Your PRINTED NAME indicates you read Chapter 2 and §3.1 of the notes: ____________.

We assume $\mathbb{F}$ is a field and V, W are vector spaces over $\mathbb{F}$.

Problem 17 Let $W = \{(x+y, y+z, 2x-w, x+y+z+w) \mid x,y,z,w \in \mathbb{F}\}$. Is W a subspace of $\mathbb{F}^4$?

Problem 18 Friedberg, Insel and Spence 5th edition, §1.2#21, page 16.

Problem 19 Let V_1, V_2 be vector spaces over $\mathbb{F}$. Suppose $W_1 \leq V_1$ and $W_2 \leq V_2$.
Prove $W_1 \times W_2 \leq V_1 \times V_2$.

Problem 20 Consider $S = \{t^2+1, t^2-1, t+1, t-1\} \subseteq P_4(\mathbb{R})$.

(a.) Show S is **not** linearly independent

(b.) Find a basis for $W = span\{S\}$

Problem 21 Let $W = \{A \in \mathbb{R}^{3\times 3} \mid tr(A) = 0 \ \& \ A^T = A\}$. Show $W \leq \mathbb{R}^{3\times 3}$ and find a basis for W. What is the dimension of W ?

Problem 22 Let $W = \{(x,y,z) \in \mathbb{Z}_5^3 \mid x - y - 2z = 0\}$.

(a.) Find a basis for W.

(b.) Calculate $dim(W)$.

(c.) Calculate the cardinality of W, we denote this $|W|$.

Problem 23 Consider $V = \mathbb{C}^{2\times 2}$ as a vector space over $\mathbb{R}$. Let $W = \{A \in V \mid A^T = -iA\}$.
Find a basis for W.

Problem 24 Let $\beta = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}$. Find the coordinates of $v = (a,b,c,d)$ with respect to β.

Problem 25 Let $\beta = \{1, (t-1), (t-1)^2, (t-1)^3\}$. Let $f(t) = a + bt + ct^2 + dt^3$ where $a,b,c,d \in \mathbb{R}$. Calculate $[f(t)]_\beta$. *Hint: use calculus.*

Problem 26 Consider $W = \{f(t) \in P_2(\mathbb{R}) \mid \int_0^1 f(t)dt = 0\}$. Show $W \leq P_2(\mathbb{R})$ and find a basis for W.

Problem 27 Suppose $W_1 = \{f(x) \in P_4(\mathbb{R}) \mid f(1) = 0\}$ and $W_2 = \{f(x) \in P_4(\mathbb{R}) \mid f(2) = 0\}$ and $W_3 = span\{x+1, x^4-1\}$

(a.) Find a basis for $W_1 \cap W_2$

(b.) Find a basis for $W_1 \cap W_3$.

Problem 28 Friedberg, Insel and Spence 5th edition, §1.5#13, page 42.

Problem 29 Friedberg, Insel and Spence 5th edition, §1.5#20, page 43.

Problem 30 Friedberg, Insel and Spence 5th edition, §1.6#4, page 55.

Problem 31 Let $S, T \in \mathcal{L}(V)$. Prove $S \circ T \in \mathcal{L}(V)$.

Problem 32 Friedberg, Insel and Spence 5th edition, §2.1#21, page 76.

Mission 2 Solution

$\boxed{P17}$ $W = \{(x+y,\ y+z,\ 2x-w,\ x+y+z+w) \mid x,y,z,w \in \mathbb{F}\}$

$\Rightarrow W = \text{span}\{(1,0,2,1),(1,1,0,1),(0,1,0,1),(0,0,-1,1)\}$

Thus $W \leq \mathbb{F}^4$ since W is formed by a span.

$\boxed{P18}$ Let V and W be vector spaces over $\mathbb{F}$.

Let $Z = \{(v,w) \mid v \in V \text{ and } w \in W\}$

define $(v_1,w_1)+(v_2,w_2) = (v_1+v_2,\ w_1+w_2)$ and $c(v_1,w_1) = (cv_1, cw_1)$

for all $(v_1,w_1),(v_2,w_2) \in Z$ and $c \in \mathbb{F}$

To prove Z is vector space we need to verify all the axioms hold. Notice $v_1+v_2 \in V$ and $w_1+w_2 \in W$ for $v_1,v_2 \in V$ and $w_1,w_2 \in W$ hence $+: Z \times Z \to Z$ and similarly since scalar mult. in V and W are single-valued it follows $(c,(v_1,w_1)) \mapsto (cv_1, cw_1)$ is a function $\cdot: \mathbb{F} \times Z \to Z$.

