Your PRINTED NAME indicates you read Chapter 4 of the notes: ____________________.

We assume $\mathbb{F}$ is a field and V, W are vector spaces over $\mathbb{F}$.

Problem 49 Let $\mathcal{A}$ be an n-dimensional vector space over $\mathbb{R}$ with a multiplication $\star : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ which is $\mathbb{R}$-bilinear, unitial and associative. Bilinearity means for all $c_1, c_2 \in \mathbb{R}$ and $x_1, x_2, y \in \mathcal{A}$:

$$(c_1x_1 + c_2x_2) \star y = c_1x_1 \star y + c_2x_2) \star y \qquad \& \qquad y \star (c_1x_1 + c_2x_2) = c_1y \star x_1 + c_2y \star x_2$$

and unital gives the existence of $1_{\mathcal{A}} \in \mathcal{A}$ such that $1_{\mathcal{A}} \star x = x = x \star 1_{\mathcal{A}}$ for each $x \in \mathcal{A}$. Finally, associativity means $(x \star y) \star z = (x \star y) \star z$ for all $x, y, z \in \mathcal{A}$. We define **left multiplication by** x as $\ell_x : \mathcal{A} \to \mathcal{A}$ by $\ell_x(y) = x \star y$ for all $y \in \mathcal{A}$. Likewise, we define **right multiplication by** x as $r_x : \mathcal{A} \to \mathcal{A}$ by $r_x(y) = y \star x$ for each $y \in \mathcal{A}$. Show:

(a.) show that $\mathcal{R}(\mathcal{A}) = \{\ell_a \mid a \in \mathcal{A}\}$ is a subspace of $\mathcal{L}(\mathcal{A})$

(b.) Show $\ell_a \circ \ell_b = \ell_{a\star b}$ and $r_a \circ r_b = r_{b\star a}$ for all $a, b \in \mathcal{A}$

(c.) Show $\ell_x \circ r_y = r_y \circ \ell_x$ for all $x, y \in \mathcal{A}$

(d.) Show $\mathcal{R}(\mathcal{A})$ is isomorphic to $\mathcal{A}$ as a unital associative algebra over $\mathbb{R}$ provided we regard the multiplication on $\mathcal{R}(\mathcal{A})$ as composition of maps.

Remark: $\mathcal{R}(\mathcal{A})$ is known as the **regular representation** of $\mathcal{A}$. In the next problem we see how to understand the regular representation in terms of matrices if we are given a basis β for $\mathcal{A}$.

Problem 50 Let $\mathcal{A}$ be an n-dimensional vector space which forms a unital associative algebra with multiplication $\star$. If β is a basis for $\mathcal{A}$ then define:

$$\mathbf{M}_{\mathcal{A}}(\beta) = \{[\ell_x]_{\beta,\beta} \mid x \in \mathcal{A}\}$$

Show the following:

(a.) $\mathbf{M}_{\mathcal{A}}(\beta) \leq \mathbb{R}^{n\times n}$

(b.) Define $\Psi : \mathcal{A} \to \mathbf{M}_{\mathcal{A}}(\beta)$ by $\Psi(x) = [\ell_x]_{\beta,\beta}$ for each $x \in \mathcal{A}$. Show $\Psi(x\star y) = \Psi(x)\Psi(y)$ and $\Psi(1_{\mathcal{A}}) = I \in \mathbb{R}^{n\times n}$

Problem 51 Let vector $v = \langle a, b, c\rangle$ and define

$$\omega_v = adx + bdy + cdz \qquad \& \qquad \Phi_v = ady \wedge dz + bdz \wedge dx + cdx \wedge dy.$$

Here we use the notation dx, dy, dz for the dual basis to the standard basis e_1, e_2, e_3 for $\mathbb{R}^3$. I usually call ω_v the **work form** and Φ_v the **flux form** corresponding to v. Show:

(a.) Show $\omega_v \wedge \omega_w = \Phi_{v\times w}$ where $v \times w$ denotes the usual cross-product of vectors in $\mathbb{R}^3$

(b.) Show $\omega_u \wedge \omega_v \wedge_w = u \bullet (v \times w) dx \wedge dy \wedge dz$

Problem 52 Consider $v = (a, b, c, d)$ and $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$. Calculate $e_1 \wedge e_2 \wedge v$ and determine what condition(s) is needed for $\{v, e_1, e_2\}$ to be linearly independent.

Problem 53 Let $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where $A \in \mathbb{F}^{m \times m}$ and $B \in \mathbb{F}^{k \times k}$. Prove that

$$\det(A \oplus B) = \det(A)\det(B)$$

using the wedge product algebra definition of the determinant.

Problem 54 Let $A \in \mathbb{F}^{n \times n}$. Prove $\det(A) = 0$ if and only if $Ax = 0$ has a nonzero solution. (you cannot simply quote Theorem 4.3.4 part (2.), however, you can use the proof there as a solution, the point of this problem is I'd like you to digest that proof and put it into your own words)

Problem 55 Friedberg, Insel and Spence 5th edition, §5.1#3d, f page 257.

Problem 56 Friedberg, Insel and Spence 5th edition, §5.1#4a, c, page 258.

Problem 57 Friedberg, Insel and Spence 5th edition, §5.1#5f, h, page 258.

Problem 58 Friedberg, Insel and Spence 5th edition, §5.1#7, page 258.

Problem 59 Friedberg, Insel and Spence 5th edition, §5.2#3a, c, e, pages 278-279.

Problem 60 Friedberg, Insel and Spence 5th edition, §5.2#8, page 279.

Problem 61 Friedberg, Insel and Spence 5th edition, §5.2#13, page 280.

Problem 62 If $p(t) = a_o + a_1 t + \cdots + a_n t^n \in \mathbb{R}[t]$ then we define $p(A) = a_o I + a_1 A + \cdots + a_n A^n$. Prove that if v is eigenvector of A with eigenvalue λ then v is also an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.

Problem 63 A square matrix A is nilpotent of degree k if $A^{k-1} \neq 0$ yet $A^k = 0$. Prove $\lambda = 0$ is the only eigenvalue of A.

Problem 64 Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$. Find the real eigenvalues and eigenvectors of A. Also, calculate A^n.