Your PRINTED NAME indicates you read Chapter 3 of the notes: ____________________.

We assume $\mathbb{F}$ is a field and V, W are vector spaces over $\mathbb{F}$.

Problem 33 We let $V_1, \dots, V_k$ be subspaces of a vector space V over $\mathbb{F}$. Define

$$V_1 + \cdots + V_k = \{x_1 + \cdots + x_k \mid x_i \in V_i \text{ for } 1 \leq i \leq k\}$$

Prove $V_1 + \cdots + V_k \leq V$.

Problem 34 Let $W_1 = \text{span}\{x + x^2, 1 + x^3\}$ and $W_2 = \text{span}\{1 + x, x^2 + x^3\}$ define subspaces of $P_3(\mathbb{R})$.

(a) Find a basis for $W_1 \cap W_2$.

(b) Find a basis for $W_1 + W_2$

(c) Verify $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Problem 35 Define $T : P_3(\mathbb{R}) \to \mathbb{R}^{1\times 3}$ by $T(f(x)) = [f(1), f(2), f(1) + f(2)]$. Find a basis β for $P_3(\mathbb{R})$ and γ for $\mathbb{R}^{1\times 3}$ for which $[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$ where $p = rank(T)$.

Problem 36 (use of technology encouraged to perform matrix calculations here) Use the matrix techniques shown in Example 3.6.4 in order to illustrate the straightening theorem for $L_A : \mathbb{R}^3 \to \mathbb{R}^4$ for $A = \begin{bmatrix} 0 & 2 & -2 \\ 4 & 3 & 1 \\ 6 & 6 & 0 \\ 3 & -1 & 4 \end{bmatrix}$. In particular, find bases β, γ for which $[L_A]_{\beta,\gamma} = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$ where I_p is the $p \times p$ identity matrix and $p = rank(L_A)$. Keep Proposition 3.5.7 in mind as you find the vectors in the desired bases.

Problem 37 Consider bases $\beta = \{x^2, x, 1\}$ and $\bar{\beta} = \{1, x - 2, (x - 2)^2\}$. Let $T = d/dx$ restricted to $P_2(\mathbb{R})$.

(a.) Find the coordinate change matrix $P_{\beta,\bar{\beta}}$ for which $[v]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[v]_\beta$ for each $v \in P_2(\mathbb{R})$

(b.) find $[T]_{\beta,\beta}$

(c.) find $[T]_{\bar{\beta},\bar{\beta}}$

(d.) explain why the trace of both matrices is the same.

Problem 38 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that:

$$T(v_1) = v_1, \qquad T(v_2) = 2v_1, \qquad T(v_3) = 3v_3$$

where $v_1 = (1, 1, 0)$ and $v_2 = (1, -1, 0)$ and $v_3 = (0, 0, 1)$. Find the standard matrix of T by an appropriate use of Proposition 3.5.7.

Problem 39 Suppose $T(f(x)) = f'(x) + f''(x)$ for $f(x) \in P_2(\mathbb{R})$.

(a.) Can you find a basis β for $P_2(\mathbb{R})$ such that $[T]_{\beta,\beta} = I_3$?

(b.) Find a subspace W with basis β_W and basis γ for $P_2(\mathbb{R})$ such that $T|_W : W \to P_2(\mathbb{R})$ has $[T|_W]_{\beta_W,\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Problem 40 Consider $V = \mathbb{R}^3$ and the subspace $W = \text{span}\{(1,1,1)\}$. Find a basis and coordinate chart for V/W. Describe the geometry of the cosets in V/W

Problem 41 Let $T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$. Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ provide bases for $W_1 = \text{span}(\beta_1)$ and $W_2 = \text{span}(\beta_2)$.

(a.) show W_1 and W_2 are invariant subspaces of T,

(b.) Find $[T_{W_1}]_{\beta_1,\beta_1}$ and $[T_{W_2}]_{\beta_2,\beta_2}$

(c.) verify $[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2}$ where $\beta = \beta_1 \cup \beta_2$

Problem 42 Let $A \in \mathbb{F}^{n\times n}$ and define $T : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $T(A) = A + A^T$

(a.) show that $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ is an invariant subspace of T.

(b.) show that $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ is contained in $\text{Ker}(T)$.

(c.) Let β_s and β_a be bases for the symmetric and antisymmetric $n \times n$ matrices over $\mathbb{F}$. Form basis $\beta = \beta_s \cup \beta_a$ and find the block-structure of the matrix $[T]_{\beta,\beta}$

Problem 43 Let $V = \mathcal{F}(\mathbb{R})$ be the set of all real-valued functions of a real variable whose domain is $\mathbb{R}$. Define $(T(f))(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in \mathbb{R}$ and $f \in V$. What does the first isomorphism theorem applied to T reveal ?

Problem 44 Let V be a vector space over $\mathbb{F}$ and define $T : V \times V \to V$ by $T(x,y) = y - x$ for all $(x,y) \in V$. Apply the first isomorphism theorem to T and explain what truth it reveals.

Problem 45 Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $T(f) = D^2 f$ where $D = d/dx$. Let $\overline{T} : \mathbb{R}[x]/\text{Ker}(T) \to \mathbb{R}[x]$ be the mapping defined by:

$$\overline{T}([f]) = T(f)$$

for each $[f] = f + \text{Ker}(T)$. Prove $\bar{T}$ is an isomorphism and find the formula for

$$\overline{T}^{-1}(c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0).$$

Problem 46 Suppose $W_1, W_2 \leq V$. Prove $\text{ann}(W_1 + W_2) = \text{ann}(W_1) \cap \text{ann}(W_2)$.

Problem 47 Suppose V is a finite dimensional vector space and $V = W_1 \oplus W_2$. Does it follow that $ann(W_1) \oplus ann(W_2) = V^*$? Prove or disprove.

Problem 48 Friedberg, Insel and Spence 5th edition, §2.6#17, page 127.