$$\begin{aligned}
(v_1,w_1)+(v_2,w_2) &= (v_1+v_2,\ w_1+w_2) &&: \text{def}^{n} \text{ in } Z \\
&= (v_2+v_1,\ w_2+w_1) &&: + \text{ commutes in vector spaces } V \,\&\, W \\
&= (v_2,w_2)+(v_1,w_1)
\end{aligned}$$

Thus $+$ commutes in Z. Likewise,

$$\begin{aligned}
u+(v+w) &= (u_1,u_2)+(v_1+w_1,\ v_2+w_2) &&\leftarrow \text{def}^{n} \text{ of } + \text{ in } Z \\
&= (u_1+(v_1+w_1),\ u_2+(v_2+w_2)) \\
&= ((u_1+v_1)+w_1,\ (u_2+v_2)+w_2) &&: \text{assoc. of } + \text{ in } V, W. \\
&= (u_1+v_1,\ u_2+v_2)+(w_1,w_2) &&\leftarrow \text{def}^{n} \text{ of } + \text{ in } Z \\
&= (u+v)+w
\end{aligned}$$

P18 continued

Let $(v,w) \in Z$ then notice $(0_V, 0_W)$ yields

$$(v,w) + (0_V, 0_W) = (v + 0_V,\ w + 0_W) = (v,w)$$

where I've denoted $0_V \in V$ and $0_W \in W$ for the zero vector of V & W respective.

Hence, $(0,0) = 0_Z$ (omitting V & W on $0_V, 0_W$)

Next, if $(v,w) \in Z$ then notice $(-v,-w) \in Z$ where $-v + v = 0_V$ and $-w + w = 0_W$ hence

$$\begin{aligned}(-v,-w) + (v,w) &= (-v+v, -w+w) \cancel{= (0,0)} \\ &= (0_V, 0_W) \\ &= 0_Z.\end{aligned}$$

Thus $+$ is commutative, associative, has zero 0_Z and every element of Z has additive inverse.

Next suppose $1 \in \mathbb{F}$ then $1 \cdot v = v \ \forall v \in V$ and $1 \cdot w = w \ \forall w \in W$ hence

$$1 \cdot (v,w) = (1 \cdot v,\ 1 \cdot w) = (v,w)$$

for all $(v,w) \in Z$. If $a, b \in \mathbb{F}$ then

$(a+b)\cdot v = a \cdot v + b \cdot v$ and $(a+b)\cdot w = a \cdot w + b \cdot w$

thus

$$\begin{aligned}(a+b)\cdot(v,w) &= ((a+b)\cdot v,\ (a+b)\cdot w) \\ &= (a \cdot v + b \cdot v,\ a \cdot w + b \cdot w) \\ &= (a \cdot v,\ a \cdot w) + (b \cdot v,\ b \cdot w) \\ &= a \cdot (v,w) + b \cdot (v,w) \\ &= (a+b) \cdot (v,w)\end{aligned}$$

P18 continued

Let (v_1, w_1), $(v_2, w_2) \in Z$ and $c \in \mathbb{F}$ then

$$c \cdot [(v_1, w_1) + (v_2, w_2)] = c \cdot (v_1 + v_2, w_1 + w_2) \quad : \text{def}^{\text{n}} \text{ of } + \text{ in } Z$$

$$= (c \cdot (v_1 + v_2), c \cdot (w_1 + w_2)) \quad : \text{def}^{\text{n}} \text{ of } \cdot \text{ in } Z$$

$$= (c \cdot v_1 + c \cdot v_2, c \cdot w_1 + c \cdot w_2) \quad : \text{def}^{\text{n}} \text{ of } \cdot \text{ in } V \,\&\, W$$

$$= (c \cdot v_1, c \cdot w_1) + (c \cdot v_2, c \cdot w_2) \quad : \text{def}^{\text{n}} \text{ of } + \text{ in } Z$$

$$= c \cdot (v_1, w_1) + c \cdot (v_2, w_2).$$

I'll leave it to the reader to check

$$a \cdot (b \cdot (v, w)) = (ab) \cdot (v, w)$$

for all $a, b \in \mathbb{F}$ and $(v, w) \in Z$.

$\boxed{P19}$ Let V_1 & V_2 be vector spaces over $\mathbb{F}$ and suppose $W_1 \leq V_1$ and $W_2 \leq V_2$. We proved $V_1 \times V_2$ is vector space over $\mathbb{F}$ in <u>P18</u>.

Notice $(0,0) \in W_1 \times W_2$ thus $W_1 \times W_2 \neq \emptyset$.

