Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 3 of the notes: ________________.

Problem 19 Let $M = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 3 & 5 & 2 \end{bmatrix}$.

(a.) Calculate rref(M) via Gaussian elimination over $\mathbb{R}$,

(b.) Find the solution set of $x + y + 3z = 1$ and $3x + 3y + 5z = 2$,

(c.) Find the solution set of $x_1 + x_2 + 3x_3 + x_4 = 0$ and $3x_1 + 3x_2 + 5x_3 + 2x_4 = 0$.

Remark: the problem above intends to get you to think about how a single rref matrix may have more than one use. There are many different angles we can look at a given row-reduction.

Problem 20 Suppose $[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 3 & 1 & 0 \end{array}\right]$ is the augmented coefficient matrix of a system

$$2x + y = 1, \quad x + 2y = 2, \quad x + 3y + z = 0$$

over a field $\mathbb{F}$. Find the solution set via Gaussian elimination for:

(a.) $\mathbb{F} = \mathbb{Z}_3 = \{0, 1, 2\}$ where $2^{-1} = 2$ since $2(2) = 4 = 1$ etc.

(b.) $\mathbb{F} = \mathbb{Q}$

(c.) $\mathbb{F} = \mathbb{R}$

Problem 21 Given 2 equations in 4 unknowns over $\mathbb{Z}_5$ what are the possible sizes of the solution set?

Problem 22 Let k be a real parameter. Consider the system:

$$\begin{aligned} x + ky &= 2 \\ kx + 4y &= 1 \end{aligned}$$

Find the solution set for all three cases.

Problem 23 Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ over $\mathbb{R}$. Find elementary matrices $E_1, E_2, \dots, E_k$ for which $E_k \cdots E_2 E_1 A = I$. Also, express A as a product of elementary matrices. Is A invertible?

Problem 24 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{bmatrix}$ over $\mathbb{R}$. Find elementary matrices $E_1, E_2, \dots, E_k$ for which $E_k \cdots E_2 E_1 A = \text{rref}(A)$. If possible, write A as a product of elementary matrices. If not, explain why it is not possible.

Problem 25 Suppose $A \in \mathbb{R}^{4\times 4}$ such that vectors v_1, v_2, v_3, v_4 give

$$Av_1 = e_1 + e_2 + e_3 + e_4, \qquad Av_2 = e_2 + e_3 + e_4, \qquad Av_3 = e_3 + e_4, \qquad Av_4 = e_4.$$

Is A invertible? If so, find a formula for A^{-1} based on the given vectors v_1, v_2, v_3, v_4.

Problem 26 Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 7 \\ 0 & 1 & 2 \end{bmatrix}$. Calculate A^{-1}.

Problem 27 For which value of k is $\{(1,2,3),(1,0,1),(2,k,8)\}$ a linearly dependent set?

Problem 28 Either prove or disprove that $S = \{(1,2,3,4),(0,1,1,0),(0,0,1,1)\}$ is a LI set in $\mathbb{R}^4$

Problem 29 Let $v = (1,1,1,2)$ and $w = (2,2,5,-1)$.

(a.) is $v \in \text{span}\{(1,1,4,-3),(2,2,8,-6)\}$?

(b.) is $w \in \text{span}\{(1,1,4,-3),(2,2,8,-6),v\}$?

Problem 30 (feel free to use technology for the row reduction here) Let

$$S = \{(1,1,1,2,2,2),(4,0,0,0,0,4)\}$$

and

$$T = \{(1,1,0,0,6,7),(0,0,1,2,-4,-5),(3,3,3,3,3,3)\}$$

(a.) Is $S \subseteq \text{span}(T)$? If not, which vector(s) in S are **not** in span(T) ?

(b.) Is $T \subseteq \text{span}(S)$? If not, which vector(s) in T are **not** in span(S) ?

