Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 4 and 5 of the notes: ____________.

Problem 37 What condition is needed for $(a, b, c) \in \text{span}\{(1, 2, 0), (1, 0, 1)\}$?

Problem 38 Is $(1, 2, 3, 4, 5, 6) \in \text{span}((1, 1, 1, 1, 1, 1), (0, 1, 0, 1, 0, 2), (6, 5, 4, 3, 2, 1))$? Use technology paired with the CCP to answer this question.

Problem 39 Find a matrix A for which $rref[A] = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $col_1(A) = [7, 5, 3]^T$ and $col_2(A) = [2, 0, 2]^T$. Is your answer unique?

Problem 40 Plot the vectors $a = \langle 1, 2\rangle$ and $b = \langle -1, 3\rangle$ in the xy-plane. Calculate $\det(a|b)$ and $\det(b|a)$. Explain the significance of the sign and magnitude of your answers.

Problem 41 Let $a = (1, 2, 2)$ and $b = (1, 0, 7)$ and $c = (1, -3, -6)$. Calculate the volume of the parallel-piped with sides a, b, c.

Problem 42 Find **all** values of k for which $S(k) = \{(k, 2, 2), (2, k, 1), (3, 3, 3)\}$ a LI set.

Problem 43 Remember, we have many properties to use in addition to the cofactor formulae,

(a.) Calculate $\det(A)$ where $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 5 & 3 & 1 \end{bmatrix}$

(b.) Calculate $\det(B)$ where $B = \begin{bmatrix} 2 & 4 & 2 & 3 & 1 \\ 0 & 8 & 6 & 7 & 2 \\ 0 & 10 & 3 & 9 & 0 \\ 0 & 7 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 & 0 \end{bmatrix}$

(c.) Let A, B be as given in the previous problems. If $M = \left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & 3B \end{array}\right]$ then calculate $\det(M)$ via application of properties of determinants given in the lecture notes and the results of the previous pair of problems.

Problem 44 Let $a = (1, 1, 1)$ and $b = (0, 1, 1)$. Form $M = [a|b] \in \mathbb{R}^{3\times 2}$. Let $\widehat{M_i}$ be the submatrix of M formed by taking M and removing the i-th row of M. For example, $\widehat{M_3} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Calculate:

(a.) $\det(\widehat{M_1})$

(b.) $\det(\widehat{M_2})$

(c.) $\det(\widehat{M_3})$

Problem 45 Let $a = (1,1,1,1)$ and $b = (0,1,1,0)$ and $c = (2,3,3,2)$. Form $M = [a|b|c] \in \mathbb{R}^{4\times 3}$. Let $\widehat{M_i}$ be the submatrix of M formed by taking M and removing the i-th row of M. For example, $\widehat{M_4} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$. Calculate:

(a.) $\det(\widehat{M_1})$

(b.) $\det(\widehat{M_2})$

(c.) $\det(\widehat{M_3})$

(d.) $\det(\widehat{M_4})$

Remark: *Apparently we can use determinants to test LI of subsets of k-vectors in $\mathbb{R}^n$ where $k < n$. Based on the calculations in the two problems above, we conjecture $S \subseteq \mathbb{R}^n$ is linearly independent if and only if there exists i for which $det(\widehat{[S]}_i) \neq 0$. This means we need to calculate a number of determinants to decide the LI of a set via direct computation.*

Problem 46 In this problem I give you a brief introduction into the **exterior algebra**. These calculations can be made rigorous, but, that is beside the point here. We can define $\wedge$ of vectors. Given two vectors a, b the $a \wedge b$ is a so-called 2-vector. Likewise, $a \wedge b \wedge c$ is a 3-vector. This **wedge product** enjoys the usual distributive laws with respect to addition and scalar multiplication of vectors,

$$(sa + tb) \wedge c = sa \wedge c + tb \wedge c \qquad \& \qquad a \wedge (sb + tc) = sa \wedge b + ta \wedge c.$$

It is also assocative,

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

However, $a \wedge b = -b \wedge a$ for any pair of vectors. In particular $a \wedge a = 0$. If we have many vectors then we generate a negative sign for each transposition of vectors:

$$a \wedge b \wedge c = -a \wedge c \wedge b = c \wedge a \wedge b = -c \wedge b \wedge a = b \wedge c \wedge a = -b \wedge a \wedge c$$

For more vectors, similar calculations hold:

$$v_1 \wedge \cdots \wedge v_{k-1} \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_m = (-1)^k v_k \wedge v_1 \wedge \cdots \wedge v_{k-1} \wedge v_{k+1} \wedge \cdots \wedge v_m.$$

(a.) Let $a = e_1 + e_3$ and $b = e_1 + e_2$ and $c = 2e_1 + e_2 + e_3$. Calculate $a \wedge b \wedge c$.