Also $W_1 \leq V_1$ and $W_2 \leq V_2 \Rightarrow W_1 \times W_2 \subseteq V_1 \times V_2$.

Let $x, y \in W_1 \times W_2$ and $c \in \mathbb{F}$ then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Since $W_1 \leq V_1$ we know $cx_1 + y_1 \in W_1$ and as $W_2 \leq V_2$ we also know $cx_2 + y_2 \in W_2$. Thus,

$$\begin{aligned} cx + y &= c(x_1, x_2) + (y_1, y_2) \\ &= (cx_1 + y_1, cx_2 + y_2) \in W_1 \times W_2 \end{aligned}$$

Thus $W_1 \times W_2 \leq V_1 \times V_2$ by the subspace test.

$\boxed{P20}$ Consider $S = \{t^2+1, t^2-1, t+1, t-1\} \subseteq P_4(\mathbb{R})$

(a.) $\underbrace{(t^2+1)-(t^2-1) = 2 = (t+1)-(t-1)}_{\text{I just saw 2 as common.}}$

$$\underbrace{t+1 = (t^2+1)-(t^2-1)+(t-1)}_{\text{linear dependence in } S}$$

$\therefore$ $\underline{S \text{ not LI}}$.

(b.) Find <u>basis</u> for $W = \text{span}(S)$

I suspect $\beta = \{t^2+1, t^2-1, t-1\}$ is LI

Let's check,

$$c_1(t^2+1) + c_2(t^2-1) + c_3(t-1) = 0$$

$$(c_1+c_2)t^2 + c_3 t + (c_1-c_2-c_3)1 = 0$$

By LI of the monomial basis $\{1, t, t^2\}$ we may equate coeff. of $1, t, t^2$

$$\begin{array}{l|l} t^2 & c_1 + c_2 = 0 \\ t & c_3 = 0 \\ 1 & c_1 - c_2 - c_3 = 0 \end{array} \qquad \begin{array}{r} c_1 + c_2 = 0 \\ + \quad c_1 - c_2 = 0 \\ \hline 2c_1 = 0 \end{array}$$

Thus $c_1 = c_2 = c_3 = 0$ and we find $\text{span}(S) = \text{span}\,\beta = W$ and β LI

$\therefore$ $\beta = \{t^2+1, t^2-1, t-1\}$ is basis for W.

$\boxed{P21}$ Let $W = \{ A \in \mathbb{R}^{3\times 3} \mid tr(A) = 0,\ A^T = A \}$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \qquad tr(A) = A_{11} + A_{22} + A_{33} = 0$$

$$A^T = A \Rightarrow A_{ij} = A_{ji} \quad \text{(only interesting for } i \neq j\text{)}$$

$$A_{12} = A_{21}$$
$$A_{13} = A_{31}$$
$$A_{23} = A_{32}$$

Therefore,

$$A = \left[\begin{array}{c|c|c} -A_{22} - A_{33} & A_{12} & A_{13} \\ \hline A_{12} & A_{22} & A_{23} \\ \hline A_{13} & A_{23} & A_{33} \end{array}\right]$$

$$= \underline{A_{12}(E_{12} + E_{21}) + A_{13}(E_{13} + E_{31}) + A_{23}(E_{23} + E_{32})} +$$
$$\underline{+ A_{22}(E_{22} - E_{11}) + A_{33}(E_{33} - E_{11})} \ *$$

Thus $W = \text{span}\Big\{ \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\beta} \Big\}$

hence $W \leq \mathbb{R}^{3\times 3}$ and

it is clear β is LI from setting $* = 0$

which yields $A_{12} = 0,\ A_{13} = 0,\ A_{23} = 0,\ A_{22} = 0,\ A_{33} = 0$

thus β serves as basis for W and

$$\dim(W) = |\beta| = \boxed{5}$$

$\boxed{P22}$ (a.) $W = \{ (x,y,z) \in \mathbb{Z}_5^3 \mid x - y - 2z = 0 \}$

Hence $W = \{ (y+2z, y, z) \mid y, z \in \mathbb{Z}_5 \}$

$= \{ y(1,1,0) + z(2,0,1) \mid y, z \in \mathbb{Z}_5 \}$

$= \text{span}_{\mathbb{Z}_5} \{ (1,1,0), (2,0,1) \}$

Notice $(1,1,0) \neq k(2,0,1) \therefore \beta = \{(1,1,0),(2,0,1)\}$ is LI as thus serves as basis for $W \leq \mathbb{Z}_5^3$.

