Your PRINTED NAME indicates you read Chapter 4 of the notes: ________________.

We assume $\mathbb{F}$ is a field and V, W are vector spaces over $\mathbb{F}$.

Problem 49 Let $\mathcal{A}$ be an n-dimensional vector space over $\mathbb{R}$ with a multiplication $\star : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is $\mathbb{R}$-bilinear, unitial and associative. Bilinearity means for all $c_1, c_2 \in \mathbb{R}$ and $x_1, x_2, y \in \mathcal{A}$:

$$(c_1x_1 + c_2x_2) \star y = c_1x_1 \star y + c_2x_2) \star y \qquad \& \qquad y \star (c_1x_1 + c_2x_2) = c_1y \star x_1 + c_2y \star x_2$$

and unital gives the existence of $1_{\mathcal{A}} \in \mathcal{A}$ such that $1_{\mathcal{A}} \star x = x = x \star 1_{\mathcal{A}}$ for each $x \in \mathcal{A}$. Finally, associativity means $(x \star y) \star z = (x \star y) \star z$ for all $x, y, z \in \mathcal{A}$. We define **left multiplication by** x as $\ell_x : \mathcal{A} \to \mathcal{A}$ by $\ell_x(y) = x \star y$ for all $y \in \mathcal{A}$. Likewise, we define **right multiplication by** x as $r_x : \mathcal{A} \to \mathcal{A}$ by $r_x(y) = y \star x$ for each $y \in \mathcal{A}$. Show:

(a.) show that $\mathcal{R}(\mathcal{A}) = \{\ell_a \mid a \in \mathcal{A}\}$ is a subspace of $\mathcal{L}(\mathcal{A})$

(b.) Show $\ell_a \circ \ell_b = \ell_{a\star b}$ and $r_a \circ r_b = r_{b\star a}$ for all $a, b \in \mathcal{A}$

(c.) Show $\ell_x \circ r_y = r_y \circ \ell_x$ for all $x, y \in \mathcal{A}$

(d.) Show $\mathcal{R}(\mathcal{A})$ is isomorphic to $\mathcal{A}$ as a unital associative algebra over $\mathbb{R}$ provided we regard the multiplication on $\mathcal{R}(\mathcal{A})$ as composition of maps.

Remark: $\mathcal{R}(\mathcal{A})$ is known as the **regular representation** of $\mathcal{A}$. In the next problem we see how to understand the regular representation in terms of matrices if we are given a basis β for $\mathcal{A}$.

Problem 50 Let $\mathcal{A}$ be an n-dimensional vector space which forms a unital associative algebra with multiplication $\star$. If β is a basis for $\mathcal{A}$ then define:

$$\mathbf{M}_{\mathcal{A}}(\beta) = \{[\ell_x]_{\beta,\beta} \mid x \in \mathcal{A}\}$$

Show the following:

(a.) $\mathbf{M}_{\mathcal{A}}(\beta) \leq \mathbb{R}^{n\times n}$

(b.) Define $\Psi : \mathcal{A} \to \mathbf{M}_{\mathcal{A}}(\beta)$ by $\Psi(x) = [\ell_x]_{\beta,\beta}$ for each $x \in \mathcal{A}$. Show $\Psi(x\star y) = \Psi(x)\Psi(y)$ and $\Psi(1_{\mathcal{A}}) = I \in \mathbb{R}^{n\times n}$

Problem 51 Let vector $v = \langle a, b, c\rangle$ and define

$$\omega_v = adx + bdy + cdz \qquad \& \qquad \Phi_v = ady \wedge dz + bdz \wedge dx + cdx \wedge dy.$$

Here we use the notation dx, dy, dz for the dual basis to the standard basis e_1, e_2, e_3 for $\mathbb{R}^3$. I usually call ω_v the **work form** and Φ_v the **flux form** corresponding to v. Show:

(a.) Show $\omega_v \wedge \omega_w = \Phi_{v\times w}$ where $v \times w$ denotes the usual cross-product of vectors in $\mathbb{R}^3$

(b.) Show $\omega_u \wedge \omega_v \wedge_w = u \bullet (v \times w)dx \wedge dy \wedge dz$

Problem 52 Consider $v = (a, b, c, d)$ and $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$. Calculate $e_1 \wedge e_2 \wedge v$ and determine what condition(s) is needed for $\{v, e_1, e_2\}$ to be linearly independent.

Problem 53 Let $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where $A \in \mathbb{F}^{m \times m}$ and $B \in \mathbb{F}^{k \times k}$. Prove that

$$\det(A \oplus B) = \det(A)\det(B)$$

using the wedge product algebra definition of the determinant.

Problem 54 Let $A \in \mathbb{F}^{n \times n}$. Prove $\det(A) = 0$ if and only if $Ax = 0$ has a nonzero solution. (you cannot simply quote Theorem 4.3.4 part (2.), however, you can use the proof there as a solution, the point of this problem is I'd like you to digest that proof and put it into your own words)

Problem 55 Friedberg, Insel and Spence 5th edition, §5.1#3d, f page 257.

Problem 56 Friedberg, Insel and Spence 5th edition, §5.1#4a, c, page 258.

Problem 57 Friedberg, Insel and Spence 5th edition, §5.1#5f, h, page 258.

