Same rules as Homework 1.

Problem 47 Your signature below indicates you have:

(a.) I read what Cook has posted of Chapter 3 of the Lecture Notes: ____________________.

(b.) I read most of Sections 5-10 of Curtis: ____________________.

Problem 48 Prove (4.) of Theorem 3.1.13 in my Lecture notes. You are free to assume that (1.), (2.) , (3.) and the Law of Cancellation are already established facts.

Problem 49 Let $W = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$. Show that $W \leq \mathbb{F}^{n\times n}$. (aww man, another subspace test problem)

Problem 50 Suppose $\beta = \{v_1, \dots, v_n\}$ forms a basis for the vector space V over the field $\mathbb{F}$. Show that $[cx+y]_\beta = c[x]_\beta + [y]_\beta$ for all $x, y \in V$ and $c \in \mathbb{F}$.

Problem 51 Find a basis β for $W = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$.
Also, calculate $[A]_\beta$ given $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 2 \end{bmatrix}$.
There are infinitely many correct answers here, sorry Daniel.

Problem 52 Find a basis for β for $W = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ as a vector space over $\mathbb{F}$. Calculate $\dim_{\mathbb{F}}(W)$. (you can set $\mathbb{F} = \mathbb{R}$ if you wish to keep it real)

Problem 53 Let $W = \{p(t) \in \mathbb{R}[t] \mid p^{(n)}(1) = 0 \text{ for } n = 3, 4, 5, \dots\}$. Prove W is a subspace of $\mathbb{R}[t]$ by explicitly finding S for which $W = \text{span}(S)$.

Problem 54 Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$. Find bases for the column, row and null space of A. For the column and row space use actual columns and rows of A to form the bases.

Problem 55 Let $A = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{1} \\ \bar{9} & \bar{1} \end{bmatrix} \in (\mathbb{Z}/11\mathbb{Z}.)^{3\times 2}$. Find bases for the column, row and null space of A.

Problem 56 Consider $W = \text{span}_{\mathbb{C}}\{1, x, x^2\}$ as a vector space over $\mathbb{R}$. Find a basis for W.

Problem 57 Let $\beta = \{x^2, (1+x)^2\}$ and define $W = \text{span}(\beta)$ as a subspace of $P_2(\mathbb{R})$. If $f(x) = ax^2 + bx + c \in W$ then find $[f(x)]_\beta$.

Problem 58 Curtis §5 #3 on page 37. Your solution should focus on the LI of the proposed basis. Prove $\{1, x, x^2, \dots, x^n\}$ is a LI set.

Problem 59 Curtis §6 #5 on page 48. (linear dependence in polynomials)

Problem 60 Curtis §7 #4 on page 52. (subspace intersection calculation)

Problem 61 Curtis §7 #5 on page 52. (subspace intersection proof question)

Problem 62 Curtis §8 #2 on page 61. (solvability of system)

Problem 63 Curtis §8 #4 on page 62 (I'd like you to use my Proposition 3.8.4 or something similar)

Problem 64 Curtis §9 #6 on page 69 (a result in the algebraic geometry of lines)

Problem 65 Curtis §10 #2 on page 73 (interesting reverse of our usual calculation)

Problem 66 Curtis §10 #7 on page 74 (hyperplanes as solution sets)

Problem 67 Curtis §10 #8 on page 74 (basis for intersection of subspace)

Problem 68 Prove Theorem 3.7.4 in my notes. In particular, show the following is true: If V is a vector space over a field $\mathbb{F}$ and $U \leq V$ and $W \leq V$ then $U \cap W \leq V$.

Problem 69 Prove Theorem 3.7.6 in my notes. In particular, show the following is true: If V is a vector space over a field $\mathbb{F}$ and $U \leq V$ and $W \leq V$ then $U + W \leq V$. Furthermore, $U + W$ is the smallest subspace which contains $U \cup W$.

Problem 70 Prove Proposition 3.4.4 of the Lecture Notes: that is, prove the following: Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$. S is a linearly independent set of vectors iff if there exist $a_i, b_i \in \mathbb{F}$ for $i \in \mathbb{N}_k$ for which

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = b_1v_1 + b_2v_2 + \cdots + b_kv_k$$

then $a_i = b_i$ for each $i = 1, 2, \ldots, k$. In other words, we can equate coefficients of a set of vectors iff the set of vectors is a LI set.

MISSION 4 SOLUTION

$\boxed{P48}$ See my notes.

$\boxed{P49}$ $W = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$. Show $W \leq \mathbb{F}^{n\times n}$

Observe $0^T = 0 = -0 \;\therefore\; 0 \in W \neq \emptyset$. Let $A, B \in W$ and note $(A+B)^T = A^T + B^T = -A - B = -(A+B) \;\therefore\; A+B \in W$. Likewise, if $\alpha \in \mathbb{F}$ and $A \in W$ then by prop. of transpose we also find $(\alpha A)^T = \alpha A^T = \alpha(-A) = -(\alpha A) \;\therefore\; \alpha A \in W$. Thus $W \leq \mathbb{F}^{n\times n}$ by the subspace test Th$^{\underline{m}}$. //

$\boxed{P50}$ Let $\beta = \{v_1, \dots, v_n\}$ form a basis for $V(\mathbb{F})$. If $x, y \in V$ then $\exists\, \alpha_i, \beta_i \in \mathbb{F}$ for $i \in \mathbb{N}_n$ such that $x = \sum_{i=1}^n \alpha_i v_i$ and $y = \sum_{i=1}^n \beta_i v_i$. By def$^{\underline{n}}$ of coordinate chart, $[x]_\beta = (\alpha_1, \dots, \alpha_n)$ and $[y]_\beta = (\beta_1, \dots, \beta_n)$. Notice,

$$
\begin{aligned}
cx + y &= c\sum_{i=1}^n \alpha_i v_i + \sum_{i=1}^n \beta_i v_i \\
&= \sum_{i=1}^n (c\alpha_i + \beta_i) v_i
\end{aligned}
$$