(b.) $\dim(W) = |\beta| = \boxed{2}$

(c.) $|W| = 25$ since y, z serve as free parameters which each range over 5 distinct values in $\mathbb{Z}_5$.

$\boxed{P23}$ $V = \mathbb{C}^{2\times 2}$ over $\mathbb{R}$ and $W = \{A \in V \mid A^T = -iA\}$

Find basis for W

$$A = \left[\begin{array}{c|c} x_{11} + iy_{11} & x_{12} + iy_{12} \\ \hline x_{21} + iy_{21} & x_{22} + iy_{22} \end{array}\right] \quad (x_{ij}, y_{ij} \in \mathbb{R} \ \forall i,j)$$

$$\underbrace{-iA = \left[\begin{array}{c|c} y_{11} - ix_{11} & y_{12} - ix_{12} \\ \hline y_{21} - ix_{21} & y_{22} - ix_{22} \end{array}\right] = \left[\begin{array}{c|c} x_{11} + iy_{11} & x_{21} + iy_{21} \\ \hline x_{12} + iy_{12} & x_{22} + iy_{22} \end{array}\right]}_{\text{this gives \underline{eight} real equations}}$$

$(1,1)$:	$y_{11} = x_{11}$ & $-x_{11} = y_{11}$	$\Rightarrow$	$\underline{x_{11} = 0}$.
$(1,2)$:	$y_{12} = x_{21}$ & $-x_{12} = y_{21}$	$\Rightarrow$	$\underline{x_{21} = 0}$.
$(2,1)$:	$y_{21} = x_{12}$ & $-x_{21} = y_{12}$	$\Rightarrow$	$\underline{x_{12} = 0}$.
$(2,2)$:	$y_{22} = x_{22}$ & $-x_{22} = y_{22}$	$\Rightarrow$	$\underline{x_{22} = 0}$.

P23 continued

we found $X_{ij} = 0 \quad \forall i,j$ thus

$$A = \left[\begin{array}{c|c} i\,y_{11} & i\,y_{12} \\ \hline i\,y_{21} & i\,y_{22} \end{array}\right]$$

I'll apply the defining condition $A^T = -iA$ once more since I believe we've not exhausted the implications of the eight eqns from the last page.

$$A^T = \left[\begin{array}{c|c} i\,y_{11} & i\,y_{21} \\ \hline i\,y_{12} & i\,y_{22} \end{array}\right] = -iA = \left[\begin{array}{c|c} y_{11} & y_{12} \\ \hline y_{21} & y_{22} \end{array}\right]$$

But, $y_{ij} \in \mathbb{R} \quad \forall i,j$ by construction thus $y_{11} = 0$, $y_{12} = 0$, $y_{21} = 0$, $y_{22} = 0$

Ha. Well, this is easier than I thought. $W = \{0\}$ thus $\beta = \phi$ is the basis (span $\phi = \{0\}$, $\beta = \phi$ LI)

$\boxed{P24}$ $\beta = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}$

Find coordinates of $v = (a,b,c,d)$ with respect to β.

Let $[v]_\beta = (c_1, c_2, c_3, c_4)$ then

$$(a,b,c,d) = c_1(1,0,0,0) + c_2(1,1,0,0) + c_3(1,1,1,0) + c_4(1,1,1,1)$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 + c_4 \\ c_2 + c_3 + c_4 \\ c_3 + c_4 \\ c_4 \end{bmatrix}$$

Therefore,

$$\begin{aligned} c_4 &= d \\ c_3 &= c - c_4 = c - d \\ c_2 &= b - c_3 - c_4 = b - (c-d) - d = b - c \\ c_1 &= a - c_2 - c_3 - c_4 = a - (b-c) - (c-d) - d = a - b \end{aligned}$$

Thus

$$\boxed{[v]_\beta = (a-b,\ b-c,\ c-d,\ d)}$$

Let me <u>check</u> my answer,

$$(a-b)(1,0,0,0) + (b-c)(1,1,0,0) + (c-d)(1,1,1,0) + d(1,1,1,1) =$$

$$= (a-b+b-c+c-d+d,\ b-c+c-d+d,\ c-d+d,\ d)$$

$$= (a, b, c, d)$$

$$= v$$ (woo hoo)

$\boxed{P25}$ $f(t) = a + bt + ct^2 + dt^3$

I showed how $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(t-1)^n$ could be used to calculate $[f(t)]_\beta$ where $\beta = \{1, t-1, (t-1)^2, (t-1)^3\}$ in-class