Problem 31 Consider $x + 2y + 3z = 0$ and $x - y + z = 0$. Find a linearly independent set S such that the solution set of the pair of given equations is span(S).

Problem 32 Consider $S = \{(1,2,3,4),(0,1,1,7)\}$. Find a system of equations in x_1, x_2, x_3, x_4 for which span(S) is the solution set.

Problem 33 Let $V = \text{span}\{(1,1,1),(2,4,2)\}$ and $W = \text{span}\{(3,2,2),(1,0,1)\}$. Find a LI set S such that span$(S) = V \cap W$. *Hint: it's easier to combine equations than spans.*

Problem 34 Suppose A, B are invertible matrices of the same size. Show $M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is invertible.

Problem 35 We define $I \in R^{n\times n}$ by $I_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. If $x \in R^{n\times 1}$ then show $Ix = x$. Then, use transposition to derive $vI = v$ for all $v \in R^{1\times n}$.

Problem 36 Let $f(x) = Ax^3 + Bx^2 + Cx + D$ be a cubic polynomial for which $f(-1) = 0$ and $f'(-1) = 3$ and $f''(-1) = 8$ and $f'''(-1) = 6$. Find the values of A, B, C, D.

MISSION 2 SOLUTION

$\boxed{P19}$ $M = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 3 & 5 & 2 \end{bmatrix}$

(a.) $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 3 & 5 & 2 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & -4 & -4 \end{bmatrix} \xrightarrow{r_2/-4} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 - 3r_2} \boxed{\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \text{rref}(M)}$

(b.) $\begin{aligned} x + y + 3z &= 1 \\ 3x + 3y + 5z &= 2 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right] = M$

Since $\text{rref}(M) = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ we find the given system is equivalent to $\left\{ \begin{aligned} x + y &= -2 \\ z &= 1 \end{aligned} \right\}$. Thus $x = -2 - y$, $z = 1$ with y free. Hence, $\boxed{\text{sol}^n \text{ set} = \{(-2-y, y, 1) \mid y \in \mathbb{R}\}}$

(c.) $\left(\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= 0 \\ 3x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \end{aligned} \right)_*$ has aug. coeff matrix of $\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 3 & 3 & 5 & 2 & 0 \end{array}\right]$ and it's simple to see

$\text{rref} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 3 & 3 & 5 & 2 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \begin{aligned} x_1 &= -x_2 + 2x_4 \\ x_3 &= -x_4 \end{aligned}$

Thus, $\boxed{\{(-x_2 + 2x_4, x_2, -x_4, x_4) \mid x_2, x_4 \in \mathbb{R}\}.}$ Sol^n set for $*$

$\boxed{P20}$ Consider $\begin{matrix} 2x+y=1 \\ x+2y=2 \\ x+3y+z=0 \end{matrix} \longrightarrow [A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 3 & 1 & 0 \end{array}\right]$

(a.) over $\mathbb{F} = \mathbb{Z}_3 = \{0, 1, 2\}$,

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 3 & 1 & 0 \end{array}\right] \xrightarrow{2r_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 3 & 1 & 0 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[r_1 - 2r_3]{r_1 - 2r_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} x + z = 0 \\ y + z = 1 \end{matrix}$$

Hence, sol^n set $= \{(-z, 1-z, z) \mid z \in \mathbb{Z}_3\}$

aka sol^n set $= \{(0,1,0), (-1,0,1), (-2,-1,2)\}$

better yet, $\boxed{\text{sol}^n \text{ set} = \{(0,1,0), (2,0,1), (1,2,2)\}}$

(b.) over $\mathbb{F} = \mathbb{Q}$,

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 3 & 1 & 0 \end{array}\right] \xrightarrow[2r_3]{2r_2} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 2 & 4 & 0 & 4 \\ 2 & 6 & 2 & 0 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 5 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{r_2/3} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & 2 & -1 \end{array}\right] \xrightarrow[r_3 - 5r_2]{r_1 - r_2} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -6 \end{array}\right] \xrightarrow[r_3/2]{r_1/2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]}_{x=0,\ y=1,\ z=-3}$$

thus $\boxed{\text{sol}^n \text{ set is } \{(0, 1, -3)\}}$

(c.) same calculation as (b.), again $\underline{\{(0,1,-3)\} \text{ is sol}^n \text{ set}}$.

