Your PRINTED NAME indicates you read Chapter 3 of the notes: ____________________.

Problem 21 Let $S : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $S(f(x)) = f''(0) + f'(0)x + f(0)x^2$.

(a.) if $\beta = \{1, x, x^2\}$ then find $[S]_{\beta,\beta}$

(b.) Find the formula for $S^2(ax^2 + bx + c)$ where $S^2 = S \circ S$.

Problem 22 Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $T(A) = BA$ where $\det(B) \neq 0$. Show T is a linear transformation and prove or disprove that T is injective.

Problem 23 Consider $S : \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$ defined by $S(A) = A^T$. Prove S is an isomorphism.

Problem 24 Suppose $\Phi_\beta \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = (a + b, b + c, c + d, a - b - c)$ defines a coordinate map on $\mathbb{R}^{2\times 2}$. Find β.

Problem 25 (20pts) Let $V = \mathbb{C} \times P_1(\mathbb{R})$ be the real vector space formed by pairs of the form $(a + ib, cx + d)$.

(a.) Find a basis β for V as a real vector space.

(b.) If possible, find an isomorphism from V to $\mathbb{R}^{n\times n}$ for appropriate n.

Problem 26 Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $T(v_1) = 3v_1$ and $T(v_2) = 2v_2$ and $T(v_3) = v_3$ for nonzero vectors v_1, v_2, v_3. Prove:

(a.) $\{v_1, v_2, v_3\}$ is linearly independent.

(b.) If $\beta = \{v_1, v_2, v_3\}$ then find $[T]_{\beta,\beta}$ and find $[T]$ in terms of the given data.

(c.) Calculate $\mathrm{tr}([T])$ and $\det([T])$.

Problem 27 Suppose $T : P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ is defined by $T(1) = E_{11} + E_{22}$ and $T(x) = 2E_{11} + 2E_{22}$ and $T(x^2) = E_{12} + E_{21}$ extended linearly.

(a.) Find the formula for $T(ax^2 + bx + c)$.

(b.) If $\beta = \{x^2, x, 1\}$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ then find $[T]_{\beta,\gamma}$

(c.) Find a basis for $Ker(T)$.

(d.) Construct bases δ for $P_2(\mathbb{R})$ and σ for $\mathbb{R}^{2\times 2}$ for which $[T]_{\delta,\sigma} = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$ where k is the rank of T.

Problem 28 Let $V = \text{span}_{\mathbb{R}}\{1, x, y, x^2, xy, y^2\}$ and $W \leq \mathbb{R}^{4\times 4}$ be the set of antisymmetric matrices. Define $T : V \to W$ by

$$T(f(x,y)) = \begin{bmatrix} 0 & f(0,0) & f(1,0) & f(0,1) \\ -f(0,0) & 0 & f_y(0,1) & 0 \\ -f(1,0) & -f_y(0,1) & 0 & -f_x(1,0) \\ -f(0,1) & 0 & -f_x(1,0) & 0 \end{bmatrix}$$

Let $\beta = \{1, x, y, x^2, y^2, xy\}$ and $\gamma = \{E_{ij} - E_{ji} \mid 1 \leq i < j \leq 4\}$ given the lexographic ordering beginning with $E_{12} - E_{21}$ and ending with $E_{34} - E_{43}$.

(a.) Calculate $[T]_{\beta,\gamma}$,

(b.) Find a basis for $Ker(T)$,

(c.) What is the dimension of $T(V)$?

Problem 29 Consider $V = \mathbb{R}[x]$. Find linear transformations on V such that

(a.) $T : V \to V$ is injective, but not surjective

(b.) $T : V \to V$ is surjective, but not injective.

Problem 30 Let U, V, W be vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove:

(a.) If $S \circ T$ is surjective then S is surjective,

(b.) If $S \circ T$ is injective then T is injective.