Thus $[cx+y]_\beta = (c\alpha_1 + \beta_1, \dots, c\alpha_n + \beta_n)$.
Finally, note,

$$(c\alpha_1 + \beta_1, \dots, c\alpha_n + \beta_n) = c(\alpha_1, \dots, \alpha_n) + (\beta_1, \dots, \beta_n)$$

$$\Rightarrow \underline{[cx+y]_\beta = c[x]_\beta + [y]_\beta \quad \forall x, y \in V,\ c \in \mathbb{F}} \quad //$$

Could write less.

$\boxed{P50}$ Concise sol$^{\underline{n}}$

Let $\beta = \{v_i\}_{i=1}^{n}$ form basis for V. Let $x, y \in V(\mathbb{F})$ and suppose $\exists\, \alpha_i, \beta_i \in \mathbb{F}$ such that $x = \sum_{i=1}^{n} \alpha_i v_i$ & $y = \sum_{i=1}^{n} \beta_i v_i$

Note, $x + cy = \sum_{i=1}^{n} \alpha_i v_i + c \sum \beta_i v_i = \sum_{i=1}^{n} (\alpha_i + c\beta_i) v_i$ by properties of finite sums. Thus $([x+cy]_\beta)_i = \underline{\alpha_i + c\beta_i}_{\,\mathrm{I}}$ $\forall i \in \mathbb{N}_n$

But, $([x]_\beta + c[y]_\beta)_i = ([x]_\beta)_i + c([y]_\beta)_i = \underline{\alpha_i + c\beta_i}_{\,\mathrm{II}}$ by def$^{\underline{n}}$ of coord. map $\therefore$ comparing I and II we find

$[x + cy]_\beta = [x]_\beta + c[y]_\beta$ $\forall c \in \mathbb{F}$ and $x, y \in V$.

Remark: maybe the less concise sol$^{\underline{n}}$ is just as good here.

$\boxed{P51}$ $W = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$ find basis β for W & calculate $[A]_\beta$ for $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = -A^T$$

Thus, $a = -a$, $b = -d$, $c = -g$, $e = -e$, $i = -i$, $f = -h$

These yield $a = e = i = 0$ and $d = -b$, $g = -c$, $h = -f$

$$A = \begin{bmatrix} 0 & a & c \\ -a & 0 & f \\ -c & -f & 0 \end{bmatrix} = a\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{g_1} + c\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{g_2} + f\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{g_3}$$

Let $\boxed{\beta = \{g_1, g_2, g_3\}}$ notice that span $\beta = W$ by calculation above and from

$$c_1 g_1 + c_2 g_2 + c_3 g_3 = \begin{bmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

$\therefore$ β is LI & span $\beta = W$ $\therefore$ β is BASIS for W.

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} = 1\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow \boxed{[A]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$\boxed{P52}$ $W = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ find β for W & $\dim_{\mathbb{F}} W = ?$

$A \in W \Rightarrow A^T = A \Rightarrow A_{ij} = A_{ji} \quad \forall i,j \in \mathbb{N}_n.$

Consider, $A = \sum_{i,j=1}^{n} A_{ij}E_{ij}$ hence, think about it,

$$A = \sum_{i<j} A_{ij}E_{ij} + \sum_{i=j} A_{ij}E_{ij} + \sum_{i>j} A_{ij}E_{ij}$$

$$= \sum_{i<j} A_{ij}E_{ij} + \sum_{i>j} A_{ji}E_{ij} + \sum_{i=1}^{n} A_{ii}E_{ii}$$

$$= \underbrace{\sum_{k<l} A_{kl}E_{kl}}_{\substack{i=k\\ j=l}} + \underbrace{\sum_{l>k} A_{kl}E_{lk}}_{\substack{i=l\\ j=k}} + \sum_{i=1}^{n} A_{ii}E_{ii}$$

$$= \sum_{k<l} A_{kl}(E_{kl}+E_{lk}) + \sum_{i=1}^{n} A_{ii}E_{ii} \qquad (*)$$

Let $\beta = \{E_{kl}+E_{lk} \mid 1 \le k < l \le n\} \cup \{E_{ii}\}_{i=1}^{n}$

We have $\beta \subseteq W$ hence $\operatorname{span}\beta \subseteq W$ $\left((E_{kl}+E_{lk})^T = E_{kl}+E_{lk} \text{ \& } E_{ii}^T = E_{ii} \right)$

and $W \subseteq \operatorname{span}\beta$ by our calculation $(*)$

Thus, $\underline{W = \operatorname{span}\beta}$. If we set

$$\sum_{k<l} c_{kl}(E_{kl}+E_{lk}) + \sum_{i=1}^{n} c_{ii}E_{ii} = 0$$

$$\Rightarrow \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{12} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{1n} & \cdots & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \Rightarrow \begin{matrix} c_{kl} = 0 & k<l \\ c_{ii} = 0 & \forall i \in \mathbb{N}_n \end{matrix}$$

Thus β is LI. (you could argue this from $(*)$ also!)