$\boxed{P21}$ Given 2 equations in 4 unknowns over $\mathbb{Z}_5$ we either have 0, 1 or 2 independent conditions on x_1, x_2, x_3, x_4 in $\mathbb{Z}_5$ which indicates either $4-0$, $4-1$, $4-2 = 4, 3, 2$ free variables in the sol^n set. Since these free variables are taken from $\mathbb{Z}_5$ we find either 5^4, 5^3 or 5^2 sol^n's in the sol^n set: $\boxed{625,\ 125 \text{ or } 25 \text{ sol}^n\text{'s.}}$

P22

$$\begin{aligned} x + ky &= 2 \quad ① \\ kx + 4y &= 1 \quad ② \end{aligned}$$

Solve ① for $x = 2 - ky$.

Hence ② yields $k(2 - ky) + 4y = 1$

Thus, $\underline{(k^2 - 4)y = 2k - 1}_{*}$

If $k = 2$ then $*$ provides $0 = 2(2) - 1 = 3$.
If $k = -2$ then $*$ provides $0 = -4 - 1 = -5$.
Therefore, $k = \pm 2$ yield no sol^{n}'s. However, if $k \neq \pm 2$ then from $*$ we find $\underline{y = \frac{2k-1}{k^2-4}}$

and $x = 2 - k\left(\frac{2k-1}{k^2-4}\right) = \frac{2k^2 - 8 - 2k^2 + k}{k^2-4} = \frac{k-8}{k^2-4}$

Thus $\boxed{\left\{\left(\frac{k-8}{k^2-4}, \frac{2k-1}{k^2-4}\right)\right\} \text{ is the sol}^{n} \text{ set for } k \neq \pm 2}$

Also, ϕ is the sol^{n} set for $k = \pm 2$.

__Remark__: the coefficient matrix $\begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix}^{-1} = \frac{1}{4-k^2}\begin{bmatrix} 4 & -k \\ -k & 1 \end{bmatrix}$ provided $k \neq \pm 2$. So we could derive the sol^{n} by multiplication $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ which means,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4-k^2}\begin{bmatrix} 4 & -k \\ -k & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{4-k^2}\begin{bmatrix} 8-k \\ -2k+1 \end{bmatrix}$$

(which is what we already found.)

Don't get the wrong idea, just because the matrix inverse formula fails it does not forbid sol^{n}'s exist. For example, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ has sol^{n} $x = 1, y = 1$ (and many more)

$\boxed{\text{P23}}$ (over $\mathbb{R}$)

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{r_4 - r_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{r_3 + 5r_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{-r_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

and $E_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ yield $E_5 E_4 E_3 E_2 E_1 A = rref(A) = I$.

Thus $A^{-1} = E_5 E_4 E_3 E_2 E_1$ with E_1, E_2, E_3, E_4, E_5 as given above.

<u>Remark</u>: grader, beware, answer <u>NOT</u> unique, could do row-ops in different order so to derive different presentation of A^{-1} as product of elementary matrices.

$\boxed{\text{P24}}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{rref(A)}$

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then it follows that,

$$\underline{E_4 E_3 E_2 E_1 A = rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Hence A^{-1} d.n.e. since $rref(A) \neq I$ and so we cannot write A is product of elementary matrices since A^{-1} exists iff A is product of elem. matrices.