(b.) Let $a = e_1 + e_2 + e_3$ and $b = e_1 + e_2$ and $c = e_3$. Calculate $a \wedge b \wedge c$

(c.) Show that if $S = \{v_1, v_2, \dots, v_k\}$ is linearly dependent then $v_1 \wedge v_2 \wedge \cdots \wedge v_k = 0$

Remark: in fact, if $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$ then $\{v_1, v_2, \dots, v_k\}$ is LI. I haven't asked you to prove that as it requires me to discuss more of the construction of $\wedge$ than I am currently interested in describing. We may return to this topic once we have a few more tools.

Problem 47 Now I'll focus on $\mathbb{R}^3$ where the algebra is most familar. We define correspondence between 2-vectors and vectors by:

$$\Phi_{\langle v_1, v_2, v_3 \rangle} = v_1 e_2 \wedge e_3 + v_2 e_3 \wedge e_1 + v_3 e_1 \wedge e_2.$$

In case you have not had Math 231 (again, so sorry if you were cheated out of this important course in your education) you should know the **dot-product** of two vectors gives a number whose size roughly describes how parallel the given vectors are:

$$\langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$$

and the **cross-product** is a vector which points in the direct perpendicular to the given vectors according to the **right-hand-rule**. The vector $a \times b$ is longest when a and b are perpendicular and $a \times b = 0$ when a and b are colinear. Anyway, all you need here is that:

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

(a.) Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ and show $a \wedge b = \Phi_{a\times b}$.

(b.) Consider vectors $a, b, c \in \mathbb{R}^3$, show $a \wedge b \wedge c = a \bullet (b \times c) e_1 \wedge e_2 \wedge e_3$.

Remark: Can you see how the components of $a \times b$ (and hence the coefficients of $a \wedge b$) relate to subdeterminants of $[a|b]$?

Problem 48 **Background:** One of my favorite applications of Cramer's Rule is found in advanced calculus. In particular, when dealing with several nonlinear equations in multiple unknowns then the question arises when you can solve for certain variables in terms of the other variables. Furthermore, if you can solve for the variable then what are the partial derivatives of the given dependent variable in terms of the independent variables ? This question arises in many contexts of applied mathematics and especially thermodynamics.

The calculational procedure is fairly simple:

(1.) take the total differential of the given equations which constrain the variables,

(2.) solve for the differentials of the desired dependent variables.

This is a problem of linear algebra because (1.) returns an equation which is linear in the differentials. Moreover, (2.) is nicely accomplished by Cramer's Rule.

I'll state your problem: find partial derivatives of u in terms of x, y given that:

$$\begin{aligned} u^2 + v^2 - x^2 - y^2 + z^2 &= 1, \\ v + z - xy &= 0 \\ xv + yz &= 0 \end{aligned}$$

I'll do the calculus part: (I take the total differential of the given constraint equations)

$$\begin{aligned} 2udu + 2vdv - 2xdx - 2ydy + 2zdz &= 0, \\ dv + dz - ydx - xdy &= 0 \\ vdx + xdv + zdy + ydz &= 0 \end{aligned}$$

Complete the calculation via the following steps:

(a.) rearrange the equation with differentials into the form

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} du \\ dv \\ dz \end{bmatrix} = \begin{bmatrix} (*)dx + (*)dy \\ (*)dx + (*)dy \\ (*)dx + (*)dy \end{bmatrix}$$

(here the $*'s$ simply denote particular expressions involving u, v, x, y, z)

(b.) solve for du via Cramer's Rule on the system you found in (a.)

(c.) Calculus tells us that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$. Compare with your result in (b.) to calculate the desired partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Problem 49 Let $A = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ where $M = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$. Find all $x \in \mathbb{C}$ for which the matrix $A - xI$ is **not** invertible.

Problem 50 We say square matrices A and B are **similar** if there is an invertible matrix P for which $B = PAP^{-1}$. Suppose A has $\det(A) = 2$ whereas B has $\det(B) = 7$. Is it possible that A is similar to B? Explain your claim.