Now, <u>COUNT</u>. In total n^2-n are off-diagonal

But only need $\frac{1}{2} \Rightarrow \underbrace{\frac{n^2-n}{2}}_{k<l} + \underbrace{n}_{ii} = \boxed{\frac{n^2+n}{2} = \dim_{\mathbb{F}}(W)}$

$+ (\#\text{ of } \{E_{ii}\}_{i=1}^{n})$

$\boxed{PS3}$ $W = \{ p(t) \in \mathbb{R}[t] \mid p^{(n)}(1) = 0 \text{ for } n = 3, 4, 5, \ldots \}$

By Taylor's Th$^{\underline{m}}$, at $a = 1$,

$$p(t) = \sum_{n=0}^{\infty} \frac{p^{(n)}(1)}{n!} (t-1)^n$$

$$= p(1) + p'(1)(t-1) + \tfrac{1}{2} p''(1) (t-1)^2$$

Thus, $\boxed{W = \operatorname{span}\{1, t-1, (t-1)^2\} \leq \mathbb{R}[t]}$.

Alternate Sol$^{\underline{n}}$

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_n t^n$$

$$p'(t) = c_1 + 2c_2 t + 3c_3 t^2 + \cdots + n c_n t^{n-1}$$

$$p''(t) = 2c_2 + 6c_3 t + \cdots + n(n-1) c_n t^{n-2}$$

$$p'''(t) = 6c_3 + 24 c_4 t + \cdots + n(n-1)(n-2) c_n t^{n-3} = 0$$

$$\Rightarrow c_3 = c_4 = \cdots = c_n = 0$$

$$\therefore \quad p(t) = c_0 + c_1 t + c_2 t^2 \in \operatorname{span}\{1, t, t^2\}$$

We can also argue $\underline{W = \operatorname{span}\{1, t, t^2\} \leq \mathbb{R}[t]}$.

Q: WHICH WAY IS BEST ?

A: I DON'T KNOW.... GUESS METHOD 1 IS NICE IF YOU KNOW YOUR CALCULUS II.

$\boxed{P54}$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{r_4 - 3r_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_3/2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rref}(A)}$$

We find $\text{Col}(A) = \text{span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}}_{\text{BASIS for Col}(A).}$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A^T)$$

pivot columns of A^T give us which rows of A to use

$\text{Row}(A) = \text{span}\underbrace{\{ [1,0,0,3], [1,0,2,3] \}}_{\text{BASIS FOR Row}(A)}$

Remark: YOU COULD ANSWER THIS W/O CALCULATION! I MERELY CALCULATE TO ILLUSTRATE A METHOD WHICH WORKS FOR MUCHO-LESS-NICE-CHOICE OF A...

$x \in \text{Null}(A) \Rightarrow Ax = 0$

for $x = (x_1, x_2, x_3, x_4)$ from $\text{rref}(A|0) = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

we read $x_1 + x_2 = 0$ & $x_3 + x_4 = 0$

$\therefore \quad x = (-x_2, x_2, -x_4, x_4) = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

It follows $\text{Null}(A) = \text{span}\underbrace{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}}_{\text{BASIS FOR Null}(A).}$

P55 $A = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{1} \\ \bar{9} & \bar{1} \end{bmatrix} \in (\mathbb{Z}/11\mathbb{Z})^{3\times 2}$ find basis for Column, Row & Null space of A.

Observe $Col_1(A) \neq k\, Col_2(A) \therefore \{col_1(A), col_2(A)\}$ LI

Hence $\underline{\{(\bar{1}, \bar{2}, \bar{9}), (\bar{0}, \bar{1}, \bar{1})\}}$ forms basis for $Col(A)$.

Likewise $[\bar{1}, \bar{0}] \neq k\, [\bar{2}, \bar{1}]$ for any $k \in \mathbb{Z}/11\mathbb{Z}$

$\therefore$ $\underline{\{[\bar{1}, \bar{0}], [\bar{2}, \bar{1}]\}}$ forms basis for $Row(A)$.

since we know $\dim(Row(A)) = \dim(Col(A)) = 2$.

Remark: I did all this w/o calculation so far! I might be able to guess basis for Null(A) But, I'll behave and exhibit the method,

$$A = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{1} \\ \bar{9} & \bar{1} \end{bmatrix} \xrightarrow[r_3 - 9r_1]{r_2 - 2r_1} \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} \Rightarrow \underline{Av = 0 \Leftrightarrow v = 0}$$

Well, this was dumb. I didn't need to do this!

$rank(A) + \nu(A) = 2$ and $rank(A) = 2 \therefore \nu(A) = 0$

which means $Null(A) = \{0\} \Rightarrow$ $\boxed{\emptyset \text{ is basis for null space of A}}$

(or you can say $\nexists$ basis for Null(A)

I'm not sure about how our def^{n}'s fall on this question ☺)

$\boxed{P56}$ $W = \text{span}_{\mathbb{C}}\{1, x, x^2\}$ as v-space over $\mathbb{R}$

$g(x) \in W \Rightarrow \exists \underbrace{a_1 + ib_1, a_2 + ib_2, a_3 + ib_3}_{a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}} \in \mathbb{C}$

such that, $g(x) = (a_1 + ib_1)(1) + (a_2 + ib_2)x + (a_3 + ib_3)x^2$

$= a_1 + ib_1 + a_2 x + b_2(ix) + a_3 x^2 + b_3(ix^2)$

Hence $g(x) \in \text{span}_{\mathbb{R}}\{1, i, x, ix, x^2, ix^2\}$

Moreover, $c_1(1) + c_2(i) + c_3(x) + c_4(ix) + c_5(x^2) + c_6(ix^2) \underset{(*)}{=} 0$

Set $x = 0$, $c_1 + ic_2 = 0 \Rightarrow \underline{c_1 = 0}$ & $\underline{c_2 = 0}$.