P25 Given V_1, V_2, V_3, V_4 s.t. $AV_1 = (1,1,1,1)$, $AV_2 = (0,1,1,1)$ $AV_3 = (0,0,1,1)$ and $AV_4 = (0,0,0,1)$ we show A^{-1} exists. Notice, to find A^{-1} it suffices to solve

$$Ay_1 = e_1 \quad , \quad Ay_2 = e_2 , \quad Ay_3 = e_3 , \quad Ay_4 = e_4$$

as to construct $A^{-1} = [y_1 | y_2 | y_3 | y_4]$. We clearly may use $y_4 = V_4$. Then, notice:

$$A(V_3 - V_4) = AV_3 - AV_4 = (0,0,1,1) - (0,0,0,1) = (0,0,1,0).$$

$$A(V_2 - V_3) = AV_2 - AV_3 = (0,1,1,1) - (0,0,1,1) = (0,1,0,0).$$

$$A(V_1 - V_2) = AV_1 - AV_2 = (1,1,1,1) - (0,1,1,1) = (1,0,0,0)$$

Thus $\underline{A^{-1} = \left[V_1 - V_2 \,\middle|\, V_2 - V_3 \,\middle|\, V_3 - V_4 \,\middle|\, V_4 \right]}$.

P26 Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 7 \\ 0 & 1 & 2 \end{bmatrix}$. Consider,

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 7 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_2]{r_1 - 2r_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[r_2 - r_3]{r_1 - r_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \xrightarrow{r_1/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -4 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\underbrace{\hphantom{\begin{matrix} 3/2 & -1/2 & -1/2 \end{matrix}}}_{A^{-1}}$$

Thus,

$$\boxed{A^{-1} = \begin{bmatrix} 3/2 & -1/2 & -1/2 \\ -4 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}}$$

we can check, $A^{-1}A = I. = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\boxed{P27}$ For which value of k is $\{(1,2,3), (1,0,1), (2,k,8)\}$ is linearly dependent?

Consider $c_1(1,2,3) + c_2(1,0,1) + c_3(2,k,8) = 0 = (0,0,0)$ if there exist nonzero sol^{n}'s (not all $c_1, c_2, c_3 = 0$) then we have linear dependence of the given set of vectors. Consider then

$$c_1 + c_2 + 2c_3 = 0$$
$$2c_1 + kc_3 = 0$$
$$3c_1 + c_2 + 8c_3 = 0$$

hence, omitting the zero-column as is logically permissable here,

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & k \\ 3 & 1 & 8 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 8 \\ 2 & 0 & k \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & -2 & k-4 \end{bmatrix} \xrightarrow{r_2/-2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & k-4 \end{bmatrix}$$

$$\xrightarrow[r_3 + 2r_2]{r_1 - r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & k-6 \end{bmatrix} \xrightarrow[r_2 + \frac{1}{k-6} r_3]{r_1 - \frac{3}{k-6} r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hookrightarrow \left.\begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}\right\} \begin{matrix} \text{means} \\ \text{set is} \\ LI. \end{matrix}$$

$(k \neq 6)$

We find $\boxed{k = 6}$ yields $c_1 = -3c_3$, $c_2 = c_3$ for any $c_3 \in \mathbb{R}$

In contrast, if $k \neq 6$ then only the trivial combination of $(1,2,3), (1,0,1), (2,k,8)$ yields $(0,0,0)$.

$\boxed{P28}$ $S = \{(1,2,3,4), (0,1,1,0), (0,0,1,1)\}$. Is S LI?

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{r_2 - 2r_1 \\ r_3 - 3r_1 \\ r_4 - 4r_1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_4 - r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the above calculation shows $c_1(1,2,3,4) + c_2(0,1,1,0) + c_3(0,0,1,1) = 0$ has only the $c_1 = c_2 = c_3 = 0$ sol^{n}. Thus $\boxed{S \text{ is } LI}$

$\boxed{P29}$ Let $v = (1,1,1,2)$ & $w = (2,2,5,-1)$

(a.) to decide if $v \in \text{span}\{(1,1,4,-3), (2,2,8,-6)\}$ we calculate,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 4 & 8 & 1 \\ -3 & -6 & 2 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 - r_1} \\ \xrightarrow{r_3 - 4r_1} \\ \xrightarrow{r_4 + 3r_1} \end{array} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 5 \end{array}\right] \Rightarrow$$

no sol^n's to $c_1(1,1,4,-3) + c_2(2,2,8,-6) = v$ exist. Thus

$\underline{v \notin \text{span}\{(1,1,4,-3), (2,2,8,-6)\}}$.

