Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 6 of the notes: ____________________.

Problem 55 In each of the following, give at least one reason why W is not a subspace of V over $\mathbb{F}$

- **(a.)** $W = \{(x, y, z) \mid x, y, z \geq 0\}$ in $V = \mathbb{R}^3$ over $\mathbb{F} = \mathbb{R}$,
- **(b.)** $W = \{1 + ax + bx^2 \mid a, b \in \mathbb{R}\}$ in $V = \mathbb{R}[x]$ over $\mathbb{F} = \mathbb{R}$,
- **(c.)** $W = \{(x, y) \mid x, y \in \mathbb{R}\}$ in $V = \mathbb{C}^2$ over $\mathbb{F} = \mathbb{C}$,
- **(d.)** $W = \{A \in \mathbb{R}^{n\times n} \mid A^3 + 3A^2 + A = 0\}$ over $\mathbb{F} = \mathbb{R}$,
- **(e.)** $W = \mathbb{R} - \mathbb{Q}$ in $V = \mathbb{R}$ over $\mathbb{Q}$.

Problem 56 Consider $W = \{f(x) \in \mathbb{R}[x] \mid f(3) = 0\}$ show $W \leq \mathbb{R}[x]$.

Problem 57 Let $W = \{(a + 2b + 3c, b + c, c - 2a) \mid a, b, c \in \mathbb{R}\}$. Prove W is a subspace of $\mathbb{R}^3$.

Problem 58 Consider $W = \{A \in \mathbb{C}^{2\times 2} \mid \text{trace}(A + \overline{A}) = 0\}$ where $\text{trace}(A) = A_{11} + A_{22}$ and $\overline{A}$ denotes the complex conjugate of A. Show W is a subspace of $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$.

Problem 59 Show $\{E_{11}, E_{12} + E_{21}, E_{12} - E_{21}\}$ is a linearly independent subset of $\mathbb{R}^{2\times 2}$.

Problem 60 Let $\beta = \{(1, 2), (1, 3)\}$.

- **(a.)** Calculate $[(a, b)]_\beta$.
- **(b.)** If $[(a, b)]_\beta = (3, 4)$ then find (a, b).

Problem 61 Let $\beta = \{(1, 1, 2, 2), (3, 2, 2, 1), (0, 0, 1, 0)\}$ provide a basis for $W = \text{span}(\beta)$. Determine what condition is needed for $(a, b, c, d) \in W$ and given that condition find the coordinate vector of (a, b, c, d) with respect to β.

Problem 62 Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define $W = \{M \in \mathbb{R}^{2\times 2} \mid MJ = JM\}$.

- **(a.)** show W is a subspace of $\mathbb{R}^{2\times 2}$
- **(b.)** Find a basis β for W and calculate $\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right]_\beta$ in terms of a and b.

Problem 63 Suppose S is a subset of $\mathbb{R}^n$. Define $S^\perp = \{v \in \mathbb{R}^n \mid v \bullet s = 0 \text{ for all } s \in S\}$.

- **(a.)** Prove $S^\perp$ is a subspace of $\mathbb{R}^n$.
- **(b.)** Find a basis for $S^\perp$ given $S = \{(1, 2, 2, 1), (0, 1, 0, 1)\}$.

Problem 64 Let $W_1 = \text{span}\{1 - x^2, x - x^3\}$ and $W_2 = \{a + bx + cx^2 + dx^3 \mid a + b + c - 2d = 0\}$. Find a basis for $W_1 \cap W_2$.

Problem 65 Suppose $\beta = \{E_{11}, iE_{11}, E_{12}, iE_{12}, E_{21}, iE_{21}, E_{22}, iE_{22}\}$ is given as a basis for $\mathbb{C}^{2\times 2}$. If $[A]_\beta = (1, 2, 3, 4, 5, 6, 7, 8)$ then what is A ?

Problem 66 Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 3 & 4 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix}$. Find bases for

- **(a.)** the column space of A (use columns of A itself to form your answer)
- **(b.)** the row space of A (use rows of A itself to form your answer)
- **(c.)** the null space of A

Problem 67 **Prove:** If V be a vector space with $U \leq V$ and $W \leq V$ then $U \cap W \leq V$.

Problem 68 Equating coefficients: Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$.
Prove: $S = \{v_1, \ldots, v_k\}$ is a linearly independent set of vectors iff if there exist $a_i, b_i \in \mathbb{F}$ for $i \in \mathbb{N}_k$ for which

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k$$

then $a_i = b_i$ for each $i = 1, 2, \ldots, k$.

Problem 69 The **trace** of a square matrix is simply the sum of its diagonal entries; $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in R^{n \times n}$ where R is a commutative ring with identity. Prove the following:

- **(a.)** $\text{trace}(I_n) = n$ where I_n is the $n \times n$ identity matrix.
- **(b.)** $\text{trace}(cA + B) = c\text{trace}(A) + \text{trace}(B)$ for all $A, B \in R^{n \times n}$ and $c \in R$,
- **(c.)** $\text{trace}(AB) = \text{trace}(BA)$ for all $A \in R^{p \times n}$ and $B \in R^{n \times p}$.

Problem 70 We see how the trace connects to the study of dimension in Lecture. It should be noted this trace has many other uses. This problem asks you to understand an application of the trace to the study of the **commutator**: $[A, B] = AB - BA$ for square matrices A, B. **Explain how the trace allows us to say:**

$$[A, B] \neq kI$$

for any square matrices A, B **and scalar** $k \neq 0$.

Problem 71 Differential equations provides interesting examples of vector spaces. In particular, consider the differential equation $y'' - 3y' + 2y = 0$. We say f is a solution of the differential equation if $f'' - 3f' + 2f = 0$. Let W be the set of solutions for $y'' - 3y' + 2y = 0$. Prove W is a subspace of the set of functions on $\mathbb{R}$.

Problem 72 Consider the rather simple differential equation $y'' = 0$ where $y' = \frac{dy}{dx}$. Find a basis for the solution space. What is the dimension of the solution space?