Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 6 of the notes: ______________.

Problem 55 In each of the following, give at least one reason why W is not a subspace of V over $\mathbb{F}$

(a.) $W = \{(x, y, z) \mid x, y, z \geq 0\}$ in $V = \mathbb{R}^3$ over $\mathbb{F} = \mathbb{R}$,

(b.) $W = \{1 + ax + bx^2 \mid a, b \in \mathbb{R}\}$ in $V = \mathbb{R}[x]$ over $\mathbb{F} = \mathbb{R}$,

(c.) $W = \{(x, y) \mid x, y \in \mathbb{R}\}$ in $V = \mathbb{C}^2$ over $\mathbb{F} = \mathbb{C}$,

(d.) $W = \{A \in \mathbb{R}^{n \times n} \mid A^3 + 3A^2 + A = 0\}$ over $\mathbb{F} = \mathbb{R}$,

(e.) $W = \mathbb{R} - \mathbb{Q}$ in $V = \mathbb{R}$ over $\mathbb{Q}$.

Problem 56 Consider $W = \{f(x) \in \mathbb{R}[x] \mid f(3) = 0\}$ show $W \leq \mathbb{R}[x]$.

Problem 57 Let $W = \{(a + 2b + 3c, b + c, c - 2a) \mid a, b, c \in \mathbb{R}\}$. Prove W is a subspace of $\mathbb{R}^3$.

Problem 58 Consider $W = \{A \in \mathbb{C}^{2\times 2} \mid \text{trace}(A + \overline{A}) = 0\}$ where $\text{trace}(A) = A_{11} + A_{22}$ and $\overline{A}$ denotes the complex conjugate of A. Show W is a subspace of $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$.

Problem 59 Show $\{E_{11}, E_{12} + E_{21}, E_{12} - E_{21}\}$ is a linearly independent subset of $\mathbb{R}^{2\times 2}$.

Problem 60 Let $\beta = \{(1, 2), (1, 3)\}$.

(a.) Calculate $[(a, b)]_\beta$.

(b.) If $[(a, b)]_\beta = (3, 4)$ then find (a, b).

Problem 61 Let $\beta = \{(1, 1, 2, 2), (3, 2, 2, 1), (0, 0, 1, 0)\}$ provide a basis for $W = \text{span}(\beta)$. Determine what condition is needed for $(a, b, c, d) \in W$ and given that condition find the coordinate vector of (a, b, c, d) with respect to β.

Problem 62 Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define $W = \{M \in \mathbb{R}^{2\times 2} \mid MJ = JM\}$.

(a.) show W is a subspace of $\mathbb{R}^{2\times 2}$

(b.) Find a basis β for W and calculate $\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right]_\beta$ in terms of a and b.

Problem 63 Suppose S is a subset of $\mathbb{R}^n$. Define $S^\perp = \{v \in \mathbb{R}^n \mid v \bullet s = 0 \text{ for all } s \in S\}$.

(a.) Prove $S^\perp$ is a subspace of $\mathbb{R}^n$.

(b.) Find a basis for $S^\perp$ given $S = \{(1, 2, 2, 1), (0, 1, 0, 1)\}$.

Problem 64 Let $W_1 = \text{span}\{1 - x^2, x - x^3\}$ and $W_2 = \{a + bx + cx^2 + dx^3 \mid a + b + c - 2d = 0\}$. Find a basis for $W_1 \cap W_2$.

Problem 65 Suppose $\beta = \{E_{11}, iE_{11}, E_{12}, iE_{12}, E_{21}, iE_{21}, E_{22}, iE_{22}\}$ is given as a basis for $\mathbb{C}^{2\times 2}$. If $[A]_\beta = (1, 2, 3, 4, 5, 6, 7, 8)$ then what is A ?

