Same rules as Homework 1.

Problem 39 Your signature below indicates you have:

(a.) I read what Cook has posted of Chapter 3 of the Lecture Notes: ____________.

(b.) I read Sections 3 and 4 of Curtis: ____________.

Problem 40 Exercise 3 from §4 of Curtis (include explanation for why certain cases are not subspaces)

Problem 41 Let $W = \{(z, w) \in \mathbb{C}^2 \mid (2+i)z + iw = 0\}$. Is W a subspace of $\mathbb{C}^2$? Prove or disprove.

Problem 42 Prove the set of invertible $n \times n$ matrices over $\mathbb{R}$ do not form a subspace for any $n \in \mathbb{N}$.

Problem 43 Let B be a square $n \times n$ matrix of real numbers. Show $W = \{A \in \mathbb{R}^{n \times n} \mid BA = 0\} \leq \mathbb{R}^{n \times n}$.

Problem 44 Consider the differential equation $x^2y'' + \sin(x)y' + 3y = 0$. Show that the solution set forms a subspace of twice differentiable functions from $(0, \infty)$ to $\mathbb{R}$; that is, if the solution set is W, then $W \leq C^2((0, \infty), \mathbb{R})$. Notice, it is easy to show that W is nonempty since the zero function is twice differentiable and clearly solves the given differential equation. You are also given that $C^2((0, \infty), \mathbb{R})$ is a vector space with respect to the usual addition and scalar multiplication of functions.

Problem 45 Suppose V and W are vector spaces over a field $\mathbb{F}$ and $V_1 \leq V$ and $W_1 \leq W$. Prove that $V_1 \times W_1 \leq V \times W$ where $V \times W$ is the vector space with scalar multiplication and vector addition defined by $c(v, w) = (cv, cw)$ and $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ for all $c \in \mathbb{F}$ and $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$.

Problem 46 Exercise 10 from §4 of Curtis

MISSION 3 SOLUTION

$\boxed{\text{PROBLEM 40}}$ #3 of §4 CURTIS,

(a.) polynomial functions form subspace of $C(\mathbb{R})$ as $0 \in \mathbb{R}[t]$ and if $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and $g(t) = \sum_{i=0}^{\infty} b_i t^i$ where $a_i, b_i = 0$ for all but finitely many i then

$$cf(t) + g(t) = c\sum_{i=0}^{\infty} a_i t^i + \sum_{i=0}^{\infty} b_i t^i = \sum_{i=0}^{\infty}(ca_i + b_i)t^i$$

and we note $ca_i + b_i = 0$ for all but finitely many i. Thus the sum & scalar multiple of a polynomial is once more a polynomial $\therefore$ $\mathbb{R}[t] \leq C(\mathbb{R})$.

Remark: a simple remark sufficed as sol$^{\underline{n}}$ here. I told the class to focus on the cases that fail to be subspace.

(b.) $W = \{f \in C(\mathbb{R}) \mid f(\tfrac{1}{2}) \in \mathbb{Q}\}$. Notice $g \in W$ $\Rightarrow \sqrt{2}\, g(\tfrac{1}{2}) \notin \mathbb{Q} \Rightarrow \sqrt{2}\, g \notin W \therefore W \not\leq C(\mathbb{R})$.

(c.) $W = \{f \in C(\mathbb{R}) \mid f(\tfrac{1}{2}) = 0\}$. Let $f, g \in W$ and $c \in \mathbb{R}$ then note $(f+cg)(\tfrac{1}{2}) = f(\tfrac{1}{2}) + cg(\tfrac{1}{2}) = 0 + c(0) = 0$. Thus $f + cg \in W$ and as $f_o(x) = 0\ \forall x \in \mathbb{R} \Rightarrow f_o \in W$ So $W \neq \phi$ and $W \leq C(\mathbb{R})$ by subspace test th$^{\underline{m}}$.

(d.) If $\int_0^1 f(t)\,dt = 1$ and $\int_0^t g(t)\,dt = 1$ then

$$\int_0^1 (f(t) + g(t))\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = 1 + 1 = 2$$

Thus $W = \{f \in C(\mathbb{R}) \mid \int_0^1 f(t)\,dt = 1\}$ is not closed under addition $\therefore W \not\leq C(\mathbb{R})$.