Problem 51 Show $\{e_1, e_1 + e_2, e_1 + e_2 + e_3, \dots, e_1 + e_2 + \cdots + e_n\}$ is a LI subset of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Problem 52 Prove part (4.) of Theorem 2.3.11 in my notes.

Problem 53 Find a careful description of $S = \{A \in \mathbb{R}^{n\times n} \mid AB = BA \text{ for all } B \in \mathbb{R}^{n\times n}\}$.

Problem 54 Find all cubic polynomials whose graphs contain the points $(1, 2), (2, 2)$.

MISSION 3 MATH 321

$\boxed{P37}$ What condition is needed for $(a,b,c) \in \text{span}\{(1,2,0),(1,0,1)\}$?

① Can we solve $c_1(1,2,0) + c_2(1,0,1) = (a,b,c)$? Consider,

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \\ 0 & 1 & c \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-2a \\ 0 & 1 & c \end{array}\right] \xrightarrow{r_2 + 2r_3} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 0 & b-2a+2c \\ 0 & 1 & c \end{array}\right]$$

$$\xrightarrow{r_1 - r_3} \left[\begin{array}{cc|c} 1 & 0 & a-c \\ 0 & 0 & b-2a+2c \\ 0 & 1 & c \end{array}\right] \Rightarrow \boxed{\text{need } b-2a+2c=0 \text{ for consistency.}} \ *$$

Moreover, given $b-2a+2c=0$ we find $c_1 = a-c$, $c_2 = c$.

② Alternatively, we want linear dependence of the set $\{(1,2,0),(1,0,1),(a,b,c)\} = S$ hence require $\det[S]=0$ namely

$$\det\begin{bmatrix} 1 & 1 & a \\ 2 & 0 & b \\ 0 & 1 & c \end{bmatrix} = 1(-b) - 1(2c) + a(2) = \underbrace{-b+2a-2c=0}_{\substack{\text{same as } * \\ \text{just multiply by } -1.}}$$

$\boxed{P38}$ Is $\overbrace{(1,2,3,4,5,6)}^{v} \in \text{span}\overbrace{\{(1,1,1,1,1,1),(0,1,0,1,0,2),(6,5,4,3,2,1)\}}^{S}$?

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 1 & 1 & 5 & 2 \\ 1 & 0 & 4 & 3 \\ 1 & 1 & 3 & 4 \\ 1 & 0 & 2 & 5 \\ 1 & 2 & 1 & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore 7(1,1,1,1,1,1) - (6,5,4,3,2,1) = (1,2,3,4,5,6)$

Indeed $v \in \text{span}(S)$.

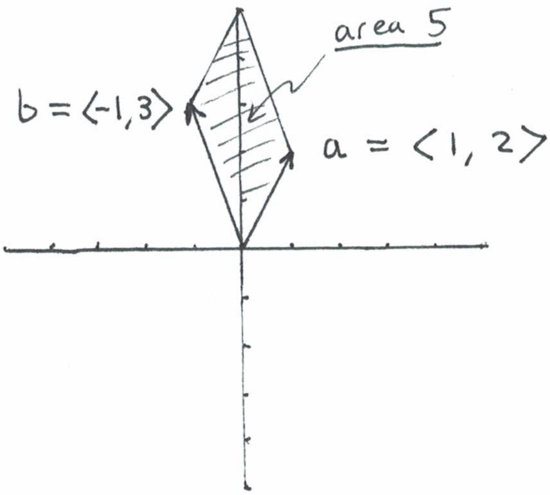
$\boxed{P39}$ Given $\underbrace{\text{rref}(A)}_{R} = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\text{col}_1(A) = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$, $\text{col}_2(A) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

we use CCP to transfer $\text{col}_3(R) = 2\,\text{col}_1(R) + 4\,\text{col}_2(R)$, $\text{col}_4(R) = 3\,\text{col}_1(R) - \text{col}_2(R)$ and $\text{col}_5(R) = 0$ to corresponding lin. dep. amongst columns of A,

$$A = \left[\begin{array}{c|c|c|c|c} \begin{matrix}7\\5\\3\end{matrix} & \begin{matrix}2\\0\\2\end{matrix} & 2\begin{bmatrix}7\\5\\3\end{bmatrix} + 4\begin{bmatrix}2\\0\\2\end{bmatrix} & 3\begin{bmatrix}7\\5\\3\end{bmatrix} - \begin{bmatrix}2\\0\\2\end{bmatrix} & \begin{matrix}0\\0\\0\end{matrix} \end{array}\right] = \underline{\begin{bmatrix} 7 & 2 & 22 & 19 & 0 \\ 5 & 0 & 10 & 15 & 0 \\ 3 & 2 & 14 & 7 & 0 \end{bmatrix}}$$

Since rref(A) is uniquely determined by its pivots and the 3 given lin. dep for columns 3, 4 and 5 we find the CCP uniquely specifies A via its 1st two columns.

