Same instructions as Mission 1. Thanks!

Note: unless directly instructed otherwise, you are to assume the usual vector space structures given for each point-set in the problems below. The standard definitions of addition and scalar multiplication are all given in the Lecture Notes.

Problem 61 Your signature below indicates you have:

(a.) I read Chapter 6 of Cook's lecture notes: ____________________.

Problem 62 Let $W = \{(x,y,z) \mid 2x+3z=0, \ \& \ x+y+z=0\}$ is $W \leq \mathbb{R}^3$? Prove or disprove.

Problem 63 (a) Suppose $W = \{f(x) \in \mathbb{R}[x] \mid f(0) = 3\}$ is this a subspace of $\mathbb{R}[x]$ given the usual point-wise defined addition of functions ? Prove or disprove.

(b) Suppose $W = \{f(x) \in \mathbb{R}[x] \mid f(3) = 0\}$ is this a subspace of $\mathbb{R}[x]$ given the usual point-wise defined addition of functions ? Prove or disprove.

Problem 64 Consider the vector space $\mathbb{R}[t]$. Is $V = \{c_1(t^2+1)+c_2(2t+3) \mid c_1, c_2 \in \mathbb{R}\} \leq \mathbb{R}[t]$? Prove or disprove.

Problem 65 Let M be a particular $n \times n$ real matrix. Suppose $W = \{A \in \mathbb{R}^{n\times n} \mid MA - AM = 0\}$. Is $W \leq \mathbb{R}^{n\times n}$? Prove or disprove.

Problem 66 Let $W = \{f \in C^\infty(\mathbb{R}) \mid f'' + f = 0\}$. Is $W \leq C^\infty(\mathbb{R})$? Here I use $C^\infty(\mathbb{R})$ to denote the set of smooth real-valued functions of a real variable.

Problem 67 Let $W = \{L : \mathbb{R}^n \to \mathbb{R}^m \mid \exists A \in \mathbb{R}^{m\times n} \text{ such that } L(x) = Ax \ \forall x \in \mathbb{R}^n\}$. Show that W is a subspace of the vector space of all functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Problem 68 Is $\{1, t+2, t-2\}$ a LI subset of $\mathbb{R}[t]$?

Problem 69 Is $\{x+2, x^2-1\}$ a spanning set for $P_2 \leq \mathbb{R}[x]$?
If not, find the set of all $f(x) \notin \text{span}\{x+2, x^2-1\}$.

Problem 70 Let $V = V_1 \times V_2$ where $V_1 = \{f(x) \in P_2 \mid f(1) = 0\}$ and $V_2 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$. Find a linearly independent spanning set for V.

Problem 71 Consider $S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 6 & 6 \\ 5 & 5 \end{bmatrix}, \begin{bmatrix} 11 & 10 \\ 7 & 6 \end{bmatrix} \right\}$. Show that S is linearly dependent.

Problem 72 Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $W = \{A \in \mathbb{R}^{2\times 2} \mid [A, M] = AM - MA = 0\}$. Find a set S for which $\text{span}(S) = W$.

Problem 73 Suppose $\mathbb{F}$ is a field. Let $M \in \mathbb{F}^{n\times n}$, define the **trace** of M by $\text{tr}(M) = \sum_{i=1}^{n} M_{ii}$. Suppose $A \in \mathbb{F}^{m\times p}$ and $B \in \mathbb{F}^{p\times m}$ show that $\text{tr}(AB) = \text{tr}(BA)$.

Comment: technically, this belongs with Part I of this course, but, I delayed the proof until this Mission as we are about to use it in a central theorem of this course. I wanted you to appreciate the truth of the identity when we use it this week

Problem 74 Prove Proposition 6.4.4; that is, prove a LI set allows us to equate coefficients.

Problem 75 Let V be a vector space and suppose $S \subseteq V$ is a LI set. If $T \subseteq S$ then prove that T is LI set.

