Your PRINTED NAME indicates you read Chapter 3 of the notes: ____________.

We assume $\mathbb{F}$ is a field and V, W are vector spaces over $\mathbb{F}$.

Problem 33 We let $V_1, \dots, V_k$ be subspaces of a vector space V over $\mathbb{F}$. Define

$$V_1 + \cdots + V_k = \{x_1 + \cdots + x_k \mid x_i \in V_i \text{ for } 1 \leq i \leq k\}$$

Prove $V_1 + \cdots + V_k \leq V$.

Problem 34 Let $W_1 = \text{span}\{x + x^2, 1 + x^3\}$ and $W_2 = \text{span}\{1 + x, x^2 + x^3\}$ define subspaces of $P_3(\mathbb{R})$.

(a) Find a basis for $W_1 \cap W_2$.

(b) Find a basis for $W_1 + W_2$

(c) Verify $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Problem 35 Define $T : P_3(\mathbb{R}) \to \mathbb{R}^{1\times 3}$ by $T(f(x)) = [f(1), f(2), f(1) + f(2)]$. Find a basis β for $P_3(\mathbb{R})$ and γ for $\mathbb{R}^{1\times 3}$ for which $[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$ where $p = rank(T)$.

Problem 36 (use of technology encouraged to perform matrix calculations here) Use the matrix techniques shown in Example 3.6.4 in order to illustrate the straightening theorem for $L_A : \mathbb{R}^3 \to \mathbb{R}^4$ for $A = \begin{bmatrix} 0 & 2 & -2 \\ 4 & 3 & 1 \\ 6 & 6 & 0 \\ 3 & -1 & 4 \end{bmatrix}$. In particular, find bases β, γ for which $[L_A]_{\beta,\gamma} = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$ where I_p is the $p \times p$ identity matrix and $p = rank(L_A)$. Keep Proposition 3.5.7 in mind as you find the vectors in the desired bases.

Problem 37 Consider bases $\beta = \{x^2, x, 1\}$ and $\bar{\beta} = \{1, x-2, (x-2)^2\}$. Let $T = d/dx$ restricted to $P_2(\mathbb{R})$.

(a.) Find the coordinate change matrix $P_{\beta,\bar{\beta}}$ for which $[v]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[v]_\beta$ for each $v \in P_2(\mathbb{R})$

(b.) find $[T]_{\beta,\beta}$

(c.) find $[T]_{\bar{\beta},\bar{\beta}}$

(d.) explain why the trace of both matrices is the same.

Problem 38 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that:

$$T(v_1) = v_1, \qquad T(v_2) = 2v_1, \qquad T(v_3) = 3v_3$$

where $v_1 = (1, 1, 0)$ and $v_2 = (1, -1, 0)$ and $v_3 = (0, 0, 1)$. Find the standard matrix of T by an appropriate use of Proposition 3.5.7.

Problem 39 Suppose $T(f(x)) = f'(x) + f''(x)$ for $f(x) \in P_2(\mathbb{R})$.

(a.) Can you find a basis β for $P_2(\mathbb{R})$ such that $[T]_{\beta,\beta} = I_3$?

(b.) Find a subspace W with basis β_W and basis γ for $P_2(\mathbb{R})$ such that $T|_W : W \to P_2(\mathbb{R})$ has $[T|_W]_{\beta_W,\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Problem 40 Consider $V = \mathbb{R}^3$ and the subspace $W = \text{span}\{(1,1,1)\}$. Find a basis and coordinate chart for V/W. Describe the geometry of the cosets in V/W

Problem 41 Let $T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$. Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ provide bases for $W_1 = \text{span}(\beta_1)$ and $W_2 = \text{span}(\beta_2)$.

(a.) show W_1 and W_2 are invariant subspaces of T,

(b.) Find $[T_{W_1}]_{\beta_1,\beta_1}$ and $[T_{W_2}]_{\beta_2,\beta_2}$

(c.) verify $[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2}$ where $\beta = \beta_1 \cup \beta_2$

Problem 42 Let $A \in \mathbb{F}^{n\times n}$ and define $T : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $T(A) = A + A^T$

(a.) show that $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ is an invariant subspace of T.

(b.) show that $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ is contained in $\text{Ker}(T)$.

(c.) Let β_s and β_a be bases for the symmetric and antisymmetric $n \times n$ matrices over $\mathbb{F}$. Form basis $\beta = \beta_s \cup \beta_a$ and find the block-structure of the matrix $[T]_{\beta,\beta}$

Problem 43 Let $V = \mathcal{F}(\mathbb{R})$ be the set of all real-valued functions of a real variable whose domain is $\mathbb{R}$. Define $(T(f))(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in \mathbb{R}$ and $f \in V$. What does the first isomorphism theorem applied to T reveal ?