I'll use my other trick here. Add zero

$$f(t) = a + b(t-1+1) + c(t-1+1)^2 + d(t-1+1)^3$$

$$= a + b(t-1) + b + c(t-1)^2 + 2c(t-1) + c + d(t-1)^3 + 3d(t-1)^2 + 3d(t-1) + d$$

$$= a+b+c+d + (b+2c+3d)(t-1) + (c+3d)(t-1)^2 + d(t-1)^3$$

Thus, $[f(t)]_\beta = \begin{bmatrix} a+b+c+d \\ b+2c+3d \\ c+3d \\ d \end{bmatrix}$

You can compare against

$$f(t) = f(1) + f'(1)(t-1) + \tfrac{1}{2}f''(1)(t-1)^2 + \tfrac{1}{6}f'''(1)(t-1)^3$$

$$= a+b+c+d + (b+2c+3d)(t-1) + \tfrac{1}{2}(2c+6d)(t-1)^2 + \tfrac{1}{6}6d(t-1)^3$$

(CALCULUS WAY IS EASIER THAN MY ADD ZERO TRICK HERE)

$\boxed{P26}$ $W = \{ f(t) \in P_2(\mathbb{R}) \mid \int_0^1 f(t)\,dt = 0 \}$

Let $at^2 + bt + c \in W$ then

$$\int_0^1 (at^2 + bt + c)\,dt = 0$$

$$\Rightarrow \left(\tfrac{1}{3}at^3 + \tfrac{1}{2}bt^2 + ct\right)\Big|_0^1 = \frac{a}{3} + \frac{b}{2} + c = 0$$

Thus $\underline{c = -a/3 - b/2}$. We find

$$at^2 + bt + (-a/3 - b/2) = a\left(t^2 - \tfrac{1}{3}\right) + b\left(t - \tfrac{1}{2}\right)$$

hence $W = \text{span}\{t^2 - 1/3,\ t - 1/2\} \leq P_2(\mathbb{R})$

as we've shown W is a span.

– (of course you could also prove $W \leq P_2(\mathbb{R})$ by the subspace test if you prefer, but since I'm tasked with finding basis this path is natural) –

Notice $\boxed{\beta = \{t^2 - 1/3,\ t - 1/2\}}$ is LI since clearly $t^2 - \frac{1}{3} \neq k(t - 1/2)$ for some $k \in \mathbb{R}$.

Thus β is basis for W.

$\boxed{P27}$ $W_1 = \{ f(x) \in P_4(\mathbb{R}) \mid f(1) = 0 \}$

$W_2 = \{ f(x) \in P_4(\mathbb{R}) \mid f(2) = 0 \}$

$W_3 = \text{span}\{ x+1, \, x^4 - 1 \}$

(a.) Find basis for $W_1 \cap W_2$

if $f(x) \in W_1 \cap W_2$ then $f(x) \in W_1$ and $f(x) \in W_2$ thus $f(1) = 0$ and $f(2) = 0$. By factor Th^{m} since $\deg(f(x)) \leq 4$ we deduce

$$f(x) = (x-1)(x-2)(Ax^2 + Bx + C)$$

Thus $W_1 \cap W_2 = \text{span}\{ x^2(x-1)(x-2), \, x(x-2)(x-1), \, (x-1)(x-2) \}$

$$\boxed{\beta = \{ x^4 - 3x^3 + 2x^2, \, x^3 - 3x^2 + 2x, \, x^2 - 3x + 2 \}}$$

I've shown span $\beta = W_1 \cap W_2$, it remains to prove LI of β. Consider

$$c_1(x^4 - 3x^3 + 2x^2) + c_2(x^3 - 3x^2 + 2x) + c_3(x^2 - 3x + 2) = 0$$

Equate coefficients of

x^4 | $c_1 = 0$.

x^3 | $-3c_1 + c_2 = 0 \quad \therefore \quad c_2 = 0$.

x^2 | $2c_1 - 3c_2 + c_3 = 0 \quad \therefore \quad c_3 = 0$.

Thus β is LI and so β serves as basis for $W_1 \cap W_2$.