(b.) to decide if $w \in \text{span}\{(1,1,4,-3), (2,2,8,-6), v\}$ consider,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & 8 & 1 & 5 \\ -3 & -6 & 2 & -1 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 - r_1} \\ \xrightarrow{r_3 - 4r_1} \\ \xrightarrow{r_4 + 3r_1} \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 5 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To solve $\underline{c_1(1,1,4,-3) + c_2(2,2,8,-6) + c_3 v = w}_{*}$

we find $c_1 = 1 - 2c_2$, $c_3 = 1$ solves $*$ for any choice of c_2

For example, set $c_2 = 0$ hence use $c_1 = 1$, $c_3 = 1$ to get

$$\underbrace{(1,1,4,-3) + (1,1,1,2) = (2,2,5,-1)}$$

explicitly demonstrates $\underline{(2,2,5,-1) \in \text{span}\{(1,1,4,-3), (2,2,8,-6), v\}}$.

$\boxed{P30}$ $S = \{(1,1,1,2,2,2), (4,0,0,0,0,4)\}$

$T = \{(1,1,0,0,6,7), (0,0,1,2,-4,-5), (3,3,3,3,3,3)\}$

(a.) to decide if $S \subseteq \text{span}(T)$ we calculate,

$$\text{rref}\big[[T]\big|[S]\big] = \text{rref}\left[\begin{array}{ccc|cc} 1 & 0 & 3 & 1 & 4 \\ 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 6 & -4 & 3 & 2 & 0 \\ 7 & -5 & 3 & 2 & 4 \end{array}\right] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

from the above calculation we see $(4,0,0,0,0,4) \notin \text{span}(T)$

whereas $(1,1,0,0,6,7) + (0,0,1,2,-4,-5) = (1,1,1,2,2,2)$

P30 continued,

(b.) $\text{rref}\left[[S]\,|\,[T]\right] = \text{rref}\left[\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 3 \\ 2 & 0 & 0 & 2 & 3 \\ 2 & 0 & 6 & -4 & 3 \\ 2 & 4 & 7 & -5 & 3 \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ } mean span eqⁿs are inconsistent.

it follows that <u>none</u> of the vectors in T are found within span(S).

P31 Consider $x + 2y + 3z = 0$ and $x - y + z = 0$. Find a LI set S such that span(S) = solⁿ set of the given eqⁿs.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right] \xrightarrow{r_2/-3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2/3 & 0 \end{array}\right]$$

$\xrightarrow{r_1 - 2r_2} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \end{array}\right]$ thus $x = \frac{-5}{3}z$, $y = \frac{-2}{3}z$

hence $(x, y, z) = \left(-\frac{5}{3}z, \frac{-2}{3}z, z\right) = z\left(-\frac{5}{3}, \frac{-2}{3}, 1\right)$

thus solⁿ set $= \text{span}\{(-5/3, -2/3, 1)\} = \underbrace{\text{span}\{(-5, -2, 3)\}}_{\text{for the fractional averse}}$

P32 Consider $S = \{(1, 2, 3, 4), (0, 1, 1, 7)\}$ find System of eqⁿs in x_1, x_2, x_3, x_4 for which span(S) is solⁿ set

