Your PRINTED NAME indicates you read Chapter 2 of the notes: ____________________.

Assume $\mathbb{F}$ is a field and $n \in \mathbb{N}$. Notice the notation $V(\mathbb{F})$ indicates $V$ is a vector space over the field $\mathbb{F}$. For example, $\mathbb{C}^{3\times 3}(\mathbb{R})$ is a real vector space whereas $\mathbb{C}^{3\times 3}(\mathbb{C})$ is a complex vector space.

**Problem 11** In each of the following, give at least one reason why $W$ is not a subspace of $V$ over $\mathbb{F}$

**(a.)** $W = \{(x, y, z) \mid x, y, z \geq 0\}$ in $V = \mathbb{R}^3$ over $\mathbb{F} = \mathbb{R}$,

**(b.)** $W = \{1 + ax + bx^2 \mid a, b \in \mathbb{R}\}$ in $V = \mathbb{R}[x]$ over $\mathbb{F} = \mathbb{R}$,

**(c.)** $W = \{(x, y) \mid x, y \in \mathbb{R}\}$ in $V = \mathbb{C}^2$ over $\mathbb{F} = \mathbb{C}$,

**(d.)** $W = \mathbb{R} - \mathbb{Q}$ in $V = \mathbb{R}$ over $\mathbb{Q}$.

**Problem 12** Consider $S = \{1, x + 1, x^2 + 2x + 1, x^3 + 3x^2 + 3x + 1\}$. Prove $S$ is a linearly independent spanning set for $P_3(\mathbb{R})$ ( these polynomials up to third order with real coefficients).

**Problem 13** Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define $W = \{M \in \mathbb{R}^{2\times 2} \mid MJ = JM\}$.

**(a.)** show $W$ is a subspace of $\mathbb{R}^{2\times 2}$

**(b.)** Find a basis $\beta$ for $W$ and calculate $\left[\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]\right]_\beta$ in terms of $a$ and $b$.

**Problem 14** Consider $W = \{A \in \mathbb{C}^{3\times 3} \mid A^T = -A,\ A_{12} + iA_{23} = 0\}$. Prove $W$ is a subspace of $\mathbb{C}^{3\times 3}(\mathbb{R})$. Find a basis $\beta$ for $W$ and write the formula for $\Phi_\beta : W \to \mathbb{R}^n$ for appropriate $n$.

**Problem 15** Consider $W = \{(a, a + b, a + b + c, a + b + c) \mid a, b, c \in \mathbb{F}\}$. Prove $W$ is a subspace of $\mathbb{F}^4$ and prove $dim(W) = 3$. What is the cardinality of $W$ ? (break into cases as needed).

**Problem 16** Define subspaces of $P_3(\mathbb{R})$ as follows:

$$U = \{f(x) \in P_3(\mathbb{R}) \mid f(2) = 0\} \qquad \& \qquad V = \text{span}\{3 + x, 3 - x\}.$$

Find a basis for $U \cap V$. Use Theorem 2.7.5 to decide what $U + V$ must be on the basis of dimensional arguments.

**Problem 17** Let $V = C^0(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Define

$$W = \left\{ f \in W \;\middle|\; \lim_{x\to 1} f(x) = 0 \right\}.$$

Show $W \leq V$ and prove that $W$ is infinite dimensional.

**Problem 18** If $V_1$ and $V_2$ are vector spaces over $\mathbb{F}$ then $V = V_1 \times V_2$ defines a vector space over $\mathbb{F}$ by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad \& \qquad c(x_1, x_2) = (cx_1, cx_2)$$

for all $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ and $c \in \mathbb{F}$. Let $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$. Prove:

**(a.)** If $S_1$ and $S_2$ are LI then $(S_1 \times \{0\}) \cup (\{0\} \times S_2)$ is LI subset of $V_1 \times V_2$

**(b.)** If $V_1 = \text{span}(S_1)$ and $V_2 = \text{span}(S_2)$ then $\text{span}\left((S_1 \times \{0\}) \cup (\{0\} \times S_2)\right) = V_1 \times V_2$.

**Problem 19** Consider $A \in \mathbb{R}^{m\times n}$. Define

$$W = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax = 0 \ \& \ \exists(z \in \mathbb{R}^n)(Az = y)\}$$

Prove $W \leq \mathbb{R}^n \times \mathbb{R}^m$. Calculate $dim(W)$.

**Problem 20** Let $W = \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}$. Prove $W \leq \mathbb{R}^{n\times n}$ and calculate $dim(W)$.