Problem 76 Suppose β is a basis of a finite dimensional vector space V. If $[v]_\beta = (1, 2, 3)$ and $\beta = \{f_1, f_2, f_3\}$. What is v ?

Problem 77 Let $v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let $\beta = \{v_1, v_2, v_3\}$ and find $[v]_\beta$ for $v = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$.

Problem 78 Consider $S = \{(1, 1, 1), (1, 2, 3), (2, 3, 4)\}$. Find a basis for $W = \text{span}(S)$. Also, find a basis for $\mathbb{R}^3$ formed by adjoining a vector v to a subset of S.

Problem 79 Define $V_p = \{(p, v) \mid v \in \mathbb{R}^n\}$. Furthermore, define addition in V_p as follows:

$$(p, v) + (p, w) = (p, v + w), \qquad \& \qquad c \cdot (p, v) = (p, cv)$$

where $v+w$, cv are defined as usual in $\mathbb{R}$. It is not difficult to show V_p is a vector space over $\mathbb{R}$ (you do not have to prove it). Let us define a dot-product on V_p in a similar manner:

$$(p, v) \bullet (p, w) = v \bullet w = v_1 w_1 + \cdots + v_n w_n.$$

Let $w = (1, 1, \ldots, 1) \in \mathbb{R}^n$ and define $S = \{(p, v) \in V_p \mid (p, v) \bullet (p, w) = 0\}$. **Show that** $S \leq V_p$.

Problem 80 Let $A \in \mathbb{R}^{m \times n}$ be a given, fixed, matrix. Also, let $b \in \mathbb{R}^m$ be a given, fixed, column vector. Define a non-standard vector space structure on $\mathbb{R}^n$ for which the solution set of $Ax = b$ forms a subspace. You do not have to check all the vector space axioms, but you should show how 0 and additive inverses are defined in your construction. Also, you should include a proof the solution set is a subspace given your modified vector addition.

Please use the symbols $\oplus$ and $\centerdot$ to define the non-standard vector addition; you need to find definitions for $x \oplus y$ and $c \centerdot x$ for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

LINEAR ALGEBRA : MISSION 4 SOLUTION :

$\boxed{P62}$ $W = \{(x,y,z) \mid 2x+3z = 0 \ \& \ x+y+z=0\}$

$= \text{Null}\left(\begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}\right) \leq \mathbb{R}^3$. as we proved null space is a subspace; we showed $A \in \mathbb{R}^{m\times n} \Rightarrow \text{Null}(A) \leq \mathbb{R}^n$.

[Remark: to use the subspace test directly here is much more of a pain.]

$\boxed{P63}$ (a.) $W = \{f(x) \in \mathbb{R}[x] \mid f(0) = 3\}$. Consider $f(x), g(x) \in W$ then $(f+g)(0) = f(0) + g(0) = 3+3 = 6 \neq 3 \therefore$ ~~f~~ $f(x)+g(x) \notin W$ hence $W \not\leq \mathbb{R}[x]$ as it is not closed under addition. (MANY OTHER ANSWERS POSSIBLE)

(b.) $W = \{f(x) \in \mathbb{R}[x] \mid f(3) = 0\}$. Let $f(x), g(x) \in W$ and $c \in \mathbb{R}$ then consider,

$$\left[cf(x) + g(x)\right]\Big|_{x=3} = c \cdot f(3) + g(3) = c(0)+0 = 0.$$

Hence $cf(x)+g(x) \in W$. Notice $f(x) = x-3$ has $f(3) = 3-3 = 0 \therefore W \neq \phi$. Furthermore, by $c=1$ we obtain $f(x)+g(x) \in W$ and by $g(x) = 0$ we obtain $cf(x) \in W$ thus by subspace th$^{\underline{m}}$ we conclude $W \leq \mathbb{R}[x]$.

$\boxed{P64}$ $V = \{c_1(t^2+1) + c_2(2t+3) \mid c_1, c_2 \in \mathbb{R}\}$

$\therefore V = \text{span}\{t^2+1, 2t+3\} \leq \mathbb{R}[t]$ by span th$^{\underline{m}}$.