Problem 44 Let V be a vector space over $\mathbb{F}$ and define $T : V \times V \to V$ by $T(x,y) = y - x$ for all $(x,y) \in V$. Apply the first isomorphism theorem to T and explain what truth it reveals.

Problem 45 Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $T(f) = D^2 f$ where $D = d/dx$. Let $\overline{T} : \mathbb{R}[x]/\text{Ker}(T) \to \mathbb{R}[x]$ be the mapping defined by:

$$\overline{T}([f]) = T(f)$$

for each $[f] = f + \text{Ker}(T)$. Prove $\bar{T}$ is an isomorphism and find the formula for

$$\overline{T}^{-1}(c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0).$$

Problem 46 Suppose $W_1, W_2 \leq V$. Prove $\text{ann}(W_1 + W_2) = \text{ann}(W_1) \cap \text{ann}(W_2)$.

Problem 47 Suppose V is a finite dimensional vector space and $V = W_1 \oplus W_2$. Does it follow that $ann(W_1) \oplus ann(W_2) = V^*$? Prove or disprove.

Problem 48 Friedberg, Insel and Spence 5th edition, §2.6#17, page 127.

MISSION 3 SOLUTION

P33 Let $V_1, V_2, \ldots, V_k \le V$ over a field $\mathbb{F}$, V a vector space.

Define $V_1 + V_2 + \cdots + V_k = \{x_1 + x_2 + \cdots + x_k \mid x_i \in V_i \text{ for } 1 \le i \le k\}$.

Observe $0 \in V_i$ for each $i = 1, 2, \ldots, k$ thus $0 = 0 + 0 + \cdots + 0 \in V_1 + \cdots + V_k \neq \emptyset$.

Let $x, y \in V_1 + V_2 + \cdots + V_k$ and $c \in \mathbb{F}$ then $\exists\, x_i, y_i \in V_i$ for $i = 1, 2, \ldots, k$ such that $x = x_1 + \cdots + x_k$ and $y = y_1 + \cdots + y_k$.

Moreover, since $x_i, y_i \in V_i$ then $\underline{cx_i + y_i \in V_i \text{ as } V_i \le V}\ *$

Consequently,

$$\begin{aligned} cx + y &= c(x_1 + \cdots + x_k) + y_1 + \cdots + y_k \\ &= cx_1 + y_1 + cx_2 + y_2 + \cdots + cx_k + y_k \in V_1 + \cdots + V_k \text{ by } *. \end{aligned}$$

Thus $cx, x+y \in V_1 + \cdots + V_k$ $\therefore$ $V_1 + \cdots + V_k \le V$ by subspace test. //

P34 $W_1 = \text{span}\{x + x^2, 1 + x^3\}$ and $W_2 = \text{span}\{1 + x, x^2 + x^3\}$ define subspaces of $P_3(\mathbb{R})$

(a.) find basis for $W_1 \cap W_2$. Let $v \in W_1 \cap W_2$ then $\exists\, a, b, c, d \in \mathbb{R}$ for which $v = \underbrace{a(x + x^2) + b(1 + x^3) = c(1 + x) + d(x^2 + x^3)}_{\text{equate coefficients of } 1, x, x^2, x^3}$

$$\left.\begin{array}{rl} 1: & b = c \\ x: & a = c \\ x^2: & a = d \\ x^3: & b = d \end{array}\right\} \quad \begin{array}{l} a = b = c = d \\ \therefore\ v = a(x + x^2 + 1 + x^3) \\ \text{or } v = a(1 + x + x^2 + x^3). \end{array}$$

$$\therefore\ \boxed{\beta_{W_1 \cap W_2} = \{1 + x + x^2 + x^3\}}$$

P34 continued

(b.) find basis for $W_1 + W_2$. Since span $\beta_1 = W_1$ and span $\beta_2 = W_2$ for $\beta_1 = \{x+x^2, 1+x^3\}$ and $\beta_2 = \{1+x, x^2+x^3\}$ we find $\beta_1 \cup \beta_2$ will give span$(\beta_1 \cup \beta_2) = W_1 + W_2$. However, $\beta_1 \cup \beta_2$ is not LI so we need to select a LI subset of $\beta = \beta_1 \cup \beta_2$. Using $\gamma = \{1, x, x^2, x^3\}$ we can study $\beta = \{x+x^2, 1+x^3, 1+x, x^2+x^3\}$

$$\Phi_\gamma(\beta) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref}\left[\Phi_\gamma(\beta)\right] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ 1st three vectors in β serve as LI set

$$\boxed{\tilde{\beta} = \{x+x^2, 1+x^3, 1+x\} \text{ gives basis for } W_1 + W_2}$$

(I think any three vectors in β will suffice to give $\tilde{\beta}$ LI)

(c.) $\dim(W_1 + W_2) = |\tilde{\beta}| = 3$

$\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = |\beta_1| + |\beta_2| + |\beta_{W_1 \cap W_2}| = 2+2-1 = 3$

$\therefore$ $\underline{\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)}$ //

P35 Let $T : P_3(\mathbb{R}) \longrightarrow \mathbb{R}^{1\times 3}$ be defined by $T(f(x)) = [f(1), f(2), f(1)+f(2)]$

If $f(x) \in Ker(T)$ then $f(1) = 0$, $f(2) = 0$ and $f(1)+f(2) = 0$.