Problem 58 Friedberg, Insel and Spence 5th edition, §5.1#7, page 258.

Problem 59 Friedberg, Insel and Spence 5th edition, §5.2#3a, c, e, pages 278-279.

Problem 60 Friedberg, Insel and Spence 5th edition, §5.2#8, page 279.

Problem 61 Friedberg, Insel and Spence 5th edition, §5.2#13, page 280.

Problem 62 If $p(t) = a_o + a_1 t + \cdots + a_n t^n \in \mathbb{R}[t]$ then we define $p(A) = a_o I + a_1 A + \cdots + a_n A^n$. Prove that if v is eigenvector of A with eigenvalue λ then v is also an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.

Problem 63 A square matrix A is nilpotent of degree k if $A^{k-1} \neq 0$ yet $A^k = 0$. Prove $\lambda = 0$ is the only eigenvalue of A.

Problem 64 Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$. Find the real eigenvalues and eigenvectors of A. Also, calculate A^n.

MISSION 4 SOLUTION

$\boxed{P49}$ Let A be m-dim'l algebra over $\mathbb{R}$ with multiplication $*: A \times A \longrightarrow A$ which is bilinear and associative with unity 1_A. We define $l_x(y) = x * y$ and $r_x(y) = y * x$

(a.) $\mathcal{R}(A) = \{ l_a \mid a \in A \}$. We seek to show $\mathcal{R}(A) \leq \mathcal{L}(A)$. Let $x, y \in A$ and $c \in \mathbb{R}$ if $l_a \in \mathcal{R}(A)$ then

$$\begin{aligned} l_a(cx+y) &= a*(cx+y) \\ &= c\, a*x + a*y \\ &= c\, l_a(x) + l_a(y) \quad \therefore l_a \in \mathcal{L}(A) \end{aligned}$$

We find $\mathcal{R}(A) \subseteq \mathcal{L}(A)$. Also, since $l_{1_A}(x) = 1_A * x = x$ we see $Id_A \in \mathcal{R}(A) \neq \emptyset$. Suppose $l_a, l_b \in \mathcal{R}(A)$ and let $c \in \mathbb{R}$. If $x \in A$ then

$$\begin{aligned} (c l_a + l_b)(x) &= c\, l_a(x) + l_b(x) \\ &= c\, a*x + b*x \\ &= (ca+b)*x \\ &= l_{ca+b}(x) \quad \therefore c l_a + l_b = l_{ca+b} \in \mathcal{R}(A). \end{aligned}$$

Thus $\mathcal{R}(A) \leq \mathcal{L}(A)$ by subspace test.//

(b.) $$\begin{aligned} (l_a \circ l_b)(x) = l_a(l_b(x)) = l_a(b*x) &= a*(b*x) \quad \text{associativity} \\ &= (a*b)*x \\ &= l_{a*b}(x) \end{aligned}$$

Thus $l_a \circ l_b = l_{a*b}$ as the above holds $\forall x \in A$.

Likewise, $(r_a \circ r_b)(x) = r_a(r_b(x)) = r_a(x*b) = (x*b)*a = x*(b*a)$
$= r_{b*a}(x)$.

Thus $r_a \circ r_b = r_{b*a}$.

P49

(c.) $(l_x \circ r_y)(z) = l_x(r_y(z))$

$$= l_x(z * y)$$
$$= x * (z * y)$$
$$= (x * z) * y$$

associativity

$$= r_y(x * z)$$
$$= r_y(l_x(z))$$
$$= \underline{(r_y \circ l_x)(z)} \; * \quad \Rightarrow \underline{l_x \circ r_y = r_y \circ l_x}.$$

since $*$ holds $\forall z \in \mathcal{A}$.

(d.) Show $\mathcal{R}(A)$ isomorphic to A

$\Psi : A \longrightarrow \mathcal{R}(A)$ define by $\Psi(a) = l_a$.

Notice $\Psi(ca+b) = \underbrace{l_{ca+b} = c\,l_a + l_b}_{\text{shown in (a.)}} = c\,\Psi(a) + \Psi(b)$.

Thus Ψ is linear mapping. Notice $\Psi^{-1}(l_a) = a$ is clear thus Ψ is a bijection.

If you doubt me, we find Ψ is 1-to-1 since,

$$\text{Ker}(\Psi) = \{a \in A \mid l_a = 0\}$$
$$= \{a \in A \mid a * x = 0 \;\; \forall x \in A\}$$
$$= \{0\} \quad (\text{choose } x = 1_A \text{ to see } a = 0)$$

If $l_a \in \mathcal{R}(A)$ then $\Psi(a) = l_a \therefore \Psi$ onto.

Finally, notice $\Psi(1_A) = l_{1_A} = Id_A$ as desired and $\underline{\Psi(a * b) = l_{a*b} = l_a \circ l_b = \Psi(a) \circ \Psi(b)}$. (by (b.))

thus Ψ is an algebra isomorphism, it is a linear bijection which preserves the product.