Problem 66 Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 3 & 4 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix}$. Find bases for

(a.) the column space of A (use columns of A itself to form your answer)

(b.) the row space of A (use rows of A itself to form your answer)

(c.) the null space of A

Problem 67 **Prove:** If V be a vector space with $U \leq V$ and $W \leq V$ then $U \cap W \leq V$.

Problem 68 Equating coefficients: Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$.
Prove: $S = \{v_1, \ldots, v_k\}$ is a linearly independent set of vectors iff if there exist $a_i, b_i \in \mathbb{F}$ for $i \in \mathbb{N}_k$ for which

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = b_1v_1 + b_2v_2 + \cdots + b_kv_k$$

then $a_i = b_i$ for each $i = 1, 2, \ldots, k$.

Problem 69 The **trace** of a square matrix is simply the sum of its diagonal entries; trace$(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in R^{n\times n}$ where R is a commutative ring with identity. Prove the following:

(a.) trace$(I_n) = n$ where I_n is the $n \times n$ identity matrix.

(b.) trace$(cA + B) = c$trace$(A) +$ trace(B) for all $A, B \in R^{n\times n}$ and $c \in R$,

(c.) trace$(AB) =$ trace(BA) for all $A \in R^{p\times n}$ and $B \in R^{n\times p}$.

Problem 70 We see how the trace connects to the study of dimension in Lecture. It should be noted this trace has many other uses. This problem asks you to understand an application of the trace to the study of the **commutator**: $[A, B] = AB - BA$ for square matrices A, B. **Explain how the trace allows us to say:**

$$[A, B] \neq kI$$

for any square matrices A, B and scalar $k \neq 0$.

Problem 71 Differential equations provides interesting examples of vector spaces. In particular, consider the differential equation $y'' - 3y' + 2y = 0$. We say f is a solution of the differential equation if $f'' - 3f' + 2f = 0$. Let W be the set of solutions for $y'' - 3y' + 2y = 0$. Prove W is a subspace of the set of functions on $\mathbb{R}$.

Problem 72 Consider the rather simple differential equation $y'' = 0$ where $y' = \frac{dy}{dx}$. Find a basis for the solution space. What is the dimension of the solution space?

Mission 4 Solution:

P55

(a.) $W = \{(x,y,z) \mid x,y,z \geq 0\} \subseteq \mathbb{R}^3$ over $\mathbb{R}$.

observe $(1,1,1) \in W$ yet $-(1,1,1) = (-1,-1,-1) \notin W$

thus W is <u>NOT</u> a subspace of $\mathbb{R}^3$.

(b.) $W = \{1 + ax + bx^2 \mid a, b \in \mathbb{R}\} \subseteq V = \mathbb{R}[x]$ over $\mathbb{R}$

Observe $1 + ax + bx^2 \neq 0$ for any choice of $a, b \in \mathbb{R}$

thus $0 \notin W \therefore W \not\leq \mathbb{R}[x]$.

(c.) $W = \{(x,y) \mid x, y \in \mathbb{R}\}$ in $V = \mathbb{C}^2$ over $\mathbb{C}$.

Notice $(1,2) \in W$ yet $i(1,2) = (i, 2i) \notin W$

thus W is not closed under scalar multiplication by $\mathbb{C}$

and we conclude $W \not\leq \mathbb{C}^2$

(d.) $W = \{A \in \mathbb{R}^{n \times n} \mid A^3 + 3A^2 + 2A = 0\}$ over $\mathbb{R}$.

Suppose $A \in W$. Consider $2A$,

$$(2A)^3 + 3(2A)^2 + 2(2A) = \underline{8A^3 + 12A^2 + 2^2 A}_{\,*}$$

there is no reason for $*$ to be zero. But, for a counter-example we need to do better. Notice

