Your printed name indicates you read Chapter 4 of the notes: ____________________.

**Problem 31** Let $W_1 = \{tI \mid t \in \mathbb{R}\}$ where $I$ is the $n \times n$ identity matrix and $W_2$ is defined by $W_2 = \{A \in \mathbb{R}^{n\times n} \mid A^T = A,\ tr(A) = 0\}$ and $W_3 = \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}$.

**(a.)** Show $W_1 + W_2 + W_3 = \mathbb{R}^{n\times n}$

**(b.)** Show $W_1, W_2, W_3$ are independent subspaces

**Problem 32** Let $T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$. Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ provide bases for $W_1 = \text{span}(\beta_1)$ and $W_2 = \text{span}(\beta_2)$.

**(a.)** show $W_1$ and $W_2$ are invariant subspaces of $T$,

**(b.)** Find $[T_{W_1}]_{\beta_1,\beta_1}$ and $[T_{W_2}]_{\beta_2,\beta_2}$

**(c.)** verify $[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2}$ where $\beta = \beta_1 \cup \beta_2$

**Problem 33** Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $T(A) = tr(A)I$. Show $T$ is diagonalizable.

**Problem 34** Let $V = \text{span}\{2^t, 3^t, 4^t\}$. Define $T$ by linearly extending the rules

$$T(2^t) = 2^t, \quad T(3^t) = 2(2^t) + 2(3^t), \quad T(4^t) = 3(2^t + 3^t + 4^t).$$

Find an eigenbasis $\beta$ for $T$ and show $[T]_{\beta,\beta}$ is diagonal.

**Problem 35** Suppose $A \in \mathbb{R}^{14\times 14}$ and $rk((A-7I)^5) = 4$, $rk((A-7I)^4) = 4$, $rk((A-7I)^3) = 5$, $rk((A-7I)^2) = 7$, $rk(A-7I) = 10$ and $rk((A-3I)^2) = 10$, $rk(A-3I) = 12$. Here $rk(M) = dim(Col(M)$ is the **rank** of the matrix $M$.

**(a.)** Find the Jordan form of $A$.

**(b.)** If $P_A(x) = det(xI - A)$ then find the formula for $P_A(x)$ in factored form.

**(c.)** Let $m_A(x)$ be the **minimal polynomial** of $A$; that is, let $m_A(x)$ be the monic polynomial of least degree for which $m_A(A) = 0$. Find $m_A(x)$.

**Problem 36** Let $A = J_2(2) \oplus J_2(6)$ and $B = J_2(6) \oplus M$. If $A$ and $B$ are similar then prove $M = J_2(2)$.

**Problem 37** Recall $e^{(\alpha+i\beta)t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t$. Suppose $T$ is a linear transformation on $V = \text{span}_{\mathbb{R}}\Upsilon$ where $\Upsilon = \{e^t\cos 2t, e^t \sin 2t, \cos 3t, \sin 3t\}$. Given that the complexification of $T$ has

$$T_{\mathbb{C}}(e^{(1+2i)t}) = (4+5i)e^{(1+2i)t} \qquad \& \qquad T_{\mathbb{C}}(e^{3it}) = (7+8i)e^{3it}$$

Find the real Jordan form of $T$.

**Problem 38** Suppose $A = J_3(7) \oplus R_4(1+3i)$. Calculate $e^{tA}$ and solve $\frac{dx}{dt} = Ax$.

**Problem 39** Let $T : V \to V$ have Jordan basis $\beta$. Let $S = T + cId_V$ where $c$ is a scalar. Find the Jordan form of $S$.

**Problem 40** Let $T, U \in \mathcal{L}(V)$. Prove $T$ and $U$ are simultaneously diagonalizable iff $TU = UT$. Note, **simultaneous diagonalizability** of $U$ and $T$ means there exists basis $\beta$ for which $[T]_{\beta,\beta}$ and $[U]_{\beta,\beta}$ are both diagonal.