By factor Th^{m}, $f(x) = (x-1)(x-2)(Ax+B)$ as $f(x) \in P_3(\mathbb{R})$.

Then $Ker(T) = span\{x(x-1)(x-2), (x-1)(x-2)\}$. We wish to extend the basis for $Ker(T)$ to a basis for $P_3(\mathbb{R})$.

Consider $T(1) = [1,1,2]$ $\therefore$ $1 \notin Ker(T)$

Also, $T(x) = [1,2,3]$ $\therefore$ $x \notin Ker(T)$

Thus $\underline{\beta = \{1, x, x(x-1)(x-2), (x-1)(x-2)\}}$ is basis for $P_3(\mathbb{R})$.

If $[a,b,c] \notin span\{[1,1,2], [1,2,3]\}$ then the row vectors $[a,b,c]$, $[1,1,2]$, $[1,2,3]$ are LI. One way to select $[a,b,c]$ is to calculate

$$\det\begin{bmatrix} a & b & c \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = a(3-4) - b(3-2) + c(2-1) = -a-b+c$$

Apparently we can use $[1,0,0]$ to extend basis for $Im(T)$ to all of $\mathbb{R}^{1\times 3}$. Let

$$\gamma = \{[1,1,2], [1,2,3], [1,0,0]\}$$

Then

$$[T]_{\beta,\gamma} = \big[\, [T(1)]_\gamma \mid [T(x)]_\gamma \mid [T(x(x-1)(x-2))]_\gamma \mid [T((x-1)(x-2))]_\gamma \,\big]$$

$$= \big[\, [[1,1,2]]_\gamma \mid [[1,2,3]]_\gamma \mid [0]_\gamma \mid [0]_\gamma \,\big]$$

$$= [\, e_1 \mid e_2 \mid 0 \mid 0 \,]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right] \qquad p = 2 = rank(T).$$

P36 Consider $L_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ where $A = \begin{bmatrix} 0 & 2 & -2 \\ 4 & 3 & 1 \\ 6 & 6 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

Find bases β, γ for which

$$[L_A]_{\beta,\gamma} = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } p = \text{rank}(A)$$

Use Prop. 3.5.7 to find the desired bases. (Follow Example 3.6.4)

$$\text{rref}[A|I_4] = \text{rref}\left[\begin{array}{ccc|cccc} 0 & 2 & -2 & 1 & 0 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccccccc} 1 & 0 & 1 & 0 & 0 & 1/24 & 1/4 \\ 0 & 1 & -1 & 0 & 0 & 1/8 & -1/4 \\ 0 & 0 & 0 & 1 & 0 & -1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -13/24 & -1/4 \end{array}\right]$$

$$\underbrace{}_{\text{rref}(A)} \qquad \underbrace{}_{P}$$

Then $\text{rref}(A) = PA$ by calculation above.

Next, calculate,

$$\text{rref}\left[(\text{rref}(A))^T \,\middle|\, I_3 \right] = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$$\underbrace{}_{R^T} \quad \underbrace{}_{Q}$$

We have,

$$R^T = Q\,[\text{rref}(A)]^T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $R^T = Q(PA)^T \Rightarrow R = PAQ^T$ thus, using $T = L_A$ so $[T] = A$,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[T]_{\beta,\gamma}} = \underbrace{\begin{bmatrix} 0 & 0 & 1/24 & 1/4 \\ 0 & 0 & 1/8 & -1/4 \\ 1 & 0 & -1/4 & 1/2 \\ 0 & 1 & -\frac{13}{24} & -1/4 \end{bmatrix}}_{[\gamma]^{-1}} \underbrace{\begin{bmatrix} 0 & 2 & -2 \\ 4 & 3 & 1 \\ 6 & 6 & 0 \\ 3 & -1 & 4 \end{bmatrix}}_{[T]} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}}_{[\beta]}$$

Thus $\boxed{\beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}}$ and γ requires some

calculation, $[\gamma] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 6 & 6 & 0 & 0 \\ 3 & -1 & 0 & 0 \end{bmatrix} \quad \therefore \boxed{\gamma = \left\{ \begin{bmatrix} 0 \\ 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$

$\boxed{P37}$ Let $\beta = \{x^2, x, 1\}$ and $\bar{\beta} = \{1, x-2, (x-2)^2\}$

Let $T = d/dx$ on $P_2(\mathbb{R})$.