Differentiate $(*)$ w.r.t. x, $c_3 + ic_4 + 2xc_5 + 2ixc_6 = 0 \quad (**)$

Set $x = 0$, $c_3 + ic_4 = 0 \Rightarrow \underline{c_3 = 0}$ & $\underline{c_4 = 0}$.

Differentiate $(**)$ ~~again~~, $2c_5 + 2ic_6 = 0 \Rightarrow \underline{c_5 = 0}$, & $\underline{c_6 = 0}$

$\therefore$ $\boxed{\{1, i, x, ix, x^2, ix^2\} \text{ forms basis of } W}$

$\boxed{P57}$ $\beta = \{x^2, (1+x)^2\}$ and define $W = \text{span}_{\mathbb{R}}\beta \leq P_2(\mathbb{R})$

Let $f(x) = ax^2 + bx + c \in W$ find $[f(x)]_\beta$

Comment, $x^2 \neq k(1+x^2) \Rightarrow \beta$ LI $\Rightarrow \beta$ BASIS of W as $W = \text{span}\,\beta$.

Solve, $c_1 x^2 + c_2(1+x)^2 = ax^2 + bx + c$

$\Rightarrow c_1 x^2 + c_2(1 + 2x + x^2) = ax^2 + bx + c$

$(c_1 + c_2)x^2 + 2c_2 x + c_2 = ax^2 + bx + c$

We deduce from equating coeff. of $1, x, x^2$ (use P58 if you wish)

$1: \quad c_2 = c \Rightarrow \underline{c_2 = c}$.

$x: \quad 2c_2 = b$

$x^2: \quad c_1 + c_2 = a \Rightarrow \underline{c_1 = a - c_2 = a - c}$. $\quad \therefore \boxed{[f(x)]_\beta = \begin{bmatrix} a-c \\ c \end{bmatrix}}$

(Notice, $c_2 = c = \frac{b}{2}$ so can have $[f(x)]_\beta = \begin{bmatrix} a - b/2 \\ b/2 \end{bmatrix}$ etc...

P58 I'll focus on showing $\{1, x, x^2, .., x^n\}$ is LI set.

Consider $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0 \quad (*)$
set $x=0$ in $(*)$ to obtain $c_0 = 0$. Next differentiate
w.r.t. x to obtain from $*$,

ok sol^{n}

$$c_1 + 2c_2 x + \cdots + n c_n x^{n-1} = 0 \quad (**)$$

Set $x=0$ and obtain $c_1 = 0$. Diff. again,

$$2c_2 + \cdots + n(n-1) c_n x^{n-2} = 0$$

Set $x=0$ and obtain $2c_2 = 0$. Continuing in this fashion obtain $c_3 = 0$, $c_4 = 0$, .., $c_n = 0$
Hence $\{1, x, x^2, .., x^n\}$ is LI set.

//

Better sol^{n}

By induction we can show this claim carefully,
Let $\{1, x, x^2, .., x^n\}$ being LI form the induction claim for $n \in \mathbb{N}$
For $n=1$, $\{1, x\}$ is LI since $1 = kx \quad \forall x \in \mathbb{R}$
is clearly absurd hence $\{1, x\}$ is not lin. dep.
Suppose inductively, $\{1, x, \ldots, x^n\}$ is LI and consider
$\{1, x, \ldots, x^n, x^{n+1}\}$. Set $\underline{c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + c_{n+1} x^{n+1} = 0} \quad *$
Let $x=0$ and obtain $c_0 = 0$. from $*$. Next, we
differentiate, $\underline{c_1 + 2c_2 x + \cdots + n c_n x^{n-1} + c_{n+1}(n+1) x^n = 0} \quad **$
But, by LI of $\{1, x, .., x^n\}$ we obtain

$$c_1 = 0, \quad 2c_2 = 0, \ldots, \quad n c_n = 0, \quad (n+1) c_{n+1} = 0$$

Thus, $c_0 = 0$, $c_1 = 0$, $c_2 = 0$, $c_n = 0$, $c_{n+1} = 0$.
Thus $\{1, .., x^{n+1}\}$ is LI and we find $\{1, x, .., x^n\}$ LI $\forall n \in \mathbb{N}$ by induction.//

$\boxed{P59}$ §6 #5 on p. 48

(a.) $S = \{x^2+2x+1,\ 2x+1,\ 2x^2-2x-1\}$ LI ?

Consider,

$$a(x^2+2x+1) + b(2x+1) + c(2x^2-2x-1) = 0$$

$$\Rightarrow (a+2c)x^2 + (2a+2b-2c)x + a+b-c = 0$$

(using P58 & P70 together) $\Rightarrow$ $a+2c=0$, $2a+2b-2c=0$, $a+b-c=0$

Consider,

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -6 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{r_2 - 2r_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

Ok, $\exists$ nontrivial sol^{n}s. I just need one set $c=1$ then $a=-2$ and $b=3$. Check it,

$$\boxed{-2(x^2+2x+1) + 3(2x+1) + 2x^2-2x-1 = 0}$$

a linear dependence of S.

(b.) $U = \{1, x-1, (x-1)^2, (x-1)^3\}$ LI ?

Set $c_1 + c_2(x-1) + c_3(x-1)^2 + c_4(x-1)^3 = 0$

Set $x=1$ to obtain $c_1 = 0$. Differentiate,

$$c_2 + 2c_3(x-1) + 3c_4(x-1)^2 = 0$$

Set $x=1$ to obtain $c_2=0$. Differentiate,

$$2c_3 + 6c_4(x-1) = 0$$

Set $x=1$ to obtain $2c_3 = 0 \Rightarrow c_3 = 0$. Finally, $6c_4(x-1) = 0$ thus set $x=2 \Rightarrow 6c_4 = 0$ $\therefore c_4 = 0$. We conclude $\{1, x-1, (x-1)^2, (x-1)^3\}$ is LI.