$\boxed{P50}$ Let $\mathcal{A}$ be n-dim'l unital associative algebra over $\mathbb{R}$ with multiplication $*$. If β is basis for $\mathcal{A}$ then define

$$M_{\mathcal{A}}(\beta) = \{ [l_x]_{\beta,\beta} \mid x \in \mathcal{A} \}$$

(a.) $M_{\mathcal{A}}(\beta) \leq \mathbb{R}^{n \times n}$ (show this)

By construction $M_{\mathcal{A}}(\beta) \subseteq \mathbb{R}^{n \times n}$.

Note $[l_{1_{\mathcal{A}}}]_{\beta,\beta} = [Id_{\mathcal{A}}]_{\beta,\beta} = I \in M_{\mathcal{A}}(\beta) \neq \phi$.

Suppose $A, B \in M_{\mathcal{A}}(\beta)$ and let $c \in \mathbb{R}$.
Then $\exists\, a, b \in \mathcal{A}$ for which $A = [l_a]_{\beta,\beta}$ and $B = [l_b]_{\beta,\beta}$

hence
$$\begin{aligned} cA + B &= c[l_a]_{\beta,\beta} + [l_b]_{\beta,\beta} \\ &= [c l_a + l_b]_{\beta,\beta} \\ &= [l_{ca+b}]_{\beta,\beta} \end{aligned}$$

$\Phi_\beta(x) = [x]_\beta$ is linear map.

(by $\boxed{P49}$ calculations)

Then as $ca + b \in \mathcal{A}$ we find $cA + B \in M_{\mathcal{A}}(\beta)$.
Therefore, $M_{\mathcal{A}}(\beta) \leq \mathbb{R}^{n \times n}$ by subspace test. //

(b.) Let $\Psi : \mathcal{A} \to M_{\mathcal{A}}(\beta)$ be defined by $\Psi(x) = [l_x]_{\beta,\beta}$ for each $x \in \mathcal{A}$. Show $\Psi(a * b) = \Psi(a)\Psi(b)$ and $\Psi(1_{\mathcal{A}}) = I \in \mathbb{R}^{n \times n}$

$$\begin{aligned} \Psi(a * b) &= [l_{a*b}]_{\beta,\beta} \\ &= [l_a \circ l_b]_{\beta,\beta} \\ &= [l_a]_{\beta,\beta}[l_b]_{\beta,\beta} \\ &= \Psi(a)\Psi(b). \end{aligned}$$

$$\begin{aligned} \Psi(1_{\mathcal{A}}) &= [l_{1_{\mathcal{A}}}]_{\beta\beta} \\ &= [Id_{\mathcal{A}}]_{\beta,\beta} \\ &= I. \end{aligned}$$

P51) Let $v = \langle a, b, c \rangle$

$\omega_v = a\,dx + b\,dy + c\,dz$ & $\Phi_v = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$

(a.) $\omega_v \wedge \omega_w = (v_1\,dx + v_2\,dy + v_3\,dz) \wedge (w_1\,dx + w_2\,dy + w_3\,dz)$

$= v_1 w_2\,dx \wedge dy + v_1 w_3\,dx \wedge dz + v_2 w_1\,dy \wedge dx +$

$+ v_2 w_3\,dy \wedge dz + v_3 w_1\,dz \wedge dx + v_3 w_2\,dz \wedge dy$

$= \underbrace{(v_2 w_3 - v_3 w_2)}_{(\vec{v} \times w)_1} dy \wedge dz + \underbrace{(v_3 w_1 - v_1 w_3)}_{(v \times w)_2} dz \wedge dx + \underbrace{(v_1 w_2 - v_2 w_1)}_{(v \times w)_3} dx \wedge dy$

$= \Phi_{v \times \mathbf{w}}$

(b.) $\omega_u \wedge \omega_v \wedge \omega_w = (u_1\,dx + u_2\,dy + u_3\,dz) \wedge \Phi_{v \times w}$

$$\left.\begin{aligned} &= u_1 (v \times w)_1\,dx \wedge dy \wedge dz \\ &\quad + u_2 (v \times w)_2\,dy \wedge dz \wedge dx \\ &\quad + u_3 (v \times w)_3\,dz \wedge dx \wedge dy \end{aligned}\right\}$$

all other terms zero due to $dx \wedge dx = 0$ etc.

$= (u_1 (v \times w)_1 + u_2 (v \times w)_2 + u_3 (v \times w)_3)\,dx \wedge dy \wedge dz$

$= \underline{u \cdot (v \times w)\,dx \wedge dy \wedge dz}.$

P52 Consider $v = (a,b,c,d)$ and $e_1 = (1,0,0,0)$ and $e_2 = (0,1,0,0)$. Calculate $e_1 \wedge e_2 \wedge v$ and find conditions for $\{v, e_1, e_2\}$ LI.

$$e_1 \wedge e_2 \wedge v = e_1 \wedge e_2 \wedge (a e_1 + b e_2 + c e_3 + d e_4)$$
$$= a \underbrace{e_1 \wedge e_2 \wedge e_1}_{0} + b \underbrace{e_1 \wedge e_2 \wedge e_2}_{0} + c e_1 \wedge e_2 \wedge e_3 + d e_1 \wedge e_2 \wedge e_4$$

Thus $e_1 \wedge e_2 \wedge v = c e_1 \wedge e_2 \wedge e_3 + d e_1 \wedge e_2 \wedge e_4$ hence $e_1 \wedge e_2 \wedge v = 0$ iff $\boxed{c = 0 \text{ and } d = 0}$.