(a.)

$$f(x) = f(2) + f'(2)(x-2) + \tfrac{1}{2}f''(2)(x-2)^2$$
$$= 4a+2b+c + (4a+b)(x-2) + a(x-2)^2$$

$$\begin{tikzcd}
 & \boxed{f = ax^2+bx+c} \arrow[dl, "\Phi_\beta"'] \arrow[dr, "\Phi_{\bar{\beta}}"] & \\
\boxed{[f]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \arrow[rr, "P_{\beta,\bar{\beta}}"] & & \boxed{[f]_{\bar{\beta}} = \begin{bmatrix} 4a+2b+c \\ 4a+b \\ a \end{bmatrix}}
\end{tikzcd}$$

$$\therefore \quad \boxed{P_{\beta,\bar{\beta}} = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

(b.) $T(ax^2+bx+c) = 2ax+b$. Note, $[T]_{\beta,\beta}[V]_\beta = [T(V)]_\beta$ &

$$[T]_{\beta,\beta}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix} \Rightarrow \boxed{[T]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

(c.) $T(a + b(x-2) + c(x-2)^2) = b + 2c(x-2)$. Since $[T]_{\bar{\beta},\bar{\beta}}[V]_{\bar{\beta}} = [T(V)]_{\bar{\beta}}$,

$$[T]_{\bar{\beta},\bar{\beta}}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix} \Rightarrow \boxed{[T]_{\bar{\beta},\bar{\beta}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

(d.) $\mathrm{tr}\,[T]_{\beta\beta} = \mathrm{tr}\,[T]_{\bar{\beta},\bar{\beta}} = 0$.

Generally, $[T]_{\beta,\beta} = P^{-1}[T]_{\bar{\beta},\bar{\beta}}P$ for some $(n\times n)$ matrix P

and
$$\mathrm{tr}([T]_{\beta\beta}) = \mathrm{tr}(P^{-1}[T]_{\bar{\beta},\bar{\beta}}P)$$
$$= \mathrm{tr}(PP^{-1}[T]_{\bar{\beta},\bar{\beta}})$$
$$= \mathrm{tr}([T]_{\bar{\beta},\bar{\beta}}).$$

(for this reason we define $\mathrm{tr}(T) = \mathrm{tr}\,[T]_{\beta\beta}$ w/o ambiguity)

$\boxed{P38}$ $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfies

$$T(V_1) = V_1 \;,\quad T(V_2) = 2V_1 \;,\quad T(V_3) = 3V_3$$

where $V_1 = (1,1,0)$ and $V_2 = (1,-1,0)$ and $V_3 = (0,0,1)$

Calculate $[T]$ via Prop. 3.5.7.

$$[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta] \Rightarrow [T] = [\beta][T]_{\beta,\beta}[\beta]^{-1}$$

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ then $[\beta] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

notice $[\beta]^{-1} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [1]^{-1} \end{bmatrix} = \left[\begin{array}{c|c} \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array}\right] = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Calculate $[T]_{\beta,\beta} = \left[[T(V_1)]_\beta \,\middle|\, [T(V_2)]_\beta \,\middle|\, [T(V_3)]_\beta \right]$

$$= \left[[V_1]_\beta \,\middle|\, [2V_1]_\beta \,\middle|\, [3V_3]_\beta \right]$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Thus,

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3/2 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 3/2 & -1/2 & 0 \\ 3/2 & -1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

P39

$$T(f(x)) = f'(x) + f''(x) \quad \text{for} \quad f(x) \in P_2(\mathbb{R})$$

(a.) $T(ax^2+bx+c) = 2ax+b+2a$

$\text{Ker}(T) = \{ ax^2+bx+c \mid 2ax+b+2a = 0 \} = \text{span}\{1\}.$

Thus $[T]_{\beta\beta} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is impossible. We find $\nu(T) = \dim(\text{Ker}(T)) = 1$ thus $\text{rk}(T) = 3-1 = 2$ and $\dim(\text{col}\,[T]_{\beta\beta}) = \dim(\text{Range}(T)) = 2$ so $[T]_{\beta\beta} = I_3$ is not possible.

(b.) Let $\beta_W = \{x^2, x\}$ then $T(x^2) = 2x+2 = w_1$ and $T(x) = 1 = w_2$. Letting $\gamma = \{w_1, w_2, x^2\}$ we have basis for $P_2(\mathbb{R})$ and

$$\begin{aligned}
[T|_W]_{\beta_W,\gamma} &= \left[[T(x^2)]_\gamma \;\middle|\; [T(x)]_\gamma \right] \\
&= \left[[2x+2]_\gamma \;\middle|\; [1]_\gamma \right] \\
&= \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}.
\end{aligned}$$

<u>Remark</u>: I chose to adjoin x^2 to $\{T(x), T(x^2)\} = \{1, 2x+2\}$ since clearly $x^2 \notin \text{span}\{1, 2x+2\} \Rightarrow \{1, 2x+2, x^2\}$ is LI hence serves as basis for 3-dim'l $P_2(\mathbb{R})$. My choices for constructing β_W and γ are certainly <u>not</u> unique.