P40



① $\det[a|b] = \det\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = 5$

Notice b is CCW rotation starting from direction of a

② $\det(b|a) = \det\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = -5$

Notice we rotate b CW to get to a-direction.

The magnitude of ± 5 is 5 and that is the area of the pictured parallelogram.

P41 To find volume of parallel-piped with ~~sides~~ edges $(1,2,2) = a$, $b = (1,0,7)$, $c = (1,-3,-6)$ calculate

$$\det[a|b|c] = \det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -3 \\ 2 & 7 & -6 \end{bmatrix}$$

$$= 1(21) - 1(-12+6) + 1(14)$$

$$= \boxed{41.}$$ (incidentally, $\{a, b, c\}$ is right-handed)

P42) $S(k) = \{(k, 2, 2), (2, k, 1), (3, 3, 3)\}$. Consider,

$$\det[S(k)] = \det\begin{bmatrix} k & 2 & 3 \\ 2 & k & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= k(3k-3) - 2(6-6) + 3(2-2k)$$

$$= 3k^2 - 3k + 6 - 6k$$

$$= 3k^2 - 9k + 6$$

$$= 3(k^2 - 3k + 2)$$

$$= 3(k-1)(k-2)$$

–(we know columns of $[S(k)]$ form LI set for $\det[S(k)] \neq 0$)–

Thus, $\det[S(k)] = 0$ iff $k = 1$ or 2. Thus $\boxed{S(k) \text{ is LI for } k \neq 1, 2.}$

P43

(a.) $\det A = \det\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 5 & 3 & 1 \end{bmatrix} = 2(2-6) - 2(0-10) = \underline{12 = \det A}.$

(b.) $$\det B = \det\begin{bmatrix} 2 & 4 & 2 & 3 & 1 \\ 0 & 8 & 6 & 7 & 2 \\ 0 & 10 & 3 & 9 & 0 \\ 0 & 7 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 2 & 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 7 & 8 \\ 0 & 0 & 3 & 9 & 10 \\ 0 & 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = -2\cdot 2\cdot 3\cdot 4\cdot 5$$

flip-flop

$\therefore \underline{\det B = -240}.$

(c.) $$\det\left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & 3B \end{array}\right] = \det(2A)\det(3B) = 2^3\cdot 3^5 \det(A)\det(B)$$
$$= 2^3 \cdot 3^5 \cdot (12)(-240)$$
$$= \boxed{-5{,}598{,}720}$$

P44

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(a.) $\det(\hat{M}_1) = \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$

(b.) $\det(\hat{M}_2) = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1$

(c.) $\det(\hat{M}_3) = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1$

P45

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$

(a.) $\det(\hat{M}_1) = \det\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$
$= 1(2-0) - 1(2-3) + 3(0-1)$
$= \underline{0}.$

(b.) $\det(\hat{M}_2) = \det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix} = \underline{0}.$

(c.) $\det(\hat{M}_3) = \det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix} = \underline{0}.$

(d.) $\det(\hat{M}_4) = \det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} = \underline{0}.$

(Indeed, $2a + b = c$ so $\{a, b, c\}$ <u>not</u> LI)

P46

(a.) $a = e_1 + e_3,\quad b = e_1 + e_2,\quad c = 2e_1 + e_2 + e_3$

$$\begin{aligned} a\wedge b\wedge c &= (e_1+e_3)\wedge(e_1+e_2)\wedge c \\ &= (\cancel{e_1\wedge e_1}^{\,0} + e_1\wedge e_2 + e_3\wedge e_1 + e_3\wedge e_2)\wedge(2e_1+e_2+e_3) \\ &= e_1\wedge e_2\wedge e_3 + e_3\wedge e_1\wedge e_2 + e_3\wedge e_2\wedge(2e_1) \\ &= e_1\wedge e_2\wedge e_3 + e_1\wedge e_2\wedge e_3 - 2\,e_1\wedge e_2\wedge e_3 \\ &= \boxed{0.} \end{aligned}$$