P53 $A \in \mathbb{F}^{m \times m}$ and $B \in \mathbb{F}^{k \times k}$ let $M = A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Let $\mathbb{F}^n = \text{span}\{e_1, \dots, e_n\}$ where $n = m+k$. Show $\det(A \oplus B) = \det A \det B$ *

$$Me_1 \wedge \cdots \wedge Me_n = \left(\sum_{i_1=1}^{m} A_{i_1 1} e_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_m=1}^{m} A_{i_m m} e_{i_m}\right) \wedge \left(\sum_{j_1=1}^{k} B_{j_1 1} e_{m+j_1}\right) \wedge \cdots$$
$$\cdots \wedge \left(\sum_{j_k=1}^{k} B_{j_k k} e_{m+j_k}\right)$$

$$= \left(\sum_{i_1=1}^{m} \cdots \sum_{i_m=1}^{m} A_{i_1 1} \cdots A_{i_m m} \underbrace{e_{i_1} \wedge \cdots \wedge e_{i_m}}_{\text{antisymmetric in } i_1, \dots, i_m \in \mathbb{N}_m}\right) \wedge \left(\sum_{j_1=1}^{k} \cdots \sum_{j_k=1}^{k} B_{j_1 1} \cdots B_{j_k k} \underbrace{e_{m+j_1} \wedge \cdots \wedge e_{m+j_k}}_{\text{antisymmetric in } j_1, \dots, j_k \in \mathbb{N}_k \text{ where } m+j_1, \dots, m+j_k \text{ ranges from } m+1 \text{ to } m+k}\right)$$

$$= \left(\sum_{i_1, \dots, i_m=1}^{m} A_{i_1 1} \cdots A_{i_m m} \epsilon_{i_1 \cdots i_m} e_1 \wedge \cdots \wedge e_m\right) \wedge$$
$$\wedge \left(\sum_{j_1, \dots, j_k=1}^{k} B_{j_1 1} \cdots B_{j_m m} \epsilon_{j_1 \cdots j_k} e_{m+1} \wedge \cdots \wedge e_{m+k}\right)$$

$$= (\det(A) e_1 \wedge \cdots \wedge e_m) \wedge (\det B \, e_{m+1} \wedge \cdots \wedge e_n)$$
$$= \underline{\det(A) \det(B)} \, e_1 \wedge \cdots \wedge e_n \underset{\text{definition of } \det(M)}{=} \underline{\det(A \oplus B)} \, e_1 \wedge \cdots \wedge e_n$$

Therefore, * follows. //

$\boxed{P54}$ Prove $\det(A) = 0 \iff Ax = 0$ for some $x \neq 0$.

$\Rightarrow$) Suppose $\det(A) = 0$. Notice by def^{n} of $\det(A)$,

$$\mathrm{col}_1(A) \wedge \cdots \wedge \mathrm{col}_n(A) = \det(A)\, e_1 \wedge \cdots \wedge e_n = 0$$

Thus $\{\mathrm{col}_1(A), \dots, \mathrm{col}_n(A)\}$ is linearly dep. set

Hence $\exists\, x \neq 0$ for which

$$x_1 \mathrm{col}_1(A) + \cdots + x_n \mathrm{col}_n(A) = 0$$

Thus $Ax = 0$ for $x \neq 0$.

$\Leftarrow$) If $Ax = 0$ for $x \neq 0$ then columns of A are linearly dep.

Thus $\mathrm{col}_1(A) \wedge \cdots \wedge \mathrm{col}_n(A) = 0$

Hence $\mathrm{col}_1(A) \wedge \cdots \wedge \mathrm{col}_n(A) = \det(A)\, e_1 \wedge \cdots \wedge e_n = 0$

$\Rightarrow$ $\underline{\det(A) = 0}$.

<u>Remark</u>: I've used the Lemma $\{v_1, \dots, v_n\}$ is Linearly dependent iff $v_1 \wedge \cdots \wedge v_n = 0$.

$\boxed{\text{PSS}}$ §5.1 # 3d, f p. 257

For $T: V \to V$ and given basis β, calculate $[T]_{\beta,\beta}$ and determine if β is eigenbasis for T.

(d.) $T(a+bx+cx^2) = (-4a+2b-2c) - (7a+3b+7c)x + (7a+b+5c)x^2$

$$\beta = \{x - x^2, \ -1 + x^2, \ -1 - x + x^2\}$$

$$\left.\begin{aligned} T(1) &= -4 - 7x + 7x^2 \\ T(x) &= 2 - 3x + x^2 \\ T(x^2) &= -2 - 7x + 5x^2 \end{aligned}\right\}$$ I find these easier to track.

$$T(x - x^2) = (2 - 3x + x^2) - (-2 - 7x + 5x^2) = 4 + 4x - 4x^2$$

$$T(-1 + x^2) = 4 + 7x - 7x^2 - 2 - 7x + 5x^2 = 2 - 2x^2$$

$$T(-1 - x + x^2) = 4 + 7x - 7x^2 - 2 + 3x - x^2 - 2 - 7x + 5x^2 = 3x - 3x^2$$

Then
$$T(x - x^2) = -4(-1 - x + x^2)$$
$$T(-1 + x^2) = -2(-1 + x^2)$$
$$T(-1 - x + x^2) = 3(x - x^2)$$

$$\boxed{[T]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{bmatrix}}$$

(Only $-1 + x^2$ is an e-vector for T with $\lambda = -2$.)

