + c_2 (2,2,2) + c_3 (3,4,k) = 0}_{*} \iff (c_1, c_2, c_3) = (0,0,0).$

Thus, consider

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 2 & k \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 \to 2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -4 & k-9 \end{bmatrix} \xrightarrow[r_3 - 2r_2]{r_1 + r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & k-9+4 \end{bmatrix}$$

$$\xrightarrow{r_2/-2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & k-5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ provided $k - 5 \neq 0$

Thus $(*)$ has only $c_1 = c_2 = c_3 = 0$ sol^n given that $k \neq 5$.
If $k = 5$ then $\exists$ nontrivial linear dep. amongst vectors in S.
For example, $(1,2,3) + (2,2,2) = (3,4,5)$.

PROBLEM 28 $S = \{s_1, s_2, s_3\}$, $T = \{t_1, t_2\} \subset \mathbb{R}^5$

$$\text{rref}\,[S|T] = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

CCP shows $s_1 + s_2 = s_3$ and $4s_1 + 3s_2 + 2t_1 = t_2$
Also $t_2 \neq ct_1$ for any c. We find,

1. S is not LI as $s_1 + s_2 = s_3$
2. T is LI as $t_2 \neq ct_1$ (for 2 vectors this is equivalent to $c_1t_1 + c_2t_2 = 0$ having ~~no~~ nontrivial sol^n's)
3. $t_1, t_2 \notin \text{span}(S)$
   as $\nexists$ way to write
   columns 4 and 5 in terms
   of columns 1,2,3 which correspond to s_1, s_2, s_3 respective.

Remark: $2t_1 - t_2 = 4s_1 + 3s_2$ can be seen from CCP.

$\boxed{\text{PROBLEM 29}}$

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ then $(col_j(AB))_i = (AB)_{ij}$ and,

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \quad = \; : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

$$= \sum_{k=1}^{n} A_{ik} (col_j(B))_k \quad : \text{def}^{\underline{n}} \text{ of column \& component}$$

$$= (A\, col_j(B))_i \quad : \text{matrix-column product def}^{\underline{n}}$$

Thus $(col_j(AB))_i = (A\, col_j(B))_i$ for $i = 1, 2, \dots, m$

Thus $col_j(AB) = A\, col_j(B)$ for each $j \in \mathbb{N}_p$

That is, $AB = [A\, col_1(B) \,|\, A\, col_2(B) \,|\, \cdots \,|\, A\, col_p(B)]$. //

$\boxed{\text{PROBLEM 30}}$. Suppose $N^3 = 0$, but $N^2 \neq 0$. Find $(I+N)^{-1}$. Consider,

$$(I+N)(aI + bN + cN^2) = aI + bN + cN^2 + aN + bN^2 + c\underbrace{N^3}_{0}$$

We want $(I+N)(I+N)^{-1} = I$ thus solve,

$$aI + (a+b)N + (b+c)N^2 = I$$

Let $a = 1$ then $a + b = 0 \Rightarrow \underline{b = -1}$. and $b + c = 0 \Rightarrow \underline{c = 1}$.

So, we make an <u>educated guess</u> $(I+N)^{-1} = I - N + N^2$,

$$(I+N)(I - N + N^2) = I - N + N^2 + N - N^2 + N^3 \stackrel{\checkmark}{=} I$$

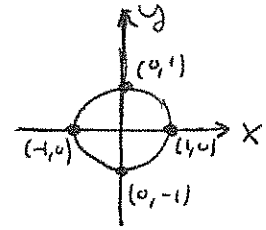
Thus $\underline{(I+N)^{-1} = I - N + N^2}$. //

— (we proved $AB = I \Rightarrow B = A^{-1}$) —
no need to check $BA = I$.

PROBLEM 31 Let $Z = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ solve $Z^2 = I$ over $\mathbb{R}$.

$$Z^2 = \left[\begin{array}{c|c} x^2+y^2 & 2xy \\ \hline 2xy & y^2+x^2 \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, solve $\overbrace{x^2+y^2 = 1}^{\text{circle}}$ and $\overbrace{2xy = 0}^{\text{coord. axes.}}$.



If $x = 0$ then $0^2+y^2 = 1 \Rightarrow y = \pm 1$

If $y = 0$ then $x^2+0^2 = 1 \Rightarrow x = \pm 1$

Hence, $\exists$ four sol^{n}s, $\boxed{(\pm 1, 0) \text{ and } (0, \pm 1)}$

meaning, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

PROBLEM 32

Let $A \in \mathbb{R}^{n\times n}$ and note $(A^p)^0 = I = A^{p(0)}$ and $(A^p)^1 = A^p = A^{p(1)}$ for arbitrary $p \in \mathbb{N}\cup\{0\}$. Hence, $(A^p)^q = A^{pq}$ for $q = 0, 1$ for fixed $p \in \mathbb{N}\cup\{0\}$. Suppose inductively $(A^p)^q = A^{pq}$ for some $q \in \mathbb{N}\cup\{0\}$. Consider,

$$\begin{aligned}
(A^p)^{q+1} &= A^p (A^p)^q && : \text{ def}^{n} \text{ of matrix power.} \\
&= A^p A^{pq} && : \text{ by induction hypothesis.} \\
&= A^{p+pq} && : \text{ by Part (1.) of Prop. 2.4.12} \\
& && \quad \text{as proved in my Lecture Notes.} \\
&= A^{p(q+1)}
\end{aligned}$$

Thus, $(A^p)^q = A^{pq}$ by induction on q for all $q \in \mathbb{N}\cup\{0\}$. But, $p \in \mathbb{N}\cup\{0\}$ was arbitrary hence $(A^p)^q = A^{pq}\ \forall p, q \in \mathbb{N}\cup\{0\}$.