Remark: ~~keep~~ keep an eye out for Null spaces and spans. If we can identify a given subset as either the Null space or span subspace th$^{\underline{m}}$ make quick work of the problem. (often subspace th$^{\underline{m}}$ ala 636 is our only hope though)

$\boxed{P65}$ Let $M \in \mathbb{R}^{n\times n}$. Let $W = \{A \in \mathbb{R}^{n\times n} \mid MA - AM = 0\}$
Consider $M(0) - 0(M) = 0 \therefore 0 \in W \neq \emptyset$. Let $A, B \in W$
and notice

$$\begin{aligned} M(A+B) - (A+B)M &= MA + MB - AM - BM \\ &= (MA - AM) + (MB - BM) \quad \text{since } A, B \in W. \\ &= 0 + 0 \end{aligned}$$

Thus $A + B \in W$. Likewise, if $A \in W$ and $c \in \mathbb{R}$ then
$M(cA) - (cA)M = c(MA - AM) = c(0) = 0 \Rightarrow cA \in W$.
Hence $W \leq \mathbb{R}^{n\times n}$ by subspace th$^{\underline{m}}$.

$\boxed{P66}$ Let $W = \{f \in C^\infty(\mathbb{R}) \mid f'' + f = 0\}$. Is $W \leq \underbrace{C^\infty(\mathbb{R})}_{\text{smooth fncts.}}$?
Notice $f(x) = 0 \ \forall x \in \mathbb{R}$ has $f'' = f = 0 \Rightarrow f'' + f = 0$
thus $0 \in W \neq \emptyset$. Consider $f, g \in W$ then

$$\begin{aligned} (f+g)'' + (f+g) &= f'' + g'' + f + g && : \text{calculus I.} \\ &= (f'' + f) + (g'' + g) && : \text{commutative addition} \\ &= 0 + 0 && : \text{as } f, g \in W. \\ &= 0 \end{aligned}$$

Thus $f, g \in W \Rightarrow f + g \in W$. Likewise, $f \in W$, $c \in \mathbb{R}$
we find $(cf)'' + cf = cf'' + cf = c(f'' + f) = c(0) = 0$.
Therefore, $f \in W$, $c \in \mathbb{R} \Rightarrow cf \in W$. Thus, by
subspace th$^{\underline{m}}$ we conclude $W \leq C^\infty(\mathbb{R})$.

Remark: Notice $\mathbb{R}^{n\times n}$ and $C^\infty(\mathbb{R})$ we known
to be vector spaces to give context to $\boxed{65}$ and $\boxed{66}$.
To apply subspace th$^{\underline{m}}$ we do need the
superset to be given some vector space structure.
I sometimes fail to comment on this as
it is very obvious from the context most places.

$\boxed{P67}$ Let $W = \{L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \mid \exists A \in \mathbb{R}^{m\times n} \text{ s.t. } L(x) = Ax \ \forall x \in \mathbb{R}^n\}$
In other words, $W = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ by the fundamental theorem of Linear Algebra. I will argue $W \leq \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$.

Observe $A = 0 \in \mathbb{R}^{m\times n}$ induces $L(x) = 0x = 0 \ \forall x \in \mathbb{R}^n$ hence the trivial transformation $L = 0 \in W \neq \emptyset$.
Suppose $L_1, L_2 \in W$ and $c \in \mathbb{R}$. Consider, by def$^{\underline{n}}$ of W, $\exists A_1, A_2$ for which

$$L_1(x) = A_1 x \quad \text{and} \quad L_2(x) = A_2 x \qquad (*)$$

$\forall x \in \mathbb{R}^n$. Consider then,

$$\begin{aligned}
(cL_1 + L_2)(x) &= (cL_1)(x) + L_2(x) && : \text{def}^{\underline{n}} \text{ of fnct. add.} \\
&= c\,L_1(x) + L_2(x) && : \text{def}^{\underline{n}} \text{ of fnct. scalar mult.} \\
&= cA_1x + A_2x && : \text{by } (*) \\
&= (cA_1 + A_2)x
\end{aligned}$$