(this makes sense, $c = a+b$, so, $a\wedge b\wedge c = a\wedge b\wedge(a+b)$
$= a\wedge b\wedge a + a\wedge b\wedge b$
$= -a\wedge a\wedge b$
$= 0.$

(b.) Consider $a = (1,1,1),\ b = (1,1,0),\ c = (0,0,1)$,

$$\begin{aligned} a\wedge b\wedge c &= (e_1+e_2+e_3)\wedge(e_1+e_2)\wedge e_3 \\ &= (e_1+e_2)\wedge(e_1+e_2)\wedge e_3 + e_3\wedge(e_1+e_2)\wedge e_3 \\ &= \boxed{0.} \end{aligned}$$

(c.) Suppose $S = \{V_1, V_2, \dots, V_k\}$ is linearly dependent.
Then $\exists V_j \in S$ and $c_1,\dots,c_k \in F$ s.t. $V_j = \sum_{i\neq j}' c_i V_i$

Observe,

$$\begin{aligned} V_1\wedge\cdots\wedge V_j\wedge\cdots\wedge V_k &= (-1)^{j-1}\, V_j\wedge V_1\wedge\cdots\wedge V_{j-1}\wedge V_{j+1}\wedge\cdots\wedge V_k \\ &= (-1)^{j-1}\sum_{i\neq j}' \underbrace{c_i V_i}_{\text{not } V_j}\wedge \underbrace{V_1\wedge\cdots\wedge V_{j-1}\wedge V_{j+1}\wedge\cdots\wedge V_k}_{k-1 \text{ vectors } \underline{\text{not}} \text{ including } V_j} \quad (*) \end{aligned}$$

But, S only has k-vectors thus $*$ reduces to zero since there must be a repeat of V_i in $V_1\wedge\cdots\wedge V_{j-1}\wedge V_{j+1}\wedge\cdots\wedge V_k$

Thus, $V_1\wedge\cdots\wedge V_k = 0$ is a necessary consequence of the linear dependence of S. //

$\boxed{P47}$ $\underline{\Phi_{\langle v_1, v_2, v_3\rangle} = v_1\, e_2 \wedge e_3 + v_2\, e_3 \wedge e_1 + v_3\, e_1 \wedge e_2}$

(a.) $a \wedge b = (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3)$

$$= a_1 b_2\, e_1 \wedge e_2 + a_2 b_1\, e_2 \wedge e_1 + a_1 b_3\, e_1 \wedge e_3 + a_3 b_1\, e_3 \wedge e_1 + a_2 b_3\, e_2 \wedge e_3 + a_3 b_2\, e_3 \wedge e_2$$

$$= (a_2 b_3 - a_3 b_2)\, e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3)\, e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1)\, e_1 \wedge e_2$$

$$= \Phi_{a \times b}.$$

(b.) $a \wedge b \wedge c = \left[(a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1) e_1 \wedge e_2\right] \wedge \left[c_1 e_1 + c_2 e_2 + c_3 e_3\right]$

$$= (a_2 b_3 - a_3 b_2)\, c_1\, e_2 \wedge e_3 \wedge e_1 + (a_3 b_1 - a_1 b_3)\, c_2\, e_3 \wedge e_1 \wedge e_2 + (a_1 b_2 - a_2 b_1)\, c_3\, e_1 \wedge e_2 \wedge e_3$$

$$= \left((a \times b)_1 c_1 + (a \times b)_2 c_2 + (a \times b)_3 c_3\right) e_1 \wedge e_2 \wedge e_3$$

$$= \underline{(a \times b) \cdot c\; e_1 \wedge e_2 \wedge e_3}.$$

<u>Remark:</u>

$$[a|b] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \longrightarrow \left.\begin{matrix} a_2 b_3 - a_3 b_2 \\ a_1 b_3 - a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{matrix}\right\}$$

sub determinants of $[a|b]$ match components of $a \wedge b$.