$A^3 + 3A^2 + 2A = A(A^2 + 3A + 2I)$. Hmm. no help. Well,

$A(A + 2I)(A + I) = 0$. Consider $A = \begin{pmatrix} 0 & & \\ & -2 & \\ & & -1 \end{pmatrix}$

then $A + 2I = \begin{bmatrix} 2 & & 0 \\ & 0 & \\ 0 & & 1 \end{bmatrix}$ and $A + I = \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{bmatrix}$

hence $A(A+2I)(A+I) = \begin{bmatrix} 0 & & 0 \\ & -2 & \\ 0 & & -1 \end{bmatrix}\begin{bmatrix} 2 & & 0 \\ & 0 & \\ 0 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

YET, $2A = \begin{bmatrix} 0 & & \\ & -4 & \\ & & -2 \end{bmatrix}$, $2A + 2I = \begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix}$, $2A + I = \begin{bmatrix} 1 & & \\ & -3 & \\ & & -1 \end{bmatrix}$

hence $(2A)^3 + 3(2A)^2 + 2(2A) = 2A(2A + 2I)(2A + I) = \begin{bmatrix} 0 & & \\ & -4 & \\ & & -2 \end{bmatrix}\begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 1 & & \\ & -3 & \\ & & -1 \end{bmatrix} \neq 0.$

PS5 continued

(d.) In case $n=3$, $A = \begin{bmatrix} 0 & & \\ & -2 & \\ & & -1 \end{bmatrix} \in W$ yet

we saw $2A = \begin{bmatrix} 0 & & \\ & -4 & \\ & & -2 \end{bmatrix} \notin W \therefore W \not\leq \mathbb{R}^{3\times 3}$

which means $W = \{A \in \mathbb{R}^{n\times n} \mid A^3 + 3A^2 + 2A = 0\}$

is not a subspace of $\mathbb{R}^{n\times n}$ for all n.

Remark: when a claim uses an $n \in \mathbb{N}$ which is unspecified like in this problem, it suffices to provide a choice of n where the claim fails. This does not prove $W = \{A \in \mathbb{R}^{n\times n} \mid A^3 + 3A^2 + 2A = 0\}$ for all n. In fact $n=1$ does take $W \leq \mathbb{R}$.

(e.) $W = \mathbb{R} - \mathbb{Q}$ in $V = \mathbb{R}$ over $\mathbb{Q}$

notice $0 \in \mathbb{Q}$ hence $0 \notin W = \mathbb{R} - \mathbb{Q} \therefore W \not\leq \mathbb{R}$.

PS6 Let $W = \{f(x) \in \mathbb{R}[x] \mid f(3) = 0\}$. Observe $f(x) = x-3 \in \mathbb{R}[x]$ and $f(3) = 3-3 = 0 \therefore x-3 \in W \neq \emptyset$. Suppose $f(x), g(x) \in W$ and let $c \in \mathbb{R}$. Consider,

$$(cf+g)(3) = cf(3) + g(3) = c(0) + 0 = 0$$

thus $cf(x) + g(x) \in W$ and it follows both $cf(x), f(x)+g(x) \in W$. Therefore $W \subseteq \mathbb{R}[x]$ is nonempty and closed under scalar multiplication and addition and we conclude $W \leq \mathbb{R}[x]$ by the subspace test Th^{m}. //

$\boxed{P57}$ Let $W = \{(a+2b+3c,\ b+c,\ c-2a) \mid a,b,c \in \mathbb{R}\}$. Then,
$W = \{a(1,0,-2) + b(2,1,0) + c(3,1,1) \mid a,b,c \in \mathbb{R}\}$ and
we find $W = \text{span}\{(1,0,-2),\ (2,1,0),\ (3,1,1)\}$ therefore
$W \leq \mathbb{R}^3$ as W is formed by a span. //

$\boxed{P58}$ $W = \{A \in \mathbb{C}^{2\times 2} \mid \text{trace}(A + \bar{A}) = 0\}$ where $\text{trace}(A) = A_{11} + A_{22}$
and $\bar{A}$ denotes complex conjugate of A; $(\bar{A})_{ij} = \overline{A_{ij}}$. Show
W is a subspace of $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$.