But, $(cA_1 + A_2) \in \mathbb{R}^{m\times n}$ hence by FTLA $cL_1 + L_2$ is a linear transformation from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ and we conclude $cL_1 + L_2 \in W$. But, $c = 1$ gives $L_1 + L_2$ and $L_2 = 0$ gives cL_1 hence $L_1, L_2 \in W$, $c \in \mathbb{R}$ $\Rightarrow L_1 + L_2, cL_1 \in W$. Thus, by subspace th$^{\underline{m}}$ we conclude $W \leq \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$. //

$\boxed{P68}$ Is $S = \{1, t+2, t-2\}$ a LI subset of $\mathbb{R}[t]$?

No. Consider $(t+2) - (t-2) = 4$ thus $1 = \frac{1}{4}(t+2) - \frac{1}{4}(t-2)$ which exhibits a manifest linear dependence of S.

– (you can find this by setting $c_1(1) + c_2(t+2) + c_3(t-2) = 0$ this gives $c_1 + 2c_2 - 2c_3 = 0$ and $c_2 + c_3 = 0$ which is two eq$^{\underline{n}}$'s in 3 unknowns $\Rightarrow \infty$ly many sol$^{\underline{n}}$'s, but give details!).

$\boxed{P69}$ Is $\{x+2, x^2-1\}$ a spanning (aka generating) set for $P_2 = \text{span}\{1, x, x^2\}$. If not find set of all $f(x) \notin \text{span}\{x+2, x^2-1\}$.

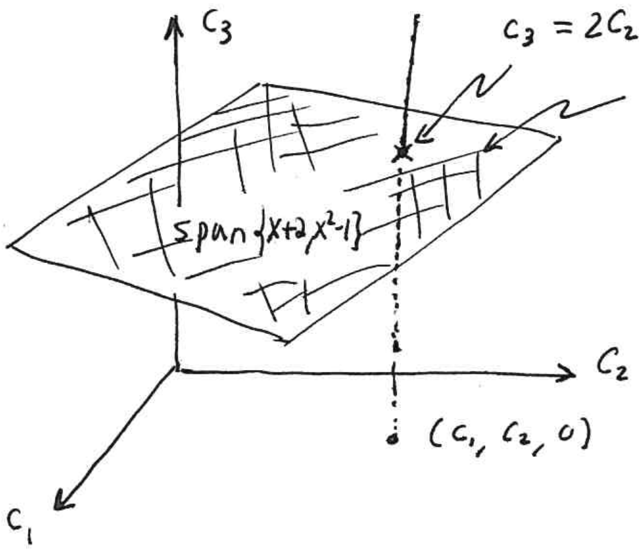
We expect the answer is No, as $\dim(P_2) = 3$ whereas $\#\{x+2, x^2-1\} = 2$. We're clearly missing something. Oh, but, what are we missing? Consider, typical element of $\text{span}\{x+2, x^2-1\}$

$$a(x+2) + b(x^2-1) = bx^2 + ax + (2a-b)$$

Thus, $f(x) \in \text{span}\{x+2, x^2-1\}$ with $f(x) = c_1x^2 + c_2x + c_3$ allows $c_1, c_2 \in \mathbb{R}$, but we are forced to set $c_3 = 2c_2 - c_1$. It follows that

$$\{c_1x^2 + c_2x + c_3 \mid c_3 \neq 2c_2 - c_1\} \cap \text{Span}\{x^2-1, x+2\} = \emptyset$$

Here's how I think about this (abstract picture of P_2)



$c_3 - 2c_2 + c_1 = 0$
plane in (c_1, c_2, c_3)-space.