P48

(a.)

$$\begin{bmatrix} 2u & 2v & 2z \\ 0 & 1 & 1 \\ 0 & x & y \end{bmatrix} \begin{bmatrix} du \\ dv \\ dz \end{bmatrix} = \begin{bmatrix} 2x\,dx + 2y\,dy \\ x\,dy + y\,dx \\ -v\,dx - z\,dy \end{bmatrix}$$

(b.)

$$du = \frac{\det(A_1)}{\det(A)} = \frac{\det \left[\begin{array}{c|cc} 2x\,dx + 2y\,dy & 2v & 2z \\ y\,dx + x\,dy & 1 & 1 \\ -v\,dx - z\,dy & x & y \end{array}\right]}{\det \begin{bmatrix} 2u & 2v & 2z \\ 0 & 1 & 1 \\ 0 & x & y \end{bmatrix}}$$

$$du = \frac{1}{2u(y-x)} \Big[(y-x)(2x\,dx + 2y\,dy) - 2$$

$$2vy\,(y\,dx + x\,dy) + (v\,dx + z\,dy)\ 2$$

$$2z\,(xy\,dx + x^2\,dy - v\,dx - z\,dy)$$

$$z - 2zx^2$$

$$v$$

$$du = \left[\frac{(y-x)2x + 2vy^2 + v + 2xyz}{2u(y-x)}\right] dx + \left[\frac{(y-x)2y + 2vyx + \cancel{z}}{2u(y-x)}\right] dy$$

(c.) Thus,

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{(y-x)2x + 2vy^2 + v + 2xyz}{2u(y-x)}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{(y-x)2y + 2yzv + z - 2zx^2 + v}{2u(y-x)}.$$

P49 $A = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ where $M = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ & $N = \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

The matrix $A - xI$ is <u>not</u> invertible if $\det(A - xI) = 0$

Thus $\det(A - xI_4) = \det\left[\begin{array}{c|c} M - xI_2 & 0 \\ \hline 0 & N - xI_2 \end{array}\right]$

$$= \det(M - xI_2)\det(N - xI_2)$$

$$= \det\begin{bmatrix} 2-x & 3 \\ -3 & 2-x \end{bmatrix} \det\begin{bmatrix} 7-x & 8 \\ 0 & 9-x \end{bmatrix}$$

$$= \left((x-2)^2 + 9\right)(x-7)(x-9)$$

Thus <u>$x = 7, 9$ or $2 \pm 3i$ make $(A - xI)^{-1}$ not exist.</u>

P50 If $B = PAP^{-1}$ for invertible P and $\det(A) = 2$ whereas $\det(B) = 7$ then we find a $\rightarrow\leftarrow$

Since $\det(B) = \det(PAP^{-1}) = \det P \det A \det P^{-1}$

$$= \det(PP^{-1}) \det A$$

$$= \det I \det A$$

$$= \det A \quad \therefore \quad 7 = 2 \quad (\text{oops!})$$

<u>We must have matching determinants for similar matrices.</u>

$\boxed{PS1}$ Consider $S = \{e_1, e_1+e_2, e_1+e_2+e_3, \dots, e_1+e_2+\cdots+e_n\}$

then $[S] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots 0 & 1 \end{bmatrix}$ ← has diagonals of 1 and is upper-triangular.

Thus $\det[S] = 1 \neq 0$ ∴ S is LI.

Alternatively, if

$$c_1 e_1 + c_2(e_1+e_2) + \cdots + c_n(e_1+e_2+\cdots+e_n) = 0$$

then

$$(c_1+c_2+\cdots+c_n, c_2+c_3+\cdots+c_n, \dots, c_n) = (0,0,\dots,0)$$

So, starting with last entry we find $c_n = 0$.

Then $(n-1)$-slot gives $c_{n-1}+c_n = 0 \Rightarrow c_{n-1} = 0$

continuing to back-substitute we find $c_1, c_2, \dots, c_n = 0$

hence S is LI.