PSS continued

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left[\begin{array}{c|c} -7a-4b+4c-4d & b \\ \hline -8a-4b+5c-4d & d \end{array}\right]$$

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

$a=1$, $c=1$ $\quad$ $a=-1$, $b=2$ $\quad$ $a=1$, $c=2$ $\quad$ $a=-1$, $d=2$

$$T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Thus,

$$[T]_{\beta,\beta} = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore β is eigenbasis with $\lambda = -3, 1, 1, 1$ eigenvalues

$\boxed{\text{PS6}}$ §5.1 #4a, c p. 258

(i.) find e-values of A

(ii) find e-vectors for each λ

(iii.) if possible find eigenbasis for A

(iv.) if such eigenbasis exists, find invertible Q and diag-matrix D for which $Q^{-1}AQ = D$.

(a.) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ for $\mathbb{F} = \mathbb{R}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = (\lambda-1)(\lambda-2) - 6$$
$$= \lambda^2 - 3\lambda - 4$$
$$= (\lambda-4)(\lambda+1) \therefore \underline{\lambda_1 = 4,\ \lambda_2 = -1}.$$

(aii.)

$$0 = (A - 4I)\vec{u}_1 = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\vec{u}_1 \Rightarrow \vec{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \therefore \mathcal{E}_{\lambda=4} = \text{span}\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$$

$$0 = (A + I)\vec{u}_2 = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\vec{u}_2 \Rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \therefore \mathcal{E}_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

(aiii.) $\beta = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is e-basis for A.

(aiv.) $Q = [\beta] = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $Q^{-1} = \dfrac{1}{-2-3}\begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$

has $\underline{Q^{-1}AQ = D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}}$.

PS6 continued / §5.1 # 4c

$$A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \quad \text{for } \mathbb{F} = \mathbb{C}$$

$$\det\begin{pmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{pmatrix} = (\lambda - i)(\lambda + i) - 2 = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

(i.) we find $\boxed{\lambda_1 = 1 \text{ and } \lambda_2 = -1}$.

(ii.) $(A - I)\vec{u}_1 = \begin{bmatrix} i-1 & 1 \\ 2 & -i-1 \end{bmatrix}\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} c}$

$$\boxed{\mathcal{E}_{\lambda=1} = \text{span}_{\mathbb{C}}\left\{\begin{bmatrix} 1 \\ 1-i \end{bmatrix}\right\}}$$

$(A + I)\vec{u}_2 = \begin{bmatrix} i+1 & 1 \\ 2 & -i+1 \end{bmatrix}\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} -1 \\ i+1 \end{bmatrix} c}$.

$$\boxed{\mathcal{E}_{\lambda=-1} = \text{span}_{\mathbb{C}}\left\{\begin{bmatrix} -1 \\ i+1 \end{bmatrix}\right\}}$$

(iii.) $\beta = \left\{\begin{bmatrix} 1 \\ 1-i \end{bmatrix}, \begin{bmatrix} -1 \\ i+1 \end{bmatrix}\right\}$ is eigenbasis for A

setting $Q = [\beta] = \begin{bmatrix} 1 & -1 \\ 1-i & 1+i \end{bmatrix}$ gives us, $2 = (1+i)(1-i)$

$Q^{-1} = \dfrac{1}{1+i+1-i}\begin{bmatrix} 1+i & 1 \\ i-1 & 1 \end{bmatrix} = \dfrac{1}{2}\left[\begin{array}{c|c} 1+i & 1 \\ i-1 & 1 \end{array}\right]$ and

$$\underline{Q^{-1}AQ = \frac{1}{2}\left[\begin{array}{c|c} 1+i & 1 \\ \hline i-1 & 1 \end{array}\right]\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}\left[\begin{array}{c|c} 1 & -1 \\ \hline 1-i & 1+i \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}.$$

$$= \frac{1}{2}\left[\begin{array}{c|c} 1+i & 1 \\ \hline i-1 & 1 \end{array}\right]\left[\begin{array}{c|c} 1 & 1 \\ \hline 1-i & -1-i \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \checkmark$$

$\boxed{\text{PS7}}$ § 5.1 # 5f, h p. 258

For each $T: V \to V$, find e-values of T and ordered basis s.t. $[T]_{\beta\beta}$ diag.