P60 §7 #4 on pg. 52

Let $S, T \leq \mathbb{R}^3$ and $\dim(S) = \dim(T) = 2$.
Prove $\dim(S \cap T) \geq 1$

I'll attempt to see this from the Th$^{\underline{m}}$,

$$\underline{\dim(S+T) + \dim(S \cap T) = \dim(S) + \dim(T)} \quad *$$

Thus, $\dim(S \cap T) = 4 - \dim(S+T)$

Since $S+T \leq \mathbb{R}^3 \Rightarrow \dim(S+T) \leq 3$ (Th$^{\underline{m}}$ in my notes)

$\therefore \quad \dim(S \cap T) = 4 - \dim(S+T) \geq 4 - 3 = 1.$

P61 §7 #5 on p. 52

Let
$$a_1 = \langle 2, 1, 0, -1 \rangle$$
$$a_2 = \langle 4, 8, -4, -3 \rangle$$
$$a_3 = \langle 1, -3, 2, 0 \rangle$$
$$a_4 = \langle 1, 10, -6, -2 \rangle$$
$$a_5 = \langle -2, 0, 6, 1 \rangle$$
$$a_6 = \langle 3, -1, 2, 4 \rangle$$

Find:
dim S
dim T
dim $T+S$
dim $T \cap S$ by $*$ above.

Let $S = \text{span}\{a_1, a_2, a_3, a_4\}$ & $T = \text{span}\{a_4, a_5, a_6\}$

I used the website www.math.odu.edu

$$\text{rref}\,[a_1\ a_2\ a_3\ a_4] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

CCP $\Rightarrow a_4 = -a_1 + a_2 - a_3$
$\{a_1, a_2, a_3\}$ LI, $\boxed{\dim S = 3}$.

$$\text{rref}\,[a_4\ a_5\ a_6] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\dim T = 3}$$ again by CCP.

$$\text{rref}\,[a_1\ a_2\ a_3\ a_4\ a_5\ a_6] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 332/3 \\ 0 & 1 & 0 & 1 & 0 & -118/3 \\ 0 & 0 & 1 & -1 & 0 & -203/3 \\ 0 & 0 & 0 & 0 & 1 & -10/3 \end{bmatrix}$$

By CCP, $\{a_1, a_2, a_3, a_5\}$ is LI $\Rightarrow \boxed{\dim(T+S) = 4}$
We calculate, $\dim(S \cap T) = \dim S + \dim T - \dim(T+S) = 6 - 4 = \boxed{2}$

P62 §8 #2 on pg. 61

For which values of α does the following system have sol$^{\underline{n}}$s?

$$3x_1 - x_2 + \alpha x_3 = 1$$
$$3x_1 - x_2 + x_3 = 5$$

Subtract eq$^{\underline{n}}$s to obtain $\alpha x_3 - x_3 = -4$

Thus $(\alpha - 1)x_3 = -4$. Oh, I'll behave,

$$\left[\begin{array}{ccc|c} 3 & -1 & \alpha & 1 \\ 3 & -1 & 1 & 5 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 3 & -1 & \alpha & 1 \\ 0 & 0 & 1-\alpha & 4 \end{array}\right] = M$$

If $\alpha \neq 1$ then, can reduce,

$$M \longrightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & \alpha/3 & 1/3 \\ 0 & 0 & 1 & 4/(1-\alpha) \end{array}\right] \quad \text{(consistent)}$$

<u>Hence for all $\alpha \neq 1$ the system has a sol$^{\underline{n}}$.</u>

Moreover, $x_3 = \dfrac{-4}{\alpha - 1}$ etc... (not that it asked for that)

P63 §8 #4 on pg. 62 of CURTIS

Prove that a system of homog. eq$^{\underline{n}}$s $\boxed{x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = 0}$ (*) in n-unknowns has a nontrivial sol$^{\underline{n}}$ iff rank $[C_1 | C_2 | \cdots | C_n] < n$.

Observe (*) is shorthand for m-eq$^{\underline{n}}$s in n-unknowns where $C_1, C_2, \dots, C_n \in \mathbb{R}^m$ (or $\mathbb{F}^m$ I suppose). Thus (*) is same as $Ax = 0$ for $A = [C_1 | C_2 | \cdots | C_n] \in \mathbb{F}^{m \times n}$.

I'll use Th$^{\underline{m}}$ 3.7.3; for $A \in \mathbb{F}^{m\times n}$, $n = \text{rank}(A) + \text{nullity}(A)$

$\text{nullity}(A) = \dim(\text{Null}(A))$ and $\text{Null}(A) = \{x \in \mathbb{F}^n \mid Ax = 0\}$.

Thus, to say $\exists x \neq 0$ sol$^{\underline{n}}$ to $Ax = 0$ is equivalent to asserting $\text{nullity}(A) \geq 1$. However, as $n = \text{rank}(A) + \text{nullity}(A)$

$$\text{rank}(A) < n \implies \text{nullity}(A) \geq 1 \quad \therefore \ Ax = 0 \text{ has nontrivial sol}^{\underline{n}}.$$

<u>Remark</u>: sorry about the less than helpful hint here.