Observe $0 \in \mathbb{C}^{2\times 2}$ has $0 + \bar{0} = 0$ and $\text{trace}(0+\bar{0}) = \text{trace}(0) = 0$
thus $0 \in W \neq \emptyset$. Let $A, B \in W$ and suppose $c \in \mathbb{R}$,

$$\overline{cA+B} = \bar{c}\bar{A} + \bar{B} = c\bar{A} + \bar{B} \quad \text{since } \bar{c} = c \text{ for } c \in \mathbb{R}.$$

Consider, using the calculation above,

$$\begin{aligned}
\text{trace}(cA + B + \overline{cA+B}) &= \text{trace}(cA + B + c\bar{A} + \bar{B}) \\
&= \text{trace}(c(A+\bar{A}) + B + \bar{B}) \\
&= \text{trace}(c(A+\bar{A})) + \text{trace}(B+\bar{B}) \quad : \text{Lemma} \\
&= c\,\text{trace}(A+\bar{A}) + \text{trace}(B+\bar{B}) \quad : \text{Lemma} \\
&= c(0) + 0 \quad : A, B \in W \\
&= 0
\end{aligned}$$

Thus $cA + B \in W \Rightarrow cA,\ A+B \in W \therefore W \leq \mathbb{C}^{2\times 2}$ over $\mathbb{R}$.

Remark: in other words, W is a subspace of $\mathbb{C}^{2\times 2}$ viewed as a real vector space.

Lemma: $\text{trace}(cA+B) = c\,\text{trace}(A) + \text{trace}(B) \quad \forall A, B \in \mathbb{C}^{2\times 2}, c \in \mathbb{C}$

Proof:
$$\begin{aligned}
\text{trace}(cA+B) &= (cA+B)_{11} + (cA+B)_{22} \quad : \text{def}^{\underline{n}} \text{ of trace} \\
&= cA_{11} + B_{11} + cA_{22} + B_{22} \quad : \text{def}^{\underline{n}} \text{ of + and scal. mult.} \\
&= c(A_{11} + A_{22}) + B_{11} + B_{22} \\
&= c\,\text{trace}(A) + \text{trace}(B). //
\end{aligned}$$

$\boxed{\text{P59}}$ Let $S = \{ E_{11}, E_{12} + E_{21}, E_{12} - E_{21} \} \subseteq \mathbb{R}^{2\times 2}$.

Suppose $c_1 E_{11} + c_2 (E_{12} + E_{21}) + c_3 (E_{12} - E_{21}) = 0$ then

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

hence,

$$\left[\begin{array}{c|c} c_1 & c_2 + c_3 \\ \hline c_2 - c_3 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

From which we find $\underline{c_1 = 0}$, $\underbrace{c_2 + c_3 = 0}_{\text{I}}$ and $\underbrace{c_2 - c_3 = 0}_{\text{II}}$

I + II : $2c_2 = 0 \therefore \underline{c_2 = 0}$

and so $c_3 = -c_2 = 0 \therefore c_1 = 0, c_2 = 0, c_3 = 0$ and we conclude S is LI subset of $\mathbb{R}^{2\times 2}$.

$\boxed{\text{P60}}$ $\beta = \{ (1,2), (1,3) \}$

(a.) $[(a,b)]_\beta = (c_1, c_2)$ where $(a,b) = c_1 (1,2) + c_2 (1,3)$

but, this means $\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}}_{[\beta]} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\beta]^{-1} \begin{bmatrix} a \\ b \end{bmatrix}$

$$\therefore [(a,b)]_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{3-2} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a - b \\ -2a + b \end{bmatrix}$$

$$\therefore \boxed{[(a,b)]_\beta = (3a - b, -2a + b)}$$

(b.) If $[(a,b)]_\beta = (3,4)$ then by def$^{\underline{n}}$ of coordinate map,

$$(a,b) = 3(1,2) + 4(3,4) = (3,6) + (12,16) = (15,22)$$

$$\therefore \boxed{(a,b) = (15,22)}$$

$\boxed{P61}$ $\beta = \{(1,1,2,2), (3,2,2,1), (0,0,1,0)\}$ provide basis of $W = \text{span}\,\beta$