(f.) $T(f(x)) = f(x) + f(2)x$ for $f(x) \in V = P_3(\mathbb{R})$

$$T(a+bx+cx^2+dx^3) = a+bx+cx^2+dx^3 + (a+2b+4c+8d)x$$
$$= a + (a+3b+4c+8d)x + cx^2 + dx^3$$

Let $\gamma = \{1, x, x^2, x^3\}$ and obrerve,

$$[T]_{\gamma,\gamma}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = [T(a+bx+cx^2+dx^3)]_\gamma = \begin{bmatrix} a \\ a+3b+4c+8d \\ c \\ d \end{bmatrix}$$

$$\therefore \quad [T]_{\gamma\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det([T]_{\gamma\gamma} - \lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 4 & 8 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\det\begin{bmatrix} 3-\lambda & 4 & 8 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= \underline{(\lambda-1)^3(\lambda-3)} \qquad \therefore \boxed{\lambda_1 = 1 \text{ and } \lambda_2 = 3}$$

$$([T]_{\gamma\gamma} - 3I)\vec{u}_4 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 8 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}\vec{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \; *$$

$$([T]_{\gamma\gamma} - I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{u} \in \text{Null}([T]_{\gamma\gamma} - I) \text{ has form } \vec{u} = (-2s-4t-8r, s, t, r)$$

$$\underline{\vec{u} = s(-2,1,0,0) + t(-4,0,1,0) + r(-8,0,0,1)} \; **$$

Recall $T(v) = \lambda v \iff [T]_{\gamma\gamma}[v]_\gamma = \lambda[v]_\gamma$ so we find $*$ and $**$ yield,

$$\boxed{\mathcal{E}_{\lambda=3} = \text{span}\{x\}, \quad \mathcal{E}_{\lambda=1} = \text{span}\{-2+x, x^2-4, x^3-8\}}$$

(f.) of PS7 (§ 5.1 # 5f) continued

$$\beta = \{x, x-2, x^2-4, x^3-8\}$$

$$[T]_{\beta,\beta} = \begin{bmatrix} 3 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Notice $T(f(x)) = f(x) + f(2)x$ has $f(2) = 0$ for $f(x) = x-2$, x^2-4 and x^3-8 thus $T(f(x)) = f(x)$ for such $f(x)$.
Likewise, $T(x) = x + 2x = 3x$ gives $\lambda = 3$ as claimed.

//

(h.) $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$

Let $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ then

$$[T]_{\gamma\gamma}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \left[T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]_\gamma = \left[\begin{bmatrix} d & b \\ c & a \end{bmatrix}\right]_\gamma = \begin{bmatrix} d \\ b \\ c \\ a \end{bmatrix} \therefore [T]_{\gamma\gamma} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det([T]_{\gamma\gamma} - \lambda I) = \det\begin{bmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)\det\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[-\lambda\det\begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{bmatrix} + 1\det\begin{bmatrix} 0 & 1-\lambda \\ 1 & 0 \end{bmatrix}\right]$$

$$= (1-\lambda)\left[-\lambda^2(\lambda-1) + 1(\lambda-1)\right]$$

$$= (1-\lambda)(1-\lambda^2)(\lambda-1) = \underline{(\lambda-1)^3(\lambda+1)}.$$

$\boxed{\lambda_1 = 1}$ algebraic mult. of 3.

$\boxed{\lambda_2 = -1}$ alg. mult. 1

PS7 continued

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (\lambda = 1)$$

$$\left.\begin{aligned} a &= d \\ b &= b \\ c &= c \\ a &= d \end{aligned}\right\} \quad \begin{pmatrix} a & b \\ c & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{\mathcal{E}_{\lambda=1} = \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}.}$$

Then for $\lambda = -1$,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix} = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} d &= -a \\ b &= -b \\ c &= -c \\ a &= -d \end{aligned} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \therefore \underline{b = c = 0} \longrightarrow d = -a$$

$$\boxed{\mathcal{E}_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}}$$

$\beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$ gives

$$\underline{[T]_{\beta,\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}.$$

$\boxed{\text{PS8}}$ §5.1 #7 p. 258

Let $T: V \longrightarrow V$ be linear trans. on finite dim'l V and let β be basis for V.
Prove λ is e-value of $T \iff \lambda$ is eigenvalue of $[T]_\beta$.

$\Rightarrow$] Suppose $\exists x \in V$, $x \neq 0$ for which $T(x) = \lambda x$ then $[T(x)]_\beta = [\lambda x]_\beta = \lambda [x]_\beta$. However, we showed that $[T(x)]_\beta = [T]_{\beta\beta}[x]_\beta$ thus

$[T]_{\beta\beta}[x]_\beta = \lambda [x]_\beta$ and $x \neq 0 \Rightarrow [x]_\beta \neq 0$

thus λ is eigenvalue of $[T]_{\beta,\beta}$.

$\Leftarrow$] If $\exists u \in \mathbb{F}^n$, $u \neq 0$ for which $[T]_{\beta,\beta} u = \lambda u$ then $\Phi_\beta^{-1}(u) = x$ has

$[x]_\beta = \Phi_\beta(\Phi_\beta^{-1}(u)) = u$ and so,

$[T]_{\beta\beta}[x]_\beta = \lambda [x]_\beta \Rightarrow [T(x)]_\beta = [\lambda x]_\beta$

Therefore, $T(x) = \lambda x$ for $x \neq 0$ and we've shown λ is e-value for T. //

$\boxed{PS9}$ § 5.2 # 3 a, c, e p. 278-279

Test for diagonalizability and if possible find β s.t. $[T]_{\beta,\beta}$ diag.