$\boxed{P40}$ Consider $V = \mathbb{R}^3$ and $W = \text{span}\{(1,1,1)\}$.
Find basis and coordinate chart for V/W.
Describe the geometry of cosets in V/W.

$\beta_W = \{(1,1,1)\}$ is basis for W

Extend to basis $\beta = \{(1,1,1), (1,0,0), (0,1,0)\}$ for V

We construct basis for V/W by

$$\beta_{V/W} = \{(1,0,0)+W,\ (0,1,0)+W\}$$
$$= \{[(1,0,0)],\ [(0,1,0)]\}$$

If $(a,b,c)+W \in V/W$ then

$$(a,b,c)+W = \underbrace{a((1,0,0)+W) + b((0,1,0)+W) + (0,0,c)+W}_{\text{not helpful, I need to get rid of the c-term.}}$$

Notice $[(a,b,c)] = [(a,b,c)-(c,c,c)] = [(a-c,\ b-c,\ 0)]$

then
$$[(a,b,c)] = (a-c,\ b-c,\ 0)+W$$
$$= (a-c)((1,0,0)+W) + (b-c)((0,1,0)+W)$$
$$= (a-c)[(1,0,0)] + (b-c)[(0,1,0)]$$

That shows $\boxed{\Phi_{\beta_{V/W}}([(a,b,c)]) = (a-c,\ b-c)}$

$\dim(V/W) = 2$ so V/W is a plane

But, from another perspective V/W is a set of parallel lines, all with direction $\langle 1,1,1\rangle$.

(the geometry question is a bit ambiguous)

P41

$T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$.

Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ be bases for W_1 & W_2 respective.

(a.) $T(1) = 1 \in W_1$

$T(x^2) = x^2 + x(2x) = 3x^2 \in W_1$

Since $T(\beta_1) \subseteq W_1 \Rightarrow T(\text{span}(\beta_1)) \subseteq W_1$ aka $T(W_1) \subseteq W_1$

hence W_1 is T-invariant.

For W_2 I'll use more straight-forward argument,

if $ax + bx^3 \in W_2 = \text{span}\,\beta_2$ then

$$T(ax+bx^3) = ax + bx^3 + x(a + 3bx^2)$$

$$= 2ax + 4bx^3 \in W_2 \quad \therefore \underline{T(W_2) \subseteq W_2}.$$

//

(b.) $\beta_1 = \{1, x^2\}$ and $T(a+bx^2) = a + 3bx^2 \Rightarrow [T_{W_1}]_{\beta_1\beta_1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\beta_2 = \{x, x^3\}$ and $T(ax+bx^3) = 2ax + 4bx^3 \Rightarrow [T_{W_2}]_{\beta_2,\beta_2} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(c.) If $\beta = \beta_1 \cup \beta_2 = \{1, x^2, x, x^3\}$

$$[T]_{\beta\beta} = \left[[T(1)]_\beta \,\middle|\, [T(x^2)]_\beta \,\middle|\, [T(x)]_\beta \,\middle|\, [T(x^3)]_\beta \right]$$

$$= \left[[1]_\beta \,\middle|\, [3x^2]_\beta \,\middle|\, [2x]_\beta \,\middle|\, [4x^3]_\beta \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \oplus \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = [T_{W_1}]_{\beta_1\beta_1} \oplus [T_{W_2}]_{\beta_2\beta_2}.$$

✓

$T: \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}^{n\times n}$ define by $T(A) = A + A^T$

(a.) $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$. If $A \in S_n$ then

$T(A) = A + A^T = A + A = 2A \in S_n$ since

$(2A)^T = 2A^T = 2A$ thus $T(S_n) \subseteq S_n$

and $T_{S_n} = 2Id_{S_n}$. *

(b.) $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ observe for $A \in A_n$

we find $T(A) = A + A^T = A - A = 0$ thus

$A \in Ker(T) \Rightarrow A_n \subseteq Ker(T)$. Btw, as $0 \in A_n$

we also may note $T(A_n) \subseteq A_n \therefore A_n$ is T-invariant and $T_{A_n} = 0$. **

(c.) If $S_n = \text{span}\, \beta_s$ and $A_n = \text{span}\, \beta_a$ recall

$S_n \oplus A_n = \mathbb{F}^{n\times n}$ thus by T-invariance of S_n & A_n

we calculate, for $\beta = \beta_s \cup \beta_a$

$$[T]_{\beta,\beta} = [T_{S_n}]_{\beta_s,\beta_s} \oplus [T_{A_n}]_{\beta_a,\beta_a}$$

$$= [2Id_{S_n}]_{\beta_s,\beta_s} \oplus [0]_{\beta_a,\beta_a}$$

$$= \left[\begin{array}{c|c} 2I_k & 0 \\ \hline 0 & 0 \end{array}\right]$$

Here we can calculate $k = \dim(S_n) = \frac{1}{2}(n^2 - n)$

since $\beta_s = \{E_{ij} + E_{ji} \mid 1 \le i < j \le n\} \cup \{E_{ii}\}_{i=1}^{n}$ serves

as basis for S_n.