—(the bulk of (c_1, c_2, c_3)-space is full of $f(x)$ not in span of $\{x+2, x^2-1\}$)—

—(such pictures are ultimately successful as P_3 is isomorphic to $\mathbb{R}^3$, but, cart, horse, day for it not the now...)—

$\boxed{\text{P70}}$ Let $V = V_1 \times V_2$ where $V_1 = \{f(x) \in P_2 \mid f(1) = 0\}$ and $V_2 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$. Find a LI spanning set for V

If $f(x) \in V_1$ then $f(x) = ax^2 + bx + c$ with $f(1) = a+b+c = 0$ hence $f(x) = ax^2 + bx - a - b = a(x^2-1) + b(x-1)$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V_2$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow b = c$

thus $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $(f(x), A) \in V$ has the following:

$$(f(x), A) = \left(a(x^2-1) + b(x-1), \begin{bmatrix} c & d \\ d & e \end{bmatrix}\right)$$

$$= a(x^2-1, 0) + b(x-1, 0) + c\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) + d\left(0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) + e\left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Observe, if I set $(f(x), A) = 0 = (0,0)$ we find

$a = 0, \ b = 0, \ c = 0, \ d = 0, \ e = 0$

thus $\boxed{\left\{(x^2-1, 0), (x-1, 0), \left(0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right\} = S'}$

is a LI set. Moreover, it is clear $\operatorname{span} S' = V$.

Let me generalize this problem a bit,

If $S_1 \subset V_1$ and $S_2 \subset V_2$ are LI sets then

~~$S_1 \times S_2$~~ $(S_1 \times \{0_2\}) \cup (\{0\} \times S_2)$ is LI set in $V = V_1 \times V_2$.

Let $S_1 = \{v_1, v_2, \dots, v_k\}$ and $S_2 = \{w_1, w_2, \dots, w_l\}$ then

$S_1 \times \{0\} = \{(v_1, 0), (v_2, 0), \dots, (v_k, 0)\}$ & $\{0\} \times S_2 = \{(0, w_1), \dots, (0, w_l)\}$

Suppose $\displaystyle\sum_{i=1}^{k} c_i (v_i, 0) + \sum_{j=1}^{l} \bar{c}_j (0, w_j) = 0 = (0,0)$

$$\Rightarrow \left(\sum_{i=1}^{k} c_i v_i, \sum_{j=1}^{l} \bar{c}_j w_j\right) = (0, 0)$$

$$\Rightarrow \sum_{i=1}^{k} c_i v_i = 0 \ \ \& \ \sum_{j=1}^{l} \bar{c}_j w_j = 0$$

$$\Rightarrow c_1 = 0, \dots, c_k = 0 \ \ \& \ \ \bar{c}_1 = 0, \dots, \bar{c}_l = 0$$

Thus $(S_1 \times \{0\}) \cup (\{0\} \times S_2)$ is a LI subset of $V_1 \times V_2$.

P71 Consider $S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 6 & 6 \\ 5 & 5 \end{bmatrix}, \begin{bmatrix} 11 & 10 \\ 7 & 6 \end{bmatrix} \right\}$. Show S is linearly dep.

Consider $c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + c_2 \begin{bmatrix} 6 & 6 \\ 5 & 5 \end{bmatrix} + c_3 \begin{bmatrix} 11 & 10 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

thus $\left[\begin{array}{c|c} c_1 + 6c_2 + 11c_3 & 2c_1 + 6c_2 + 10c_3 \\ \hline 3c_1 + 5c_2 + 7c_3 & 4c_1 + 5c_2 + 6c_3 \end{array}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\text{rref}\begin{bmatrix} 1 & 6 & 11 \\ 2 & 6 & 10 \\ 3 & 5 & 7 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = c_3 \ \& \ c_2 = -2c_3 \\ \text{any } c_3 \in \mathbb{R} \text{ gives sol}^{\underline{n}}. \end{array}$$

In particular, choose $c_3 = 1 \Rightarrow c_1 = 1,\ c_2 = -2$. Of course, you might have guessed (seen by inspection) that

$$\boxed{\begin{bmatrix} 11 & 10 \\ 7 & 6 \end{bmatrix} = 2\begin{bmatrix} 6 & 6 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$

linear dep. in S.