(a.) $T(f(x)) = f'(x) + f''(x)$ for $f(x) \in P_3(\mathbb{R})$

(c.) $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$ (for $\mathbb{R}^3 = V$)

(e.) $T(z, w) = (z + iw,\ iz + w)$ for $T: \mathbb{C}^2 \to \mathbb{C}^2$

$$T(\underbrace{a + bx + cx^2 + dx^3}_{f(x)}) = b + 2cx + 3dx^2 + 2c + 6dx$$
$$= b + 2c + (2c + 6d)x + 3dx^2$$

$\gamma = \{1, x, x^2, x^3\}$,

$$[T(f(x))]_\gamma = [T]_{\gamma\gamma}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b + 2c \\ 2c + 6d \\ 3d \\ 0 \end{bmatrix} \therefore [T]_{\gamma\gamma} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We find $\underline{\lambda = 0 \text{ with multiplicity } 4}$ since $[T]_{\gamma\gamma}$ triangular.

Notice $\mathcal{E}_{\lambda=0} = \mathrm{Ker}(T)$ and $f(x) \in \mathrm{Ker}(T)$ has $f'(x) + f''(x) = b + 2c + (2c + 6d)x + 3dx^2 = 0$

Hence, $b + 2c = 0$, $2c + 6d = 0$ and $3d = 0$ $\therefore \underline{d = 0}$.

and so $\underline{c = 0}$ and $\underline{b = 0}$, only a is free and we find $f(x) = a \in \mathrm{Ker}(T)$. That is,

$$\mathcal{E}_{\lambda=0} = \mathrm{span}\{1\} \neq P_3(\mathbb{R})$$

Thus T is not diagonalizable.

(In fact, $\exists \beta$ for which $[T]_{\beta\beta} = J_4(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

PS9 continued

(c.) $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\det(T-\lambda) = \det\begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2+1)$

thus T is not diagonalizable as

$\mathrm{char}_T(x) = \underbrace{(x^2+1)}_{\text{irreducible over } \mathbb{R}}(2-x)$ is not split over $\mathbb{R}$

(e.) $T\begin{pmatrix} z \\ w \end{pmatrix} = \begin{bmatrix} z+iw \\ iz+w \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} = A\begin{bmatrix} z \\ w \end{bmatrix}$

$\det(T-\lambda) = \det\begin{bmatrix} 1-\lambda & i \\ i & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - i^2 = \underline{(\lambda-1+i)(\lambda-1-i)}.$

Thus $\lambda_1 = 1-i$ and $\lambda_2 = 1+i$ since $\lambda_1 \neq \lambda_2$ we find at least two LI e-vectors for T thus T is diagonalizable.

$\lambda_1 = 1-i$ $\quad A-(1-i)I = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} - \begin{bmatrix} 1-i & 0 \\ 0 & 1-i \end{bmatrix} = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$

$\mathcal{E}_{1-i} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$ I see this, do you?

$\lambda_2 = 1+i$ $\quad A-(1+i)I = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix} \therefore \mathcal{E}_{1+i} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Let $\beta = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ then $[\beta]^T = [\beta]^{-1}$ and

$[\beta]^{-1}A[\beta] = [\beta]^T A [\beta] = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}.$

P60 § 5.2 # 8, p. 279

Suppose $A \in \mathbb{F}^{n \times n}$ with two distinct e-values $\lambda_1 \neq \lambda_2$ and $\dim(E_{\lambda_1}) = n-1$. Prove A diagonalizable.

Notice $E_{\lambda_1} \oplus E_{\lambda_2} \leq \mathbb{F}^n$ since eigenspaces of distinct e-values are independent. However, $\dim(E_{\lambda_2}) \geq 1$ and $\dim(E_{\lambda_1}) = n-1$ hence

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \geq 1 + n - 1 = n$$

But $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) > n$ is absurd as $E_{\lambda_1} \oplus E_{\lambda_2} \leq \mathbb{F}^n$ $\therefore$ $\underline{\dim(E_{\lambda_1}) = 1}$. Thus

$\mathbb{F}^n = E_{\lambda_1} \oplus E_{\lambda_2}$ and it follows that A is diagonalizable. (take $\beta = \{v_1\} \cup \beta_2$ where $E_{\lambda_2} = \operatorname{span} \beta_2$ then

$$[\beta]^{-1} A [\beta] = \begin{bmatrix} \lambda_2 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_1 \end{bmatrix}. \quad)$$

P61 § 5.2 #13, p. 280

$T: V \to V$, $\dim(V) < \infty$

(a.) If λ e-value for T then λ^{-1} is e-value for T^{-1}
prove $\mathcal{E}_{\lambda}(T) = \mathcal{E}_{\lambda^{-1}}(T^{-1})$

(b.) Prove if T diagonalizable then T^{-1} diagonalizable.