P42 continued

I hope you see what follows was <u>not</u> needed given the argument I shared in (c.).

$$A \in S_n \Rightarrow A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \\ A_{1n} & \cdots & & A_{nn} \end{bmatrix}$$

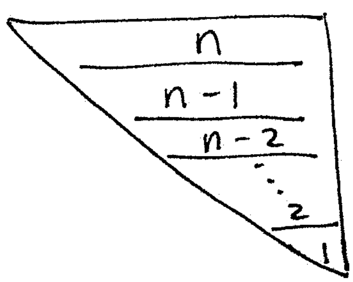
$$= \sum_{i<j} A_{ij}(E_{ij}+E_{ji}) + \sum_{i=1}^{n} A_{ii}E_{ii}$$

It follow $\beta_S = \{E_{12}+E_{21}, E_{13}+E_{31}, \dots, E_{n-1,n}+E_{n,n-1}\} \cup \{E_{11}, E_{22}, \dots, E_{nn}\}$ is LI spanning set for S_n

aka $\beta_S = \{E_{ij}+E_{ji} \mid 1 \leq i < j \leq n\} \cup \{E_{11}, \dots, E_{nn}\}$

Counting, $\#\beta_S = n + n-1 + \cdots + 2 + 1 = \frac{1}{2}n(n-1)$

↑ Gauss' Kindergarten f-la.



} independent parameters in symmetric matrix this is where my "counting" comes from.

Likewise, $A \in A_n$

$$A = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1n} \\ -A_{12} & 0 & & \vdots \\ -A_{13} & & \ddots & \\ \vdots & & & A_{n-1,n} \\ -A_{1n} & & -A_{n-1,n} & 0 \end{pmatrix} = \sum_{i<j} A_{ij}(E_{ij}-E_{ji})$$

$\Rightarrow$ <u>$\beta_a = \{E_{ij}-E_{ji} \mid 1 \leq i < j \leq n\}$</u>.

P43 Let $V = \mathcal{F}(\mathbb{R})$ and define $T: V \to V$ by

$$(T(f))(x) = \frac{1}{2}(f(x) - f(-x)) \quad \forall x \in \mathbb{R} \text{ and } f \in V.$$

Since T is linear and $Ker(T) = \{f \in V \mid f(x) = f(-x)\}$

we find $\mathcal{F}(\mathbb{R}) / Ker(T) \cong Im(T)$ shows that

$$\mathcal{F}(\mathbb{R}) / \{\text{even functions}\} \cong \{\text{odd functions}\}$$

Since $Im(T)$ is the set of odd functions since

$$(T(f))(-x) = \frac{1}{2}(f(-x) - f(-(-x))) = \frac{-1}{2}(f(x) - f(-x)) = -(T(f))(x).$$

P44 Define $T: V \times V \to V$ by $T(x,y) = y - x$ $\forall (x,y) \in V \times V$.

Notice T is a linear transformation and since $T(0,y) = y - 0 = y$ it is clear that $Im(T) = V$.

$$Ker(T) = \{(x,y) \in V \times V \mid T(x,y) = y - x = 0\}$$
$$= \{(x,x) \mid x \in V\}$$

Then

$$\boxed{\frac{V \times V}{\{(x,x) \mid x \in V\}} \cong V}$$

P45 $T: \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(f) = \frac{d^2 f}{dx^2}$

Let $\overline{T}: \mathbb{R}[x]/\ker(T) \to \mathbb{R}[x]$ be defined by

$$\overline{T}([f]) = T(f)$$

for each $[f] = f + \ker(T)$. Prove $\overline{T}$ is an isomorphism and find formula for $\overline{T}^{-1}(c_n x^n + \cdots + c_2 x^2 + c_1 x + c_0)$.