Determine what condition is needed for $(a,b,c,d) \in W$ and given that condition find coordinate vector of (a,b,c,d) w.r.t. β

If $x(1,1,2,2) + y(3,2,2,1) + z(0,0,1,0) = (a,b,c,d)$ then $[(a,b,c,d)]_\beta = (x,y,z)$. Consider,

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & a \\ 1 & 2 & 0 & b \\ 2 & 2 & 1 & c \\ 2 & 1 & 0 & d \end{array}\right] \xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - 2r_1}]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & a \\ 0 & -1 & 0 & b-a \\ 0 & -4 & 1 & c-2a \\ 0 & -5 & 0 & d-2a \end{array}\right] \xrightarrow[\substack{r_3 - 4r_2 \\ r_4 - 5r_2}]{r_1 + 3r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3b-2a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & 1 & 2a-4b+c \\ 0 & 0 & 0 & d-2a-5(b-a) \end{array}\right]$$

$$\xrightarrow{-r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3b-2a \\ 0 & 1 & 0 & a-b \\ 0 & 0 & 1 & 2a-4b+c \\ 0 & 0 & 0 & 3a-5b+d \end{array}\right]$$

Thus we need $3a - 5b + d = 0$ for $(a,b,c,d) \in W$ and from the calculation above we find (given $3a-5b+d=0$)

$$\boxed{[(a,b,c,d)]_\beta = (3b-2a,\ a-b,\ 2a-4b+c)}$$

We can check on my calculation in two ways,

1.) $(3b-2a)(1,1,2,2) + (a-b)(3,2,2,1) + (2a-4b+c)(0,0,1,0) =$

$= (\cancel{3b} - 2a + 3(a - \cancel{b}),\ 3b - \cancel{2a} + 2(\cancel{a} - b),\ 6b - 4a + 2a - 2b + 2a - 4b + c,\ 6b - 4a + a - b)$

$= (a, b, c, 5b - 3a)$

$= (a, b, c, d)$ assuming $3a - 5b + d = 0$ $\hookrightarrow$ $\underline{d = 5b - 3a}$.

2.) Can check $\det \begin{bmatrix} 1 & 3 & 0 & a \\ 1 & 2 & 0 & b \\ 2 & 2 & 1 & c \\ 2 & 1 & 0 & d \end{bmatrix} = 0$ yields $3a - 5b + d = 0$

$$\det \begin{bmatrix} 1 & 3 & a \\ 1 & 2 & b \\ 2 & 1 & d \end{bmatrix} = 1(2d - b) - 3(d - 2b) + a(1 - 4)$$
$$= -d + 5b - 3a = 0 \quad \text{yep.}$$

P62 Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $W = \{ M \in \mathbb{R}^{2\times 2} \mid MJ = JM \}$

(a.) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ implies $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \Rightarrow \begin{matrix} b = c \\ d = d \end{matrix}$$

Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is typical element of W

Consequently $M \in W \Rightarrow M = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

It follows $W = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \leq \mathbb{R}^{2\times 2}$.

(b.) Since $c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$

we find $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is LI spanning set for $W = \text{span}\,\beta$. Thus β serves as <u>basis</u> for W.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

thus $\boxed{\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right]_\beta = (a, b).}$

$\boxed{P63}$ Let $S \subseteq \mathbb{R}^n$. Define $S^\perp = \{ v \in \mathbb{R}^n \mid v \cdot s = 0 \ \forall s \in S \}$