(a.) Let $x \in \mathcal{E}_{\lambda}(T) = \operatorname{Ker}(T - \lambda)$ then $(T-\lambda)(x) = 0$ thus
$T(x) = \lambda x \Rightarrow T^{-1}(T(x)) = T^{-1}(\lambda x) \Rightarrow \underline{T^{-1}(x) = \lambda^{-1} x}$,
thus $x \in \operatorname{Ker}(T^{-1} - \lambda^{-1}) \therefore \mathcal{E}_{\lambda}(T) \subseteq \mathcal{E}_{\lambda^{-1}}(T^{-1})$.
Likewise, if $y \in \mathcal{E}_{\lambda^{-1}}(T^{-1})$ then $T^{-1}(y) = \lambda^{-1} y$
and thus $T(T^{-1}(y)) = T(\lambda^{-1} y) = \lambda^{-1} T(y)$ and
hence $y = \lambda^{-1} T(y) \therefore T(y) = \lambda y \Rightarrow y \in \mathcal{E}_{\lambda}(T)$
consequently $\mathcal{E}_{\lambda^{-1}}(T^{-1}) \subseteq \mathcal{E}_{\lambda}(T)$. We've shown
that $\mathcal{E}_{\lambda^{-1}}(T^{-1}) = \mathcal{E}_{\lambda}(T)$ by double-containment.

(b.) If T is diagonalizable then
$$V = \mathcal{E}_{\lambda_1} \oplus \mathcal{E}_{\lambda_2} \oplus \cdots \oplus \mathcal{E}_{\lambda_s}.$$
If T^{-1} exists then all $\lambda_1, \dots, \lambda_s \neq 0$ and
by (a.) we find $\mathcal{E}_{\lambda_j}(T) = \mathcal{E}_{\lambda_j^{-1}}(T^{-1})$ thus,
omitting the T^{-1} dep. we have
$$V = \mathcal{E}_{\lambda_1^{-1}} \oplus \mathcal{E}_{\lambda_2^{-1}} \oplus \cdots \oplus \mathcal{E}_{\lambda_s^{-1}} \quad (\text{for } T^{-1})$$
which shows T^{-1} is diagonalizable. //

P62 Let $P(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{R}[t]$

then $P(A) = a_0 I + a_1 A + \cdots + a_n A^n$. Suppose $Av = \lambda v$

for $v \neq 0$ and $\lambda \in \mathbb{R}$ then,

$$
\begin{aligned}
P(A)v &= (a_0 I + a_1 A + \cdots + a_n A^n)v \\
&= a_0 Iv + a_1 Av + \cdots + a_n A^{n-1} Av \\
&= a_0 v + a_1 \lambda v + \cdots + a_n A^{n-2} A \lambda v \\
&= a_0 v + a_1 \lambda v + \cdots + a_n \lambda^n v \\
&= (a_0 + a_1 \lambda + \cdots + a_n \lambda^n) v \\
&= P(\lambda) v
\end{aligned}
$$

continuing, $\underbrace{A^k v = \lambda^k v}_{*\ \text{Lemma}}$

$\therefore$ v is e-vector of $P(A)$ with e-value $P(\lambda)$. //

Lemma: $A^k v = \lambda^k v \quad \forall k \in \mathbb{N}$

Notice $k = 1$ true since $Av = \lambda v$ is given.

Suppose inductively $A^k v = \lambda^k v$ for some $k \in \mathbb{N}$.

Consider
$$
\begin{aligned}
A^{k+1} v = A(A^k v) &= A \lambda^k v \quad \text{by induct. hyp.} \\
&= \lambda^k A v \\
&= \lambda^k \lambda v \quad Av = \lambda v \text{ given} \\
&= \lambda^{k+1} v
\end{aligned}
$$

thus the claim for $k \Rightarrow$ claim true for $k+1$

Hence by PMI, $A^k v = \lambda^k v \quad \forall k \in \mathbb{N}$. //

$\boxed{P63}$ Suppose $A \in \mathbb{F}^{n \times n}$ with $A^{k-1} \neq 0$ yet $A^k = 0$. Prove $\lambda = 0$ is only e-value of A.

Suppose $\exists\, x \neq 0$ for which $Ax = \lambda x$ then by Lemma of previous problem

$$A^{k-1}x = \lambda^{k-1}x \quad \text{and} \quad A^k x = \lambda^k x$$

However, $A^k = 0$ thus $\lambda^k x = 0$ for $x \neq 0$

Consequently, $\lambda^k = 0 \Rightarrow \boxed{\lambda = 0.}$

$\boxed{P64}$

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow \det\begin{pmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = (\lambda-4)(\lambda-1)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

We find eigenvalues $\lambda = 2$ and $\lambda = 3$

$\underline{\lambda = 2|}$ $A - 2I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathcal{E}_{\lambda=2} = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$

$\underline{\lambda = 3|}$ $A - 3I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathcal{E}_{\lambda=3} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

Then $[\beta] = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ and $[\beta]^{-1} = \dfrac{1}{-1+2}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

and $[\beta]^{-1}A[\beta] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A = [\beta]\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}[\beta]^{-1}$

$$\begin{aligned}
A^n &= [\beta]\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}[\beta]^{-1}[\beta]\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}[\beta]^{-1}\cdots[\beta]\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}[\beta]^{-1} \\
&= [\beta]\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^n[\beta]^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -2^n & -2^n \\ 2(3)^n & 3^n \end{bmatrix}
\end{aligned}$$

$$\therefore \boxed{A^n = \begin{bmatrix} -2^n + 2(3)^n & -2^n + 3^n \\ 2^{n+1} - 2(3)^n & 2^{n+1} - 3^n \end{bmatrix}}$$