P64 §9 #6 on p. 69

Let $(\alpha, \beta) \neq (\gamma, \delta)$ be distinct points in $\mathbb{R}^2$.
Prove $\exists\, A, B, C$ not all zero such that $(\alpha, \beta), (\gamma, \delta)$ solve

$$Ax + By + C = 0$$

and if both points also solve $A'x + B'y + C' = 0$ then $\exists\, \lambda \in \mathbb{R}$ such that $\langle A', B', C' \rangle = \lambda \langle A, B, C \rangle$.
Given a line in $\mathbb{R}^2$ is sol$^{\underline{n}}$ set to $Ax + By + C = 0$ for some $A, B, C \in \mathbb{R}$ (not all zero), show $(\alpha, \beta), (\gamma, \delta)$ distinct lie on a unique line.

There are many methods to find A, B, C. One approach, use two-point formula for line, for $\gamma \neq \alpha$,

$$y - \delta = \left(\frac{\delta - \beta}{\gamma - \alpha}\right)(x - \gamma) = \left(\frac{\delta - \beta}{\gamma - \alpha}\right)x - \gamma\left(\frac{\delta - \beta}{\gamma - \alpha}\right)$$

Hence,

$$\left(\frac{\beta - \delta}{\gamma - \alpha}\right)x + y - \delta + \gamma\left(\frac{\delta - \beta}{\gamma - \alpha}\right) = 0$$

I'll clean this up a bit, as $\gamma \neq \alpha$ multiply by $\gamma - \alpha \neq 0$,

$$(\beta - \delta)x + (\gamma - \alpha)y - \underline{\delta(\gamma - \alpha)} + \underline{\gamma(\delta - \beta)} = 0$$

$$\boxed{\underbrace{(\beta - \delta)}_{A}x + \underbrace{(\gamma - \alpha)}_{B}y + \underbrace{\alpha\delta - \gamma\beta}_{C} = 0} \quad (*)$$

Let's check (*). Plug in $x = \alpha$, $y = \beta$

$$(\beta - \delta)\alpha + (\gamma - \alpha)\beta + \alpha\delta - \gamma\beta = 0.$$

Likewise, $x = \gamma$, $y = \delta$ solve it

$$(\beta - \delta)\gamma + (\gamma - \alpha)\delta + \alpha\delta - \gamma\beta = 0.$$

P 64 continued

Suppose $\exists A', B', C'$ for which $A'x + B'y + C' = 0$ has $(\alpha, \beta), (\gamma, \delta)$ as sol'ns [$(\alpha, \beta) \neq (\gamma, \delta)$ still imposed].

$$\left.\begin{aligned} A'\alpha + B'\beta + C' &= 0 \\ A'\gamma + B'\delta + C' &= 0 \\ A\alpha + B\beta + C &= 0 \\ A\gamma + B\delta + C &= 0 \end{aligned}\right\} \quad \text{How to relate } A, A' \;\; B, B', \;\; C, C' \text{ ?}$$

P65 § 10 #2

Find homog. system whose sol$^{\underline{n}}$ set is generated by
$\langle 3,-1,1,2\rangle$, $\langle 4,-1,-2,3\rangle$, $\langle 10,-3,0,7\rangle$, $\langle -1,1,-7,0\rangle$
Then find nonhomogeneous system whose sol$^{\underline{n}}$ set is a linear manifold with the given vectors in the directing set and with base point $\langle 1,1,1,1\rangle$

Let $V_1 = (3,-1,1,2)$
$V_2 = (4,-1,-2,3)$
$V_3 = (10,-3,0,7)$
$V_4 = (-1,1,-7,0)$
and $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^{m\times 4}$

We want to find A s.t. $AV_1 = 0$, $AV_2 = 0$, $AV_3 = 0$, $AV_4 = 0$
which is to say $a_j V_k = 0$ for $j = 1,2,\ldots,m$ and $k = 1,2,3,4$.
It's easier to solve

$$a_j V_1 = 0, \quad a_j V_2 = 0, \quad a_j V_3 = 0, \quad a_j V_4 = 0$$

$$\Rightarrow a_j [V_1 | V_2 | V_3 | V_4] = [0,0,0,0]$$

Simply transpose to find a problem we have tools to solve!

$$\begin{bmatrix} V_1^T \\ \hline V_2^T \\ \hline V_3^T \\ \hline V_4^T \end{bmatrix} a_j^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Calculate $\operatorname{rref}\begin{bmatrix} 3 & -1 & 1 & 2 \\ 4 & -1 & -2 & 3 \\ 10 & -3 & 0 & 7 \\ -1 & 1 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -10 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

If $a_j = [a_{j1}, a_{j2}, a_{j3}, a_{j4}]$ then the calculation above reveals $a_{j1} = 3a_{j3} - a_{j4}$ and $a_{j2} = 10a_{j3} - a_{j4}$

p65 continued

The sol$^{\underline{n}}$ to our problem is given by

$$a_j = [3a_{j3} - a_{j4},\ 10a_{j3} - a_{j4},\ a_{j3},\ a_{j4}]$$
$$= a_{j3}[3, 10, 1, 0] + a_{j4}[-1, -1, 0, 1]$$

Apparently, two eq$^{\underline{n}}$'s will do nicely,

$$\boxed{\begin{bmatrix} 3 & 10 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$ ← an example of the desired homogeneous sol$^{\underline{n}}$.

Has sol$^{\underline{n}}$'s which include V_1, V_2, V_3, V_4.

Finding the nonhomogeneous system with $(1,1,1,1)$ in sol$^{\underline{n}}$ set is not hard! Consider,

$$\begin{bmatrix} 3 & 10 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 1 \end{bmatrix}$$

Thus $$\boxed{\begin{bmatrix} 3 & 10 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 14 \\ 1 \end{bmatrix}}$$ ← the desired nonhomogeneous system.