(also, $cf \notin W$ for $c \neq 1$, $f \in W$ and $0 \notin W$ etc...)

(e.) $W = \{f \in C(\mathbb{R}) \mid \int_0^1 f(t)\,dt = 0\} \leq C(\mathbb{R})$ by subspace test Th$^{\underline{m}}$ again.

P40 continued

#3 p. 33 part (f.) $W = \{ f \in C(\mathbb{R}) \mid \frac{df}{dt} = 0 \}$

If $f, g \in W$ and $c \in \mathbb{R}$ then

$$\frac{d}{dt}(cf+g) = c\frac{df}{dt} + \frac{dg}{dt} = 0 \quad \therefore \quad cf+g \in W$$

and as constant functions are in $W \neq \phi$ we find $W \leq C(\mathbb{R})$.

(g.) $W = \{ f \in C(\mathbb{R}) \mid \alpha f''(t) + \beta f'(t) + \gamma f(t) = 0 \}$

Let $f, g \in W$ and $c \in \mathbb{R}$,

$$\alpha(cf+g)'' + \beta(cf+g)' + \gamma(cf+g) = ?$$

$$? = c(\alpha f'' + \beta f' + \gamma f) + \alpha g'' + \beta g' + \gamma g$$

[properties of $\frac{d}{dt}$ from calculus

$$= c(0) + (0)$$

$$= 0 \quad \therefore \quad cf+g \in W$$

and as $f \equiv 0$ solves $\alpha f'' + \beta f' + \gamma f = 0$ we see $W \neq \phi$ $\therefore$ by subspace test Th$^{\underline{m}}$, $W \leq C(\mathbb{R})$.

(h.) Notice $\alpha(0)'' + \beta(0)' + \gamma(0) = 0 \neq g$ (unless $g \equiv 0$ in which case we have subspace as in (g.) above)

$\therefore$ $0 \notin W = \{ f \in C(\mathbb{R}) \mid \alpha f'' + \beta f' + \gamma f = g \}$

$\Rightarrow W \not\leq C(\mathbb{R})$.

Remark: to grader: minimal sol$^{\underline{n}}$s suffice here. They did not have to write as much as I did here.

$\boxed{P41}$ $W = \{(z,w) \in \mathbb{C}^2 \mid (2+i)z + iw = 0\} \leq \mathbb{C}^2$?

Observe $(0,0) \in W$ since $(2+i)0 + i(0) = 0$.

Suppose $(a,b), (z,w) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}$. Consider,

$$\lambda(a,b) + (z,w) = (\lambda a + z, \lambda b + w)$$

Furthermore,

$$(2+i)(\lambda a + z) + i(\lambda b + w) =$$
$$= \lambda\left[(2+i)a + ib\right] + (2+i)z + iw$$
$$= \lambda(0) + 0$$
$$= 0.$$

Thus $\lambda(a,b) + (z,w) \in W$ and it follows $\lambda(a,b) \in W$ $\forall \lambda \in \mathbb{C}$ and $(a,b) + (z,w) \in W$ $\therefore W \leq \mathbb{C}^2$ by subspace test Th$^{\underline{m}}$.

$\boxed{P42}$ Let $Gl(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid A^{-1} \text{ exists}\}$.

Note, $0 \notin Gl(n,\mathbb{R})$ $\therefore$ $Gl(n,\mathbb{R}) \not\leq \mathbb{R}^{n\times n}$.

(also, sum of invertible need not be invertible etc...)

$\boxed{P43}$ $W = \{A \in \mathbb{R}^{n\times n} \mid BA = 0\}$ for some $B \in \mathbb{R}^{n\times n}$.

Notice $0 \in \mathbb{R}^{n\times n}$ solves $B(0) = 0$ $\therefore$ $0 \in W \neq \emptyset$.

Moreover, if $X, Y \in W$ and $c \in \mathbb{R}$ then note

$$B(cX + Y) = cBX + BY \quad : \text{matrix algebra}$$
$$= c(0) + (0) \quad : X, Y \in W$$
$$= 0$$

Thus $cX + Y \in W$ and it follows cX, $X + Y \in W$

$\therefore$ $W \leq \mathbb{R}^{n\times n}$ by Subspace Test Th$^{\underline{m}}$. //

$\boxed{P44}$ Consider $\underline{x^2y'' + \sin(x)y' + 3y = 0}$. $*$

Let W be the sol^{n} set of $*$. Observe for $y_0 \equiv 0$

$x^2y_0'' + \sin(x)\,y_0' + 3y_0 = x^2(0) + \sin(x)0 + 3(0) = 0 \therefore y_0 \in W$.