PROBLEM 33

$$S = \{ D \in \mathbb{F}^{n\times n} \mid AD = DA \quad \forall A \in \mathbb{F}^{n\times n} \}$$

Let $D = \sum_{i,j=1}^{n} D_{ij} E_{ij} \in S$ then $AD = DA \quad \forall A \in \mathbb{F}^{n\times n}$

hence $E_{kl} D = D E_{kl}$ for all $k, l \in \mathbb{N}_n$. This means,

$$E_{kl} D = \sum_{i,j} D_{ij} E_{kl} E_{ij} = \sum_{i,j} D_{ij} \delta_{li} E_{kj} = \sum_{j} D_{lj} E_{kj}$$

$$D E_{kl} = \sum_{i,j} D_{ij} E_{ij} E_{kl} = \sum_{i,j} D_{ij} \delta_{jk} E_{il} = \sum_{i} D_{ik} E_{il}$$

Thus,

$$\sum_{i=1}^{n} D_{li} E_{ki} = \sum_{i=1}^{n} D_{ik} E_{il}$$

Consider the (a,b)-th component, $(E_{ki})_{ab} = \delta_{ka} \delta_{ib}$ and $(E_{il})_{ab} = \delta_{ia} \delta_{lb}$ thus,

$$\sum_{i=1}^{n} D_{li} \delta_{ka} \delta_{ib} = \sum_{i=1}^{n} D_{ik} \delta_{ia} \delta_{lb}$$

$$\Rightarrow \quad \underline{D_{lb} \delta_{ka} = D_{ak} \delta_{lb} \quad \forall l, k, a, b \in \mathbb{N}_n} \quad (*)$$

If $l \neq b$ then $D_{lb} \delta_{ka} = 0 \Rightarrow D_{lb} = 0$ as $\exists\, k, a$ for which $\delta_{ka} \neq 0$. thus nondiagonal entries in D are zero.

Consider $l = b$ so $\delta_{lb} = 1$ we have from $(*)$

$$D_{bb} \delta_{ka} = D_{ak} \implies D_{bb} = D_{aa} \text{ for } k = a$$

But, a, b are arbitrary. Thus $D_{11} = D_{22} = \cdots = D_{nn} = \lambda$ and we conclude $\boxed{D = \lambda I \text{ for some } \lambda \in \mathbb{F}}$

PROBLEM 34

Let $v = (1,1)$ and $w = (1,-1)$
If $Av = (1,0)$ and $Aw = (0,1)$ then find A and A^{-1} explicitly

$$Av = e_1 \Rightarrow A^{-1}Av = A^{-1}e_1 \Rightarrow col_1(A^{-1}) = v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Aw = e_2 \Rightarrow A^{-1}Aw = A^{-1}e_2 \Rightarrow col_2(A^{-1}) = w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus $\boxed{A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$ and $A = (A^{-1})^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

Hence $\boxed{A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}$, Check Answer: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

PROBLEM 35 $A^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$.

$$(A^TB)^{-1} = B^{-1}(A^T)^{-1} \quad \longleftarrow \text{socks-shoes}$$

$$= B^{-1}(A^{-1})^T \quad \longleftarrow (A^T)^{-1} = (A^{-1})^T$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -2 & 2 \\ 18 & 22 \end{bmatrix}}$$

PROBLEM 36

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{-r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-r_2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = E_3$$

$$E_3 E_2 E_1 A = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} \quad (*)$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$EA = E\left[\begin{matrix} 1 & 1 \\ 3 & 2 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix}\right] = \left[E\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \middle| E\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} -2 \\ 3 \end{matrix}\right]$$

$$\therefore \boxed{\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1} = E = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}}$$

(I need to do a little calculation to be explicit however, from $(*)$ it is clear $E = E_3E_2E_1$ is $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1}$.)

PROBLEM 37

fun with column ops.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow[c_3 - c_1]{c_2 - c_1} \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & -3 \end{bmatrix} \xrightarrow{c_2/-2} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -3 \end{bmatrix} \xrightarrow[c_3 + 3c_2]{c_1 - 3c_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(2×3)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_2 - c_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_3 - c_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_2/-2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_1 - 3c_2} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_3 + 3c_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = C_5$$

$$AC_1C_2C_3C_4C_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

PROBLEM 38

Let $N, D \in \mathbb{F}^{2\times 2}$ where $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

and $N^2 = 0$ but $N \neq 0$ (nilpotent degree 2)

Let $M = \begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix}$. Is M invertible?

Prove or disprove. Discuss.

M^{-1} exists $\Rightarrow \exists X, Y, Z, W \in \mathbb{F}^{2\times 2}$ such that $M^{-1} = \left[\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right]$ and $MM^{-1} = I_4 = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & I_2 \end{array}\right]$

That is, $\begin{bmatrix} N & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{c|c} NX + 0Z & NY + 0W \\ \hline 0X + DZ & 0Y + DW \end{array}\right] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c} NX & NY \\ \hline DZ & DW \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

Need <u>all 4</u> of

$NX = I, \quad NY = 0, \quad DZ = 0, \quad DW = I.$

But, $NX = I$ fails since,

$NNX = NI \Rightarrow N^2X = N \Rightarrow \underline{0 = N}.$

But $N \neq 0$ $\therefore$ <u>M^{-1} does not exist as N^{-1} d.n.e.</u> //