Calculus I provides $\frac{d}{dx}$ is linear transformation on $\mathbb{R}[x]$ and thus $\frac{d^2}{dx^2}$ is likewise linear on $\mathbb{R}[x]$. Notice

$$\ker(T) = \{ f(x) \in \mathbb{R}[x] \mid f''(x) = 0 \} = \{ b_0 + b_1 x \mid b_0, b_1 \in \mathbb{R} \}.$$

Furthermore, notice $T\left(\frac{x^{p+2}}{(p+1)(p+2)}\right) = x^p$ since

$$\frac{d}{dx}\frac{d}{dx}\left(\frac{1}{(p+1)(p+2)} x^{p+2}\right) = \frac{p+2}{(p+1)(p+2)} \frac{d}{dx} x^{p+1} = \frac{p+1}{p+1} x^p = x^p.$$

Thus T maps onto the monomial basis for $\mathbb{R}[x]$ and it follows that T is a surjection. Therefore, $\overline{T}$ defined by $\overline{T}([f]) = T(f)$ gives an isomorphism of $\mathbb{R}(x)/\ker(T) \cong \operatorname{Im}(T) = \mathbb{R}[x]$. The formula for $\overline{T}^{-1}$ is obtained by twice integrating,

$$\overline{T}^{-1}(f(x)) = \left[\int\left(\int f(x)\,dx\right) dx \right]$$

$$= \boxed{\frac{c_n}{(n+1)(n+2)} x^{n+2} + \cdots + \frac{c_2}{12} x^4 + \frac{c_1}{6} x^3 + \frac{c_0}{2} x^2 + \ker(T)}$$

Remark: w/o using 1st isomorphism Thm you'd need to retrace the steps of our proof for that Thm.

P46 Suppose $W_1, W_2 \leq V$. Notice $W_1 + W_2 \leq V$ and $W_1, W_2 \subseteq W_1 + W_2$ since $W_1 + W_2 = \{x_1 + x_2 \mid x_1 \in W_1, x_2 \in W_2\}$ and $0 \in W_1$ and $0 \in W_2$ thus clearly $W_1, W_2 \leq W_1 + W_2$.

Suppose $\alpha \in \text{ann}(W_1 + W_2)$ then $\alpha(x) = 0 \ \forall x \in W_1 + W_2$. Thus $\alpha(x_1) = 0$ for each $x_1 \in W_1 \subseteq W_1 + W_2$ and $\alpha(x_2) = 0$ for each $x_2 \in W_2 \subseteq W_1 + W_2$. Hence $\alpha \in \text{ann}(W_1)$ and $\alpha \in \text{ann}(W_2)$ $\therefore$ $\alpha \in \text{ann}(W_1) \cap \text{ann}(W_2)$ and we've shown $\text{ann}(W_1 + W_2) \subseteq \text{ann}(W_1) \cap \text{ann}(W_2)$.

Suppose $\beta \in \text{ann}(W_1) \cap \text{ann}(W_2)$ then $\beta(x_1) = 0$ for each $x_1 \in W_1$ since $\beta \in \text{ann}(W_1)$. Likewise, $\beta \in \text{ann}(W_2)$ thus $\beta(x_2) = 0$ for each $x_2 \in W_2$. Suppose $x \in W_1 + W_2$ then $x = x_1 + x_2$ for $x_1 \in W_1, x_2 \in W_2$ and thus $\beta(x) = \beta(x_1 + x_2) = \beta(x_1) + \beta(x_2) = 0 + 0 = 0$. Consequently $\beta \in \text{ann}(W_1 + W_2)$ and we find $\text{ann}(W_1) \cap \text{ann}(W_2) \subseteq \text{ann}(W_1 + W_2)$.

$\therefore$ $\text{ann}(W_1) \cap \text{ann}(W_2) = \text{ann}(W_1 + W_2)$ //

P47 Suppose $\dim(V) = n < \infty$ and $V = W_1 \oplus W_2$.
We seek to <u>prove</u> $\mathrm{ann}(W_1) \oplus \mathrm{ann}(W_2) = V^*$

Suppose $V = W_1 \oplus W_2$. Let $\alpha \in \mathrm{ann}(W_1) \cap \mathrm{ann}(W_2)$ then $\alpha(x_j) = 0$ for all $x_j \in W_j$ for $j = 1, 2$ as $\alpha \in \mathrm{ann}(W_j)$ for $j = 1, 2$. Let $x \in V$ then $\exists\, x_1 \in W_1$ and $x_2 \in W_2$ for which $x = x_1 + x_2$ and

$$\begin{aligned} \alpha(x) &= \alpha(x_1 + x_2) \quad \text{linearity of } \alpha \\ &= \alpha(x_1) + \alpha(x_2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Thus $\alpha(x) = 0 \quad \forall x \in V \Rightarrow \alpha = 0 \in V^*$ and so we've shown $\mathrm{ann}(W_1) \cap \mathrm{ann}(W_2) = \{0\}$.

It remains to show $\mathrm{ann}(W_1) + \mathrm{ann}(W_2) = V^*$. My argument will be based on using the dual bases for the basis β_1 of W_1 and β_2 of W_2.