Suppose $y_1, y_2 \in W$ and $c \in \mathbb{R}$, note,

$$x^2(cy_1+y_2)'' + \sin(x)\,(cy_1+y_2)' + 3(cy_1+y_2) =$$

$$= c\,(\underbrace{x^2y_1'' + \sin(x)\,y_1' + 3y_1}_{0}) + \underbrace{x^2y_2'' + \sin(x)\,y_2' + 3y_2}_{0}$$

as y_1, y_2 are given to be in W and hence solve $(*)$ thus $cy_1 \in W$ and $y_1 + y_2 \in W \therefore W \leq C^2((0,\infty), \mathbb{R})$ by subspace test Th^{m}.

$\boxed{P45}$ Suppose V and W are vector spaces over $\mathbb{F}$ and $V_1 \leq V$ and $W_1 \leq W$. Define $V \times W$ by $c(v,w) = (cv, cw)$ and $(v_1, w_1) + (v_2, w_2) = (v_1+v_2, w_1+w_2)$ for all $(v,w), (v_1, w_1), (v_2, w_2) \in V \times W$ and $c \in \mathbb{F}$. Prove that $V_1 \times W_1 \leq V \times W$

Notice $(0,0) + (v,w) = (0+v, 0+w) = (v,w) \therefore (0_V, 0_W) = 0_{V\times W}$.

Since $0_V \in V_1$ as $V_1 \leq V$ and $0_W \in W_1 \leq W$ we have $(0_V, 0_W) \in V_1 \times W_1 \therefore 0_{V\times W} \in V_1 \times W_1 \neq \emptyset$. Consider, $(a,b), (c,d) \in V_1 \times W_1$ and $\alpha \in \mathbb{F}$. We have $a, c \in V_1$ and $b, d \in W_1$ thus $\alpha a + c \in V_1$ and $\alpha b + d \in W_1$ as $V_1 \leq V$ and $W_1 \leq W \Rightarrow$ closure under $+$ and scalar mult. in V_1 & W_1 respective. Thus $(\alpha a + c, \alpha b + d) \in V_1 \times W_1$.

But, $\alpha(a,b) + (c,d) = (\alpha a + c, \alpha b + d)$...

Thus $\alpha(a,b),\ (a,b) + (c,d) \in V_1 \times W_1 \Rightarrow V_1 \times W_1 \leq V \times W$

By the Subspace Test Th^{m}. //

P46 #10 from §10 of CURTIS

<u>Prove</u>: if U is linearly dependent then $V \supseteq U$ is likewise lin. dep.
<u>Question</u>: what is corresponding statement about LI sets of vectors?

If U is linearly dependent then $\exists\, u_1, \dots, u_k \in U$ and $c_1, c_2, \dots, c_k \in \mathbb{F}$, not all zero such that $\sum_{i=1}^{k} c_i u_i = 0$.
But, $U \subseteq V \Rightarrow u_1, \dots, u_k \in V$ and $\sum_{i=1}^{k} c_i u_i = 0$ is once more a non-trivial linear dependence $\therefore$ V is linearly dep.

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If S is a LI set then any subset of S is likewise LI.
Consider, $T \subseteq S$ if T has a linear dep. then
$\Rightarrow$ S has lin. dep (aka S is linearly dep.)
But, S is LI hence we obtain $\rightarrow\leftarrow$ and conclude $T \subseteq S$ has no nontrivial lin. dep. $\therefore$ T is LI.

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Alternatively, if S is LI and $T \subseteq S$. If $t_1, \dots, t_k \in T$ and $\sum_{i=1}^{k} c_i t_i = 0$ then as $t_1, \dots, t_k \in S$, since $T \subseteq S$, we have $c_1 = 0, \dots, c_k = 0$ by LI of S. Therefore T is LI.

(these are the two most obvious ways to argue subsets of LI set are LI once again)