P72 Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $W = \{ A \in \mathbb{R}^{2\times 2} \mid [A, M] = AM - MA = 0 \}$ find a spanning set S for which span$(S) = W$.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \left[\begin{array}{c|c} a + 2b & 2a + b \\ \hline c + 2d & 2c + d \end{array}\right]$

whereas $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[\begin{array}{c|c} a + 2c & b + 2d \\ \hline 2a + c & 2b + d \end{array}\right]$ equating these yield:

$$\left.\begin{array}{lll} a + 2b = a + 2c & \Rightarrow & b = c \\ c + 2d = 2a + c & \Rightarrow & a = d \\ 2a + b = b + 2d & \Rightarrow & a = d \\ 2c + d = 2b + d & \Rightarrow & c = b \end{array}\right\} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \underline{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}.$$

Thus $\boxed{\text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = W}$

$\boxed{P73}$ Let $M \in \mathbb{F}^{n \times n}$ and define $tr(M) = \sum_{i=1}^{n} M_{ii}$. Let $A \in \mathbb{F}^{m \times p}$ and $B \in \mathbb{F}^{p \times m}$ thus $AB \in \mathbb{F}^{m \times m}$ and $BA \in \mathbb{F}^{p \times p}$. Consider

$$\begin{aligned}
tr(AB) &= \sum_{i=1}^{m} (AB)_{ii} && : \text{def}^{n} \text{ of trace} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{p} A_{ij} B_{ji} && : \text{def}^{n} \text{ of matrix mult.} \\
&= \sum_{j=1}^{p} \sum_{i=1}^{m} B_{ji} A_{ij} && : \text{prop of } \textstyle\sum \text{ \& commuted \#'s} \\
&= \sum_{j=1}^{p} (BA)_{jj} && : \text{def}^{n} \text{ of matrix mult.} \\
&= tr(BA). // && : \text{def}^{n} \text{ of trace.}
\end{aligned}$$

$\boxed{P74}$ Prove LI $\Leftrightarrow$ Equating Coeff. Property for $S = \{v_1, v_2, \dots, v_n\}$. Note we use Prop. 6.4.3. to trade def^{n} of LI for the trivial linear combo. condition characterization of LI of S.

$\Rightarrow$] Assume $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \Rightarrow c_1 = 0, c_2 = 0, \dots, c_n = 0$. Consider $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$. Algebra reveals $(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_n - b_n) v_n = 0$. But, by assumption of LI we have $a_1 - b_1 = 0$, $a_2 - b_2 = 0, \dots, a_n - b_n = 0$ thus $a_1 = b_1$, $a_2 = b_2, \dots, a_n = b_n$. That is, S has the equating coeff. property.

$\Leftarrow$] Suppose $a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n \Rightarrow a_1 = b_1, \dots, b_n = a_n$. Consider $c_1 v_1 + \cdots + c_n v_n = 0$. Observe that the ~~zero~~ zero vector is also expressed as $0 \cdot v_1 + \cdots + 0 \cdot v_n = 0$ hence,

$$c_1 v_1 + \cdots + c_n v_n = 0 \cdot v_1 + \cdots + 0 \cdot v_n$$

Thus, by equating coeff of $v_1, \dots, v_n$, we find

$$c_1 = 0, \dots, c_n = 0$$

Thus $S = \{v_1, \dots, v_n\}$ is a LI set.

P75 Let V be a vector space and $S \subseteq V$ a LI set. If $T \subseteq S$ then show T is LI set.

Suppose S is LI and $T \subseteq S$. Consider $t_1, \dots, t_k \in T$ with $\sum_{i=1}^{k} c_i t_i = 0$. Observe $t_1, \dots, t_k \in S$ as ~~S~~ $T \subseteq S$ hence by LI of S we find $\sum_{i=1}^{k} c_i t_i = 0 \Rightarrow c_i = 0 \ \forall i = 1, 2, \dots, k$. Therefore, T is a LI set.