P27 continued

Remark: $W_3 = \text{span}\{x+1, x^4-1\}$ is not as nice to combine with $W_1 = \{f(x) \in P_4(\mathbb{R}) \mid f(1) = 0\}$

As a rule of thumb, when finding intersections it is nice to have each subspace described by an equation or equations. I'll show how to "fix" the given description of W_3

$$W_3 = \{ c_1(x+1) + c_2(x^4-1) \mid c_1, c_2 \in \mathbb{R} \}$$

If $ax^4+bx^3+cx^2+dx+e \in W_3$ then

$$ax^4+bx^3+cx^2+dx+e = c_1(x+1) + c_2(x^4-1)$$

Thus,

$$\left.\begin{array}{l|l} x^4 & a = c_2 \\ x^3 & b = 0 \\ x^2 & c = 0 \\ x & d = c_1 \\ 1 & e = c_1 - c_2 \end{array}\right\}$$

goal is to eliminate c_1 & c_2

$$\left.\begin{array}{l} c_1 = d. \\ c_2 = a. \end{array}\right\} \text{ done!}$$

We have a, d free and $b = c = 0$ whereas $e = c_1 - c_2 = d - a$.

$$W_3 = \{ ax^4+bx^3+cx^2+dx+e \mid b = c = 0,\ e = d-a,\ a, d \in \mathbb{R} \}$$

P27 continued

If $f(x) = ax^4+bx^3+cx^2+dx+e \in W_3 \cap W_1$ then $b = c = 0$ and $e = d-a$ since $f(x) \in W_3$. Moreover, $f(x) \in W_1$ thus $f(1) = a+b+c+d+e = 0$. I'll use row-reduction to be safe, $(a+e-d=0)$

$$\begin{array}{ccccc} a & b & c & d & e \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Oh, so, $b = c = d = 0$ and $\underline{a = -e}$.

that is $f(x) = a(x^4-1)$ and so $W_1 \cap W_3 = \text{span}\{x^4-1\}$ and $\beta = \{x^4-1\}$ serves as basis for $W_1 \cap W_3$.

(Remark: $x^4-1 = g(x)$ has $g(1)=0 \therefore x^4-1 \in W_1$ so the calculation's result is not too surprising.)

P28) § 1.5 # 13 from p. 42

Let V be vector space over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$.

(a.) Let u, v be distinct vectors in V.
Prove $\{u, v\}$ LI $\iff$ $\{u+v, u-v\}$ LI.

(b.) Let u, v, w be distinct vectors in V.
Prove $\{u, v, w\}$ is LI $\iff$ $\{u+v, u+w, v+w\}$ LI

(a.) Assume u, v distinct. Suppose $\{u, v\}$ is LI.

Consider $c_1(u+v) + c_2(u-v) = 0$ then

$(c_1 + c_2)u + (c_1 - c_2)v = 0 \Rightarrow \underline{c_1 + c_2 = 0}$ ① & $\underline{c_1 - c_2 = 0}$ ②

by LI of $\{u, v\}$. Then add and subtract

Equations ① and ② to obtain $2c_1 = 0$ & $2c_2 = 0$

thus $c_1 = c_2 = 0$ and we've shown $\{u+v, u-v\}$ LI.

//

Conversely suppose $\{u+v, u-v\}$ is LI.

Consider $c_1 u + c_2 v = 0$ then add zero,

$c_1(u + v - v) + c_2(v - u + u) = 0$

$\Rightarrow c_1(u+v) + c_2(-(u-v)) + c_2 u - c_1 v = 0$

Well, I don't see it. Let's try another path

P28 continued

Suppose $\{u+v, u-v\}$ is LI and suppose $\{u,v\}$ is not LI towards $\rightarrow\leftarrow$

Then we can write $v = ku$ for some $k \neq 0$

and so $\underline{u+v = u+ku = (1+k)u}$ $*$

where as $u-v = u-ku = (1-k)u$

Notice u, v distinct so $k \neq 1$ and we find $u = \frac{1}{1-k}(u-v)$ we substitute into $*$,

$$u+v = (1+k)\frac{1}{1-k}(u-v) = \lambda(u-v)$$

where $\lambda = \frac{1+k}{1-k}$. If $1+k \neq 0$ then $\{u+v, u-v\}$ is <u>not</u> LI since $u+v$ is scalar multiple of $u-v$. Likewise, if $1+k=0$ then $u+v=0$ thus $\{u+v, u-v\}$ is not LI as any set containing zero is not LI.