<u>Remark</u>: to find linear manifold $P + S$ as sol$^{\underline{n}}$ set we can take the generating set β for $S' = \text{span}\,\beta$ and calculate $\text{rref}[\beta]^T$. This will show us how to realize S' as a sol$^{\underline{n}}$ set of a homog. system. Then simply plug in P to find the nonhomog. system.
– (in other words, you can generalize my sol$^{\underline{n}}$ here) –

P66 §10 # 7 on pg. 74

Suppose p & q are vectors in a linear manifold V. Show the line through p & q is contained in V

We are given $V = p_o + S$ for some subspace S.

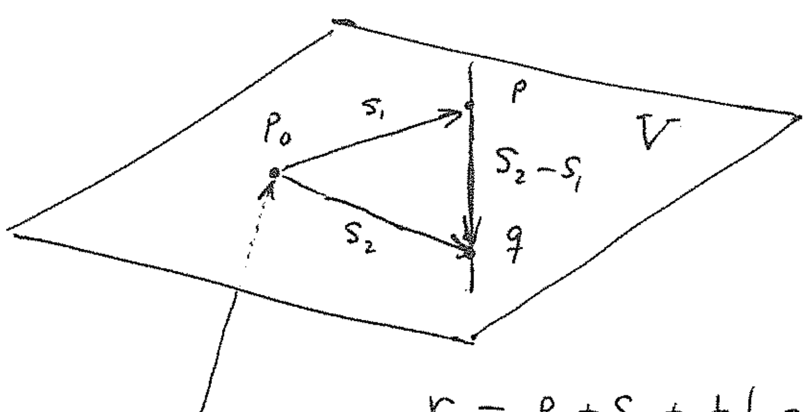
Furthermore $p, q \in V \Rightarrow \exists s_1, s_2 \in S$ s.t. $p = p_o + s_1$ & $q = p_o + s_2$.

A line through p & q is parametrized by $r = p + t(q - p)$ for $t \in \mathbb{R}$. Observe,

$$
\begin{aligned}
r &= p + t(q - p) \\
&= (1+t)p + tq \\
&= (1-t)(p_o + s_1) + t(p_o + s_2) \\
&= p_o - \cancel{tp_o} + (1-t)s_1 + \cancel{tp_o} + ts_2 \\
&= p_o + (1-t)s_1 + ts_2 \\
&= p_o + s_3 \quad \text{as } \underbrace{(1-t)s_1 + ts_2}_{*} = s_3 \in S \leftarrow \text{subspace}
\end{aligned}
$$

so we know $s_1, s_2 \in S$ implies the linear combo $*$ once more in S

Thus $r \in V$ for each $t \in \mathbb{R}$ and this shows the line connecting p & q is contained in V.



$r = p_o + s_1 + t(s_2 - s_1)$

(you can see this in picture)

P67 §10 #8 on p. 74

Consider subspaces S & T of $\mathbb{R}^n$. These are sol$^{\underline{n}}$ sets of

$$S: \quad a_1 \cdot x = 0, \dots, \quad a_r \cdot x = 0$$
$$T: \quad b_1 \cdot x = 0, \dots, \quad b_s \cdot x = 0$$

Prove that $S \cap T$ is sol$^{\underline{n}}$ set to both systems at once.

Also, find basis for $S \cap T$ where S & T are given by

$S = \text{span}\{V_1, V_2, V_3, V_4\}$

$T = \text{span}\{V_4, V_5, V_6\}$

Remark: $a_1, a_2, a_3, a_4, a_5, a_6$ in Ex #5 of §7 are not same as those given at start of this problem. I relabled as $V_1, V_2, \dots, V_6$

$V_1 = \langle 2, 1, 0, -1 \rangle$

$V_2 = \langle 4, 8, -4, -3 \rangle$

$V_3 = \langle 1, -3, 2, 0 \rangle$

$V_4 = \langle 1, 10, -6, -2 \rangle$

$V_5 = \langle -2, 0, 6, 1 \rangle$

$V_6 = \langle 3, -1, 2, 4 \rangle$

Let $x \in S \cap T$ then $x \in S$ and $x \in T$ thus $a_1 \cdot x = 0, \dots, a_r \cdot x = 0$ as $x \in S$ and $b_1 \cdot x = 0, \dots, b_s \cdot x = 0$ as $x \in T$. Thus x solves both systems at once. //

We have $S \cap T$ is the sol$^{\underline{n}}$ set of

$$\left.\begin{array}{l} a_1 \cdot x = 0 \\ a_2 \cdot x = 0 \\ \vdots \\ a_r \cdot x = 0 \\ b_1 \cdot x = 0 \\ b_2 \cdot x = 0 \\ \vdots \\ b_s \cdot x = 0 \end{array}\right\} \longrightarrow \left[\frac{A}{B}\right] x = \left[\frac{0}{0}\right]$$

In this notation $S = \text{Null}(A)$ whereas $T = \text{Null}(B)$. we've shown $S \cap T = \text{Null}\left(\frac{A}{B}\right)$. In order to use this for the Ex #5 data we need to realize S & T as sol$^{\underline{n}}$ sets,

P67 continued

$$S^{\perp} = \text{span}\{V_1, V_2, V_3, V_4\} = \text{Null}(A)$$

$$\text{rref}\begin{bmatrix} V_1^T \\ \hline V_2^T \\ \hline V_3^T \\ \hline V_4^T \end{bmatrix} = \text{rref}\begin{bmatrix} 2 & 1 & 0 & -1 \\ 4 & 8 & -4 & -3 \\ 1 & -3 & 2 & 0 \\ 1 & 10 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} a_{j1} &= \tfrac{1}{2} a_{j4} \\ a_{j2} &= 0 \\ a_{j3} &= \tfrac{-1}{4} a_{j4} \end{aligned}\right\} \quad \begin{aligned} a_j &= [a_{j1}, a_{j2}, a_{j3}, a_{j4}] \\ &= [1/2\, a_{j4},\ 0,\ \tfrac{-1}{4} a_{j4},\ a_{j4}] \\ &= a_{j4}\underbrace{[\,1/2,\ 0,\ -1/4,\ 1\,]}_{A} \end{aligned}$$