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Consider, $a(1, 2, 3, 4) + b(0, 1, 1, 7) = (x_1, x_2, x_3, x_4) \in \text{span}(S)$

we wish to eliminate a & b. The above is 4 eqⁿs in 6 unknowns

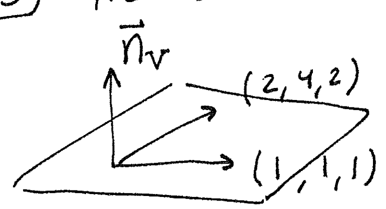
$$\left[\begin{array}{cccccc} a & b & x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 & -1 & 0 \\ 4 & 7 & 0 & 0 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & -7/17 & 1/17 \\ 0 & 1 & 0 & 0 & 4/17 & -3/17 \\ 0 & 0 & 1 & 0 & -7/17 & 1/17 \\ 0 & 0 & 0 & 1 & -10/17 & -1/17 \end{array}\right]$$ } eqⁿs w/o a & b.

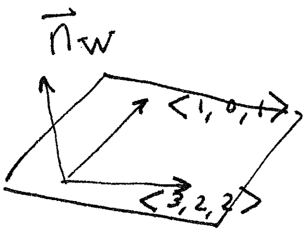
Thus $x_1 - \frac{7}{17}x_3 + \frac{1}{17}x_4 = 0$ and $x_2 - \frac{10}{17}x_3 - \frac{1}{17}x_4 = 0$

or, to clean things up a bit, $\boxed{\begin{array}{l} 17x_1 - 7x_3 + x_4 = 0 \\ 17x_2 - 10x_3 - x_4 = 0 \end{array}}$

(check my solⁿ, both $(1,2,3,4)$ and $(0,1,1,7)$ are solⁿs)

$\boxed{P33}$ the smart way to do this is via cross-products





$$\vec{n}_V = (2,4,2) \times (1,1,1) = (4-2, 2-2, 2-4) = \underbrace{(2,0,-2)}$$

perpendicular to both (2,4,2) and (1,1,1) serves as normal

$$V = \{(x,y,z) \mid 2x - 2z = 0\}$$

$$\vec{n}_W = (1,0,1) \times (3,2,2) = (0-2, 3-2, 2-0) = \underbrace{(-2,1,2)}$$

$\perp$ to both (1,0,1) & (3,2,2) so serves as normal to W which is sol$^{\text{n}}$ set of $-2x+y+2z=0$

$$W = \{(x,y,z) \mid -2x+y+2z=0\}$$

Then, if $(x,y,z) \in V \cap W$ we need $\underbrace{2x-2z=0}_{\text{Ⅰ.}}$ and $\underbrace{-2x+y+2z=0}_{\text{Ⅱ.}}$

Clearly Ⅰ. reveals $x = z$ and so Ⅱ. simplifies to $\underbrace{y = -2z+2x}_{y=0}$.

Thus $(x,y,z) \in V \cap W$ has the form $(x,0,x) = x(1,0,1)$ and we find $\boxed{V \cap W = \text{span}\{(1,0,1)\}}$.

For those w/o cross-products, follow approach of $\boxed{P32}$ solve $a(2,4,2) + b(1,1,1) = (x,y,z)$ and eliminate a & b to find cartesian eq$^{\text{n}}$ (x,y,z).

$$\begin{aligned} 2a + b - x &= 0 \\ 4a + b - y &= 0 \\ 2a + b - z &= 0 \end{aligned} \qquad \begin{aligned} 2a - y + x &= 0 \\ 2a - y + z &= 0 \\ \hline x - z &= 0 \end{aligned} \quad \therefore \ \underline{x = z}$$

Or, $a(1,0,1) + b(3,2,2) = (x,y,z)$

$$\begin{aligned} a + 3b - x &= 0 \\ 2b - y &= 0 \\ a + 2b - z &= 0 \end{aligned} \longrightarrow \begin{aligned} b - x + z &= 0 \\ b - y/2 &= 0 \\ \hline -x + y/2 + z &= 0 \end{aligned} \hookrightarrow \underline{-2x + y + 2z = 0}$$