Thus we reach a $\rightarrow\leftarrow$ and so we conclude $\{u,v\}$ is LI.//

(I'd rather not use $\rightarrow\leftarrow$ if I can help it... I'm going to try the algebra again $\lrcorner$

P28 continued

$$c_1 u + c_2 v = b_1 (u+v) + b_2 (u-v)$$
$$= (b_1 + b_2) u + (b_1 - b_2) v$$

$$\left.\begin{aligned} c_1 &= b_1 + b_2 \\ c_2 &= b_1 - b_2 \end{aligned}\right\} \text{ solve for } b_1 \text{ \& } b_2$$

$$b_1 = \frac{1}{2}(c_1 + c_2)$$
$$b_2 = \frac{1}{2}(c_1 - c_2)$$

Thus,

$$c_1 u + c_2 v = \frac{1}{2}(c_1 + c_2)(u+v) + \frac{1}{2}(c_1 - c_2)(u-v)$$

Consequently

$c_1 u + c_2 v = 0 \Rightarrow \frac{1}{2}(c_1 + c_2) = 0$ and $\frac{1}{2}(c_1 - c_2) = 0$ by LI of $\{u+v, u-v\}$

Then $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which yields $c_1 = c_2 = 0 \therefore \{u, v\}$ LI //

P28 continued

(b.) $\Rightarrow$ Suppose $\{u, v, w\}$ is LI.

Consider $c_1(u+v) + c_2(u+w) + c_3(v+w) = 0$

Hence $(c_1+c_2)u + (c_1+c_3)v + (c_2+c_3)w = 0$

Then by LI of $\{u, v, w\}$ we find

$$\underbrace{c_1 + c_2}_{\text{I}} = \underbrace{c_1 + c_3}_{\text{II}} = \underbrace{c_2 + c_3}_{\text{III}} = 0$$

Notice I and II yield $c_2 = c_3$ which gives III the form $2c_2 = 0$ $\therefore$ $c_2 = 0 \Rightarrow c_3 = 0 \Rightarrow c_1 = 0$.

Therefore $\{u+v, u+w, v+w\}$ is LI.//

$\Leftarrow$ Suppose $\{u+v, u+w, v+w\}$ is LI.

Assume $c_1 u + c_2 v + c_3 w = 0$. We wish to rewrite the equation above by finding b_1, b_2, b_3 for which

$$c_1 u + c_2 v + c_3 w = b_1(u+v) + b_2(u+w) + b_3(v+w)$$

This suggests we solve (if possible!)

$$\left.\begin{aligned} c_1 &= b_1 + b_2 \\ c_2 &= b_1 + b_3 \\ c_3 &= b_2 + b_3 \end{aligned}\right\} \text{ solve for } b_1, b_2, b_3 \;\}$$

P28 continued

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & c_1 \\ 1 & 0 & 1 & c_2 \\ 0 & 1 & 1 & c_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & c_1 \\ 0 & -1 & 1 & c_2 - c_1 \\ 0 & 1 & 1 & c_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & c_1 + (c_2 - c_1) \\ 0 & -1 & 1 & c_2 - c_1 \\ 0 & 0 & 2 & c_3 + c_2 - c_1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & c_2 \\ 0 & 1 & -1 & c_1 - c_2 \\ 0 & 0 & 1 & \frac{1}{2}(-c_1 + c_2 + c_3) \end{array}\right]$$

$$\sim \begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_2 - \frac{1}{2}(-c_1 + c_2 + c_3) \\ 0 & 1 & 0 & c_1 - c_2 + \frac{1}{2}(-c_1 + c_2 + c_3) \\ 0 & 0 & 1 & \frac{1}{2}(c_2 + c_3 - c_1) \end{array}\right] & \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \end{array}$$

$$b_1 = \frac{1}{2}(c_1 + c_2 - c_3)$$
$$b_2 = \frac{1}{2}(c_1 - c_2 + c_3)$$
$$b_3 = \frac{1}{2}(-c_1 + c_2 + c_3)$$

Hence,

$$c_1 u + c_2 v + c_3 w = \frac{1}{2}(c_1 + c_2 - c_3)(u+v) + \frac{1}{2}(c_1 - c_2 + c_3)(u+w) + \frac{1}{2}(-c_1 + c_2 + c_3)(v+w)$$

and by LI of $\{u+v, u+w, v+w\}$ we find $c_1 + c_2 - c_3 = 0$ and $c_1 - c_2 + c_3 = 0$ and $-c_1 + c_2 + c_3 = 0$ from which it follows $c_1 = c_2 = c_3 = 0 \therefore \{u, v, w\}$ LI//

$\boxed{P29}$ §1.5 #20 p. 43

Let $f, g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ be functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$ where $r \neq s$. Prove f and g are LI in $\mathcal{F}(\mathbb{R},\mathbb{R})$.

Suppose $\underline{c_1 e^{rt} + c_2 e^{st} = 0}_{\,(1)}$ for all $t \in \mathbb{R}$
then differentiating with respect to t yields
$\underline{c_1 r e^{rt} + c_2 s e^{st} = 0}_{\,(2)}$ for all $t \in \mathbb{R}$.