$\boxed{PS2}$ For $A \in \mathbb{R}^{m\times n}$, $X, Y \in \mathbb{R}^{n\times p}$ and $c_1, c_2 \in \mathbb{R}$,

$$\begin{aligned}
(A(c_1X+c_2Y))_{ij} &= \sum_{k=1}^{n} A_{ik}(c_1X+c_2Y)_{kj} && : \text{def}^{n} \text{ of mat. mult.}\\
&= \sum_{k=1}^{n} A_{ik}\left[(c_1X)_{kj} + (c_2Y)_{kj}\right] && : \text{def}^{n} \text{ of mat. } +\\
&= \sum_{k=1}^{n} A_{ik}(c_1X_{kj} + c_2Y_{kj}) && : \text{def}^{n} \text{ of scalar mult.}\\
&= \sum_{k=1}^{n} (c_1A_{ik}X_{kj} + c_2A_{ik}Y_{kj}) && : \text{arithmetic in } \mathbb{R}\\
&= c_1\sum_{k=1}^{n} A_{ik}X_{kj} + c_2\sum_{k=1}^{n} A_{ik}Y_{kj} && : \text{prop. of } \textstyle\sum\\
&= c_1(AX)_{ij} + c_2(AY)_{ij} && : \text{def}^{n} \text{ of matrix mult.}\\
&= (c_1AX + c_2AY)_{ij} && : \text{def}^{n} \text{ of mat } + \text{ \& scal. mult.}
\end{aligned}$$

Thus $A(c_1X+c_2Y) = c_1AX + c_2AY$ as the above holds $\forall i,j$.

$\boxed{\text{PS3}}$ Let $S = \{ A \in \mathbb{R}^{n \times n} \mid AB = BA \}$

Consider $A \in S$ and study $B = E_{ij}$,

$$AB = \sum_{k,l} A_{kl} \underbrace{E_{kl} E_{ij}}_{\delta_{li} E_{kj}} = \sum_k A_{ki} E_{kj}$$

$$BA = E_{ij} \sum_{k,l} A_{kl} E_{kl} = \sum_{k,l} A_{kl} \underbrace{E_{ij} E_{kl}}_{\delta_{jk} E_{il}} = \sum_l A_{jl} E_{il}$$

Thus $AB = \sum_k A_{ki} E_{kj} = \underline{\sum_k A_{jk} E_{ik}} = BA$ (swapping l for k) $*$

Consider $(AB)_{ab} = \left(\sum_k A_{ki} E_{kj} \right)_{ab} = \sum_k A_{ki} \underbrace{(E_{kj})_{ab}}_{\delta_{ka} \delta_{jb}}$

$$\therefore \quad (AB)_{ab} = A_{ai} \delta_{jb}$$

Likewise from $*$ $(BA)_{ab} = \left(\sum_k A_{jk} E_{ik} \right)_{ab} = \sum_k A_{jk} \underbrace{(E_{ik})_{ab}}_{\delta_{ia} \delta_{kb}} = A_{jb} \delta_{ia}$

To summarize, $A_{ai} \delta_{jb} = A_{jb} \delta_{ia}$

If $a \neq i$ then $\delta_{ia} = 0 \Rightarrow A_{ai} = 0$

If $a = i$ then $\underline{A_{ii} \delta_{jb} = A_{jb} \delta_{ii} = A_{jb}}$

setting $j = b$ we obtain $A_{ii} = A_{jj}$

Hence $A_{ij} = 0$ for $i \neq j$ and $A_{11} = A_{22} = \cdots = A_{nn}$

In summary $\boxed{S = \{ kI \mid k \in \mathbb{R} \}}$

$\boxed{P54}$ Find cubic polynomials with pts $(1,2)$, $(2,2)$ in their graphs.

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

subject to $f(1) = 2$ and $f(2) = 2$

Thus,

$$A + B + C + D = 2$$
$$8A + 4B + 2C + D = 2$$

So, calculate,

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 8 & 4 & 2 & 1 & 2 \end{array}\right] \xrightarrow{r_2 - 8r_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -4 & -6 & -7 & -14 \end{array}\right]$$

$$\xrightarrow{r_1 + \frac{1}{4}r_2} \left[\begin{array}{cccc|c} 1 & 0 & -1/2 & -3/4 & -3/2 \\ 0 & -4 & -6 & -7 & -14 \end{array}\right]$$

Enough. $A = \frac{1}{2}C + \frac{3}{4}D - \frac{3}{2}$

$B = \frac{-1}{4}(6C + 7D - 14)$

$$\therefore \boxed{f(x) = \left(\frac{1}{2}C + \frac{3}{4}D - \frac{3}{2}\right)x^3 - \frac{1}{4}(6C + 7D - 14)x^2 + Cx + D}$$

for any $C, D \in \mathbb{R}$

-(we technically should say "degenerate cubics" since it is possible for $\frac{1}{2}C + \frac{3}{4}D - \frac{3}{2} = 0$ which makes $\deg(f(x)) \leq 2$.)-