$\boxed{P34}$ § A, B invertible and let $M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Consider the eq$^{\underline{n}}$ $Mv = 0$ if we write $v = \begin{bmatrix} x \\ y \end{bmatrix}$ where $x, y \in \mathbb{F}^n$ and so $M \in \mathbb{F}^{(2n)\times(2n)}$ whereas $A, B \in \mathbb{F}^{n\times n}$

$$Mv = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ay \\ Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus $Ay = 0$ and $Bx = 0$ which yield $y = 0$ & $x = 0$ by invertibility of A & B. Hence $Mv = 0 \Leftrightarrow v = 0$ and so we find M^{-1} exists.

OR, can derive formula, assuming again A^{-1}, B^{-1} exist we seek C, D, E, F

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} C & D \\ E & F \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\left[\begin{array}{c|c} AE & AF \\ \hline BC & BD \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

$$\left.\begin{array}{lll} AE = I & \Rightarrow & E = A^{-1} \\ AF = 0 & \Rightarrow & F = 0 \end{array}\right\} \text{multiplied by } A^{-1} \text{ to obtain} \Rightarrow$$

$$\left.\begin{array}{lll} BC = 0 & \Rightarrow & C = 0 \\ BD = I & \Rightarrow & D = B^{-1} \end{array}\right\} \text{multiplied by } B^{-1} \text{ to obtain} \Rightarrow$$

Thus $\boxed{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}}$ (easy to check $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$).

$\boxed{P35}$ Let $I \in R^{n\times n}$ be defined by $I_{ij} = \delta_{ij}$.

Let $x \in R^{n\times 1}$ and calculate,

$$(Ix)_i = \sum_{j=1}^{n} I_{ij}\, x_j = \sum_{j=1}^{n} \underbrace{\delta_{ij}\, x_j}_{\substack{\text{only nonzero} \\ \text{for } i=j}} = x_i$$

The above holds for each $i = 1, 2, .., n$ thus $\underline{Ix = x}$.

Let $x = v^T$ for $v \in R^{1\times n}$ then $x \in R^{n\times 1}$ and

So $Ix = x \Rightarrow \underset{*}{\underline{Iv^T = v^T}}$. Transpose $*$ and use socks-shoes identity $(Iv^T)^T = (v^T)^T \Rightarrow v^{TT} I^T = v^{TT} \Rightarrow \underline{vI = v}$.

as $(I^T)_{ij} = I_{ji} = \delta_{ji} = \delta_{ij} = I_{ij}$ ($I^T = I$ clear from $\delta_{ij} = \delta_{ji}$).

Thus $vI = v$ $\forall v \in R^{1\times n}$ and $Ix = x$ $\forall x \in R^{n\times 1}$.

$\boxed{P36}$

$$\left.\begin{aligned}
f(x) &= Ax^3 + Bx^2 + Cx + D &&\Rightarrow f(-1) = -A + B - C + D \overset{①}{=} 0 \\
f'(x) &= 3Ax^2 + 2Bx + C &&\Rightarrow f'(-1) = 3A - 2B + C \overset{②}{=} 3 \\
f''(x) &= 6Ax + 2B &&\Rightarrow f''(-1) = -6A + 2B \overset{③}{=} 8 \\
f'''(x) &= 6A &&\Rightarrow f'''(-1) = 6A \overset{④}{=} 6
\end{aligned}\right\}\ \text{given in problem statement.}$$

Begin with ④ to obtain $\underline{A = 1}$.

Then ③ provides $2B = 8 + 6A = 14 \Rightarrow \underline{B = 7}$.

Next, ② yields $C = 3 + 2B - 3A = 3 + 14 - 3 = 14$ that is $\underline{C = 14}$.

Finally, ① yields $D = C - B + A = 14 - 7 + 1 = \underline{8}$

In summary, $\boxed{f(x) = x^3 + 7x^2 + 14x + 8.}$