\#

$$T = \text{span}\{V_4, V_5, V_6\} = \text{Null}(B)$$

$$\text{rref}\begin{bmatrix} V_4^T \\ \hline V_5^T \\ \hline V_6^T \end{bmatrix} = \text{rref}\begin{bmatrix} 1 & 10 & -6 & -2 \\ -2 & 0 & 6 & 1 \\ 3 & -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$$\left.\begin{aligned} b_{j1} &= -b_{j4} \\ b_{j2} &= 0 \\ b_{j3} &= \tfrac{-1}{2} b_{j4} \end{aligned}\right\} \quad \begin{aligned} b_j &= [-b_{j4},\ 0,\ \tfrac{-1}{2} b_{j4},\ b_{j4}] \\ &= b_{j4}\underbrace{[-1,\ 0,\ \tfrac{-1}{2},\ 1]}_{B} \end{aligned}$$

Thus, the $S \cap T$ is given by

$$\text{Null}\left(\frac{A}{B}\right) = \text{Null}\begin{bmatrix} 1/2 & 0 & -1/4 & 1 \\ -1 & 0 & -1/2 & 1 \end{bmatrix} = \text{Null}\begin{bmatrix} 2 & 0 & -1 & 4 \\ -2 & 0 & -1 & 2 \end{bmatrix}$$

$$\text{rref}\begin{bmatrix} 2 & 0 & -1 & 4 \\ -2 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \quad \therefore\ X \in S \cap T$$

has $X_1 = \frac{-1}{2} X_4,\ X_3 = 3X_4 \quad \therefore\ X = \begin{bmatrix} -X_4/2 \\ X_2 \\ 3X_4 \\ X_4 \end{bmatrix} = X_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + X_4\begin{bmatrix} -1/2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

P67 continued

We find $S \cap T = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$. (this is in agreement with P61)

P68 Suppose $U \leq V$ and $W \leq V$ where V is vector space over $\mathbb{F}$.
Consider $0 \in U$ and $0 \in W$ thus $0 \in U \cap W \neq \phi$.
Let $x, y \in U \cap W$ and $c \in \mathbb{F}$. Then $x, y \in U$ and $x, y \in W$. But, $U \leq V \Rightarrow cx + y \in U$ and $W \leq V \Rightarrow cx + y \in W$. Thus $cx + y \in U \cap W$ and hence $cx,\ x + y \in U \cap W$ $\therefore$ $U \cap W \leq V$ by Subspace Th^{m}.

P69 Let $V(\mathbb{F})$ be vector space and $U \leq V$ and $W \leq V$.
Consider $U + W = \{x + y \mid x \in U, y \in W\}$. Observe $0 \in U$ and $0 \in W$ thus $0 + 0 = 0 \in U + W \neq \phi$.
Consider $z, w \in U + W$ and $\alpha \in \mathbb{F}$. Then, by def^{n} of $U + W$, $\exists\, a, b \in U$ and $c, d \in W$ s.t. $z = a + c$ & $w = b + d$.
Observe, $\alpha z + w = \alpha(a + c) + b + d = \underline{(\alpha a + b) + (\alpha c + d)}$. $*$
Note, $\alpha a + b \in U$ and $\alpha c + d \in W$ as $a, b \in U \leq V$ and $c, d \in W \leq V$. Thus $*$ shows $\alpha z + w \in U + W$, Hence, $\alpha z,\ z + w \in U + W$ and we conclude $U + W \leq V$ by the Subspace Test Th^{m}. It remains to show $U + W$ is <u>smallest</u> subspace which contains $U \cup W$.

$\boxed{P69}$

Note, $U \cup W$ is clearly a subset of $U+W$ since $x \in U \cup W \Rightarrow \underline{x \in U}_{①}$ or $\underline{x \in W}_{②}$ and $x+0$ is clearly in $U+W$ in both cases. ① & ②.

Let $H \leq V$ such that $U \cup W \subseteq H$. Hence if $x \in U$ and $y \in W$ then $x, y \in H$ and thus $x+y \in H$. But, every element in $U+W$ can be written as $x+y$ for $x \in U, y \in W$ thus $U+W \subseteq H \Rightarrow U+W$ is smallest subspace containing $U \cup W$.

$\boxed{P70}$ Suppose S is LI subset of $V(\mathbb{F})$. Also, suppose $\exists a_i, b_i \in \mathbb{F}$ for $i \in \mathbb{N}_n$ and $v_1, \dots, v_n \in S$ such that

$$a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$$

Then, by algebra,

$$(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n = 0$$

hence $a_1 - b_1 = 0, \dots, a_n - b_n = 0$ by LI of S. Thus $a_1 = b_1, \dots, a_n = b_n$ and we've shown S allows for equating coefficients.

Conversely, suppose S allows us to equate coefficients. If $v_1, v_2, \dots, v_n \in S$ and $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ then note $c_1 v_1 + \cdots + c_n v_n = 0 \cdot v_1 + \cdots + 0 \cdot v_n (= 0)$ thus equating coefficients yields $c_1 = 0, \dots, c_n = 0 \therefore S$ is LI.//

<u>Remark</u>: we oft find use of equating coefficients in our study of calculus. Thinking back on calculus II there is much linear algebra in the partial-fractions technique...