P76 Suppose β is a basis of a finite-dim'l vector space V. If $[v]_\beta = (1,2,3)$ and $\beta = \{f_1, f_2, f_3\}$, what is v?

$$[v]_\beta = (1,2,3) \iff v = 1 \cdot f_1 + 2 \cdot f_2 + 3 \cdot f_3 \quad \therefore \boxed{v = f_1 + 2f_2 + 3f_3}$$

P77 Let $v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and define $\beta = \{v_1, v_2, v_3\}$. Find $[v]_\beta$ for $v = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$.

Let me solve a harder problem, find $[w]_\beta$ for $w = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

$$w = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \therefore \underline{[w]_\beta = \begin{bmatrix} b \\ c \\ a \end{bmatrix}}.$$

Applying this result to v yields,

$$\boxed{[v]_\beta = (1, 2, 3)}$$

<u>Remark</u>: the solᵗ above required us to "see" something. For those w/o this vision you can still solve by brute-force,

$$x\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix} \Longrightarrow \underline{x = 1, \ y = 2, \ z = 3}$$

actually, brute force not so bad here.

$$\therefore [v]_\beta = (1, 2, 3).$$

$\boxed{P78}$ Consider $S = \{(1,1,1), (1,2,3), (2,3,4)\}$. Let $W = \text{span}\, S$. Find basis for W, and, find basis for $\mathbb{R}^3$ by adjoining v to an appropriate <u>subset</u> of S.

Begin by noting $(1,1,1)+(1,2,3) = (2,3,4)$ yet $\beta = \{(1,1,1), (1,2,3)\}$ is clearly LI. To make S into a LI set we may simply remove either the 1^{st}, 2^{nd} or 3^{rd} element. I choose to remove $(2,3,4)$. It follows, as span S = span $\{(1,1,1), (1,2,3)\}$ that β is a LI spanning set for W; $\boxed{\beta = \{(1,1,1), (1,2,3)\}}$

To extend β to a basis for $\mathbb{R}^3$ we must find a vector outside span β. Consider

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 1 & b \\ 3 & 1 & c \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -1 & b-2a \\ 0 & -2 & c-3a \end{array}\right] \xrightarrow[r_3 - 2r_2]{r_1 + r_2} \left[\begin{array}{cc|c} 1 & 0 & a+b-2a \\ 0 & -1 & b-2a \\ 0 & 0 & \underbrace{c-3a-2(b-2a)}_{*} \end{array}\right]$$

$* \quad c - 3a - 2(b-2a) = c - 2b + a$

We have $(a,b,c) \in \text{span}\, \beta$ iff $c - 2b + a = 0$ by the calculation above. Any choice of (a,b,c) for which $* \neq 0$ will provide a reasonable vector to adjoin to β. For example, $b = c = 0$ and $a = 1$ gives $* = 1 \neq 0$

$$\beta \cup \{(1,0,0)\} \leftarrow \text{basis for } \mathbb{R}^3.$$

there are ∞ly many answers. Basically,

<u>$\beta \cup \{(a,b,c)\}$ s.t. $c - 2b + a \neq 0$</u>

(But I didn't ask $\forall$ possibilities, just one will do here.)

PROBLEM 78

Suppose S is LI subset of vector space V and $T \subseteq S$. Claim: T is LI

Suppose $c_1 t_1 + c_2 t_2 + \cdots + c_k t_k = 0$ for some $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $t_1, t_2, \ldots, t_k \in T$. Note $T \subseteq S \Rightarrow t_1, t_2, \ldots, t_k \in S$ hence by LI of S we find $c_1 = c_2 = \cdots = c_k = 0$.

Thus T is LI. //

PROBLEM 79 $S = \{(P,V) \in V_P \mid (P,V) \cdot (P,W) = 0\}$ when $W = (1,1,\ldots,1) \in \mathbb{R}^n$. Show $S \leq V_P$

Observe, $(P,V) \cdot (P,W) = V \cdot W$. Furthermore, $0 = (P,0) \in V_P$ and clearly $(P,0) \cdot (P,W) = 0 \cdot W = 0 \therefore 0 \in S \neq \emptyset$.