Notation, $\beta_1 = \{f_1, \dots, f_{n_1}\}$ and $\beta_2 = \{g_1, \dots, g_{n_2}\}$ where $n_1 + n_2 = n$. Then we define <u>dual</u> bases β_1^*, β_2^* via

$$f^i(f_j) = \delta_{ij} \quad \forall\, 1 \le i, j \le n_1$$
$$g^i(g_j) = \delta_{ij} \quad \forall\, 1 \le i, j \le n_2.$$

Technically, $f^i : W_1 \longrightarrow \mathbb{F}$ and $g^i : W_2 \longrightarrow \mathbb{F}$. We extend these maps to all of V by making the natural rules

$$f^i(g_j) = 0 \quad \text{and} \quad g^j(f_i) = 0$$

for all $1 \le i \le n_1$ and $1 \le j \le n_2$.

P47 continued

Sorry. I forget my current spelling.

Notice $f^i \in \text{Ann}(W_2)$ and $g^j \in \text{ann}(W_1)$ since $x_2 \in W_2$ has $x_2 = \sum_{j=1}^{n_2} c_j g_j$ and

$$f^i(x_2) = \sum_{j=1}^{n_2} c_j f^i(g_j) = 0$$

Likewise, $x_1 \in W_1$ has $x_1 = \sum_{i=1}^{n_1} b_i f_i$ and

$$g^j(x_1) = \sum_{i=1}^{n_1} b_i g^j(f_i) = 0$$

Thus $\beta_1^* \subseteq \text{ann}(W_2)$ and $\beta_2^* \subseteq \text{ann}(W_1)$

Hence $n_1 \leq \dim(\text{ann}(W_2))$ and $n_2 \leq \dim(\text{ann}(W_1))$.

We know

$$\dim(\text{ann}(W_1) + \text{ann}(W_2)) = \dim(\text{ann}(W_1)) + \dim(\text{ann}(W_2)) - \dim(\text{ann}(W_1) \cap \text{ann}(W_2))$$

However, $\text{ann}(W_1) \cap \text{ann}(W_2) = 0$ hence

$$\dim(\text{ann}(W_1) + \text{ann}(W_2)) = \dim(\text{ann}(W_1)) + \dim(\text{ann}(W_2)) \geq n_2 + n_1 = n$$

However, $\text{ann}(W_1) + \text{ann}(W_2) \leq V^*$ so $\dim(\text{ann}(W_1) + \text{ann}(W_2)) \leq n$ hence

$$\dim(\text{ann}(W_1) + \text{ann}(W_2)) = n$$

and it follows $\text{ann}(W_1) + \text{ann}(W_2) = V^*$ and we've shown $\text{ann}(W_1) \oplus \text{ann}(W_2) = V^*$ //

P48 §2.6 #17, p.127

Let $T: V \to V$ be linear and $W \leq V$.
Prove W is T-invariant iff $\operatorname{ann}(W)$ is T^t-invariant

$\Rightarrow$| Suppose $T(W) \subseteq W$. Recall we define $T^t: V^* \to V^*$ by $(T^t)(\alpha)(x) = \alpha(T(x))$ $\forall \alpha \in V^*$ and $x \in V$.

Let $\alpha \in \operatorname{ann}(W)$ and consider $x \in W$,

$$(T^t)(\alpha)(x) = \alpha(T(x)) = 0$$

since $T(W) \subseteq W$ gives $T(x) \in W$ and $\alpha(w) = 0$ $\forall w \in W$.

Thus $T^t(\alpha) \in \operatorname{ann}(W)$ and we've shown that $T^t(\operatorname{ann}(W)) \subseteq \operatorname{ann}(W)$. That is, $\operatorname{ann}(W)$ is a T^t-invariant subspace of V^*.

$\Leftarrow$| Suppose $\operatorname{ann}(W)$ is T^t-invariant; $T^t(\operatorname{ann}(W)) \subseteq \operatorname{ann}(W)$.
Suppose $W = \operatorname{span}\underbrace{\{g_1, .., g_k\}}_{\beta_W}$ extend to $\beta = \{g_1, .., g_k, g_{k+1}, .., g_n\}$ for V

Then $\beta_W^* = \{g^1, ..., g^k\}$ has $g^i(g_j) = \delta_{ij}$
Notice $g^1, .., g^k \notin \operatorname{ann}(W)$ since $g^i(g_i) = 1$ for $i = 1, 2, .., k$ and $g_i \in W$ for such i. However, $g^{k+1}, .., g^n \in \operatorname{ann}(W)$ since
$g^j(g_i) = 0$ as $i \neq j$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$.

Since $\operatorname{ann}(W)$ is T^t-invariant we have $T^t(\alpha) \in \operatorname{ann}(W)$ $\forall \alpha \in \operatorname{ann}(W)$.
Then $T^t(\alpha)(x) = \alpha(T(x)) = 0$ for all $\alpha \in \operatorname{ann}(W)$.
$\Rightarrow$ $g^i(T(x)) = 0$ $\forall i = k+1, .., n$ $\therefore$ $T(x) = \sum_{i=1}^{k} g^i(T(x))g_i \in W.$ //