Evaluate (1) and (2) at $t = 0$ to obtain

$$\left\{ \begin{array}{l} c_1 + c_2 = 0 \\ rc_1 + sc_2 = 0 \end{array} \right\} \longrightarrow \begin{bmatrix} 1 & 1 \\ r & s \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note $\begin{bmatrix} 1 & 1 \\ r & s \end{bmatrix}^{-1} = \dfrac{1}{s-r} \begin{bmatrix} s & -1 \\ -r & 1 \end{bmatrix}$ since $r \neq s$

and multiplying by this matrix yields $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus $c_1 = c_2 = 0$ and we find $\{e^{rt}, e^{st}\}$ is LI.

(Remark: alternatively, if $e^{rt} = ke^{st}$ then $k = \dfrac{e^{rt}}{e^{st}} = e^{(r-s)t}$ which is absurd since $r \neq s$)

P30 §1.6 # 4 p. 55

Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$ and $3x - 2$ generate $P_3(\mathbb{R})$?

Notice

$$\dim\left(\operatorname{span}\{x^3 - 2x^2 + 1,\ 4x^2 - x + 3,\ 3x - 2\}\right) \leq 3$$

However, $\dim(P_3(\mathbb{R})) = |\{1, x, x^2, x^3\}| = 4$

thus the given polynomials <u>cannot</u> span $P_3(\mathbb{R})$ as <u>all</u> bases for $P_3(\mathbb{R})$ have 4 vectors.

$\boxed{P31}$ $S, T \in \mathcal{L}(V)$

then consider $x, y \in V$ and $c \in \mathbb{F}$

$$(S \circ T)(cx + y) = S(T(cx + y)) \quad T \in \mathcal{L}(V)$$
$$= S(cT(x) + T(y)) \quad S \in \mathcal{L}(V)$$
$$= c\, S(T(x)) + S(T(y))$$
$$= c(S \circ T)(x) + (S \circ T)(y)$$

Thus $S \circ T \in \mathcal{L}(V)$.

$\boxed{P32}$ § 2.1 # 21, p. 76

$$V = \{ (a_1, a_2, \dots) \mid a_i \in \mathbb{F} \text{ for each } i \in \mathbb{N} \}$$

then $(a + b)_i = a_i + b_i \quad \forall i \in \mathbb{N}$

and $(ca)_i = c a_i \quad \forall i \in \mathbb{N}$ where $c \in \mathbb{F}$

Let us define,

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots) : \text{LEFT SHIFT}$$
$$U(a_1, a_2, \dots) = (0, a_1, a_2, \dots) : \text{RIGHT SHIFT}$$

P32 continued

(a.) Show T and U are linear.

Let $a, b \in V$ and $c \in \mathbb{F}$ then

$$ca + b = (ca_1 + b_1, ca_2 + b_2, ca_3 + b_3, \dots)$$

Thus

$$\begin{aligned} T(ca+b) &= (ca_2 + b_2, ca_3 + b_3, \dots) \\ &= c(a_2, a_3, \dots) + (b_2, b_3, \dots) \\ &= cT(a) + T(b) \end{aligned}$$

Likewise,

$$\begin{aligned} U(ca+b) &= (0, ca_1 + b_1, ca_2 + b_2, \dots) \\ &= c(0, a_1, a_2, \dots) + (0, b_1, b_2, \dots) \\ &= cU(a) + U(b) \end{aligned}$$

Thus $T, U \in \mathcal{L}(V)$.

(b.) Prove T is onto but not one-to-one.

Let $a = (a_1, a_2, \dots) \in V$ then observe

$T(0, a_1, a_2, \dots) = (a_1, a_2, \dots) = a \therefore T$ onto V.

If $T(a) = 0$ then $(a_2, a_3, \dots) = 0$ for $a = (a_1, a_2, a_3, \dots) \therefore \text{Ker}(T) = \text{span}\{(1, 0, \dots)\}$ and so T is not 1-to-1.

(for example, $T(0) = (0, 0, \dots) = T(1, 0, \dots)$.)

P32 continued

(c.) Prove U is one-to-one, but not onto

Let $a = (a_1, a_2, \dots) \in \operatorname{Ker}(U)$ then

$U(a) = (0, a_1, a_2, \dots) = 0 \Rightarrow a = 0$

thus $\operatorname{Ker}(U) = \{0\}$ and it follows that U is injective.

To see U is not onto notice $(1, 0, 0, \dots) \in V$ yet

$$U(a) = (0, a_1, a_2, \dots) = (1, 0, 0, \dots)$$

is impossible to solve as $0 \neq 1$.