Let $(P,V_1), (P,V_2) \in S$ and $c \in \mathbb{R}$. Consider,

$$c(P,V_1) + (P,V_2) = (P, cV_1 + V_2)$$

Moreover,

$$\begin{aligned}(P, cV_1 + V_2) \cdot (P,W) &= (cV_1 + V_2) \cdot W \quad \text{Lemma.} \\ &= cV_1 \cdot W + V_2 \cdot W \\ &= 0 \quad : \text{as } (P,V_1), (P,V_2) \in S.\end{aligned}$$

Thus, taking $c = 1$ and $V_2 = 0$ separately, we find S is closed under vector addition and scalar multiplication. Therefore, by subspace test Th$^{\text{m}}$, $S \leq V_P$.

Lemma for 79: If $x, y, z \in \mathbb{R}^n$ then

$$(cx + y) \cdot z = cx \cdot z + y \cdot z.$$

Proof: $(cx+y) \cdot z = \sum_{i=1}^{n} (cx+y)_i z_i$: def^{n} of $\cdot$,

$= \sum_{i=1}^{n} (cx_i + y_i) z_i$: def^{n} of $+$ for $\mathbb{R}^n$,

$= c \sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i$: prop. of $\sum$,

$= cx \cdot z + y \cdot z$ //

PROBLEM 80 Consider $x \in \mathbb{R}^n$ such that $Ax = b$. Recall that $\exists x_p, x_h \in \mathbb{R}^n$ such that $Ax_p = b$ and $Ax_h = 0$. The idea is to move addition to $x_p = P$

Think of $x = P + x_h$ as $(P, x_h) \approx (P, v)$ of 79.

Suppose, $x, y \in \mathbb{R}^n$ such that $x = P + x_h$ & $y = P + y_h$ then

$$x \oplus y = P + x_h + y_h$$

$$c \odot x = P + cx_h$$

We have $0_p = P$ as $(P+0) \oplus (P + x_h) = P + x_h \quad \forall x = P + x_h$.

Additive inverse of $x = P + x_h$ is $-x = P - x_h$ note,

$$x \oplus (-x) = (P + x_h) \oplus (P - x_h)$$

$$= P + x_h - x_h$$

$$= P.$$

$$= 0_p$$

Let us see this sol^{n} in notation like 79, I think that'll be easier,

$\boxed{\text{PROBLEM 80}}$ Sol^{n} to $Ax = b$ can be written as $x = x_p + x_h$ where $Ax_h = 0$ and $Ax_p = b$.
Let $x_p = p$ in what follows. Note p is fixed throughout.
We write $x = (p, x_h)$

$$\begin{aligned} x_1 \oplus x_2 &= (p, x_{1h}) \oplus (p, x_{2h}) \\ &= (p, x_{1h} + x_{2h}) \\ &= p + x_{1h} + x_{2h} \end{aligned}$$

$$c \boxdot x = c \boxdot (p + x_h) = c \boxdot (p, x_h) = (p, cx_h)$$

Again $(p, 0)$ serves as zero and $-(p, x_h) = (p, -x_h)$.

Subspace Sol^{n} Set ?

Let $Ax = b$ and $Ay = b$ where $x = p + x_h$ and $y = p + y_h$ then if $c \in \mathbb{R}$ we calculate,

$$\begin{aligned} A(p + cx_h + y_h) &= Ap + cAx_h + Ay_h \\ &= b + c(0) + 0 \\ &= b. \end{aligned}$$

Thus $p + cx_h + y_h$ is in the Sol^{n} Set. However $c = 1$ and $y_h = 0 \Rightarrow x + y,\ cx \in$ Sol^{n} Set thus Sol^{n} Set is a subspace of $\mathbb{R}^n$ with respect to the nonstandard addition $\oplus$ and scalar multiplication $\boxdot$ based on the particular sol^{n} p : $Ap = b$.//