(a.) Since $0 \cdot x = 0 \ \forall x \in \mathbb{R}^n$ it follows $0 \cdot s = 0 \ \forall s \in S$ thus $0 \in S^\perp \neq \emptyset$. Suppose $v_1, v_2 \in S^\perp$ and $c \in \mathbb{R}$, let $s \in S$ and calculate, by properties of dot-product,

$$(cv_1 + v_2) \cdot s = c(v_1 \cdot s) + v_2 \cdot s = c(0) + 0 = 0$$

since $v_1 \cdot s = 0$ and $v_2 \cdot s = 0$ as $v_1, v_2 \in S^\perp$. Thus $cv_1 + v_2 \in S^\perp \Rightarrow cv_1, \ v_1 + v_2 \in S^\perp \therefore S^\perp \leq \mathbb{R}^n$ by the subspace test Th^{m}. //

(b.) Let $S = \{ (1,2,2,1), (0,1,0,1) \}$ then $x = (x_1, x_2, x_3, x_4) \in S^\perp$ iff $x \cdot (1,2,2,1) = 0$, $x \cdot (0,1,0,1) = 0$ which means $x_1 + 2x_2 + 2x_3 + x_4 = 0$ and $x_2 + x_4 = 0$.

Apparently $S^\perp = \text{Null}\,[S]^T$ so calculate,

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Hence $x_1 = -2x_3 + x_4$, $x_2 = -x_4$ for $x \in S^\perp$ which means $x = (x_1, x_2, x_3, x_4) = (-2x_3 + x_4, -x_4, x_3, x_4)$

$$\Rightarrow x = x_3(-2, 0, 1, 0) + x_4(1, -1, 0, 1)$$

thus $\beta = \{ (-2,0,1,0), (1,-1,0,1) \}$ serves as basis for $S^\perp$ as $\text{span}\,\beta = S^\perp$ and clearly β is LI.

$\boxed{P64}$ $W_1 = \text{span}\{1-x^2,\ x-x^3\}$

$W_2 = \{a+bx+cx^2+dx^3 \mid a+b+c-2d=0\}$

To find basis for $W_1 \cap W_2$ we need to find a good description of $W_1 \cap W_2$. This is easy if we can recast W_1 as the sol$^{\underline{n}}$ set for an appropriate eq$^{\underline{n}}$(s),

$$a+bx+cx^2+dx^3 \in W_1 = \text{span}\{1-x^2,\ x-x^3\}$$

implies $\exists c_1, c_2 \in \mathbb{R}$ for which:

$$\begin{aligned} a+bx+cx^2+dx^3 &= c_1(1-x^2) + c_2(x-x^3) \\ &= c_1 + c_2 x - c_1 x^2 - c_2 x^3 \end{aligned}$$

Apparently, by LI of $\{1, x, x^2, x^3\}$ we find:

$$a = c_1,\quad b = c_2,\quad c = -c_1,\quad d = -c_2$$

thus setting $c_1 = c_1$ and $c_2 = c_2$ reveals,

$$a = -c \quad \text{and} \quad b = -d$$

Therefore, $W_1 = \{a+bx+cx^2+dx^3 \mid a+c=0,\ b+d=0\}$

If $a+bx+cx^2+dx^3 \in W_1 \cap W_2$ then we find:

$$\begin{aligned} a+b+c-2d &= 0 \\ a+c &= 0 \\ b+d &= 0 \end{aligned}$$

Then,

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array} \xrightarrow{r_2 - r_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[r_3 + r_2]{r_1 + r_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

Hence $a = -c$, $b = 0$, $d = 0$ follows soon after the above.

So $\quad a+bx+cx^2+dx^3 = a - ax^2 = a(1-x^2) \in W_1 \cap W_2$

In fact, $\boxed{W_1 \cap W_2 = \text{span}\{1-x^2\}}$

P65 Let $\beta = \{E_{11}, iE_{11}, E_{12}, iE_{12}, E_{21}, iE_{21}, E_{22}, iE_{22}\}$ provide a basis for $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$. If $[A]_\beta = (1,2,3,4,5,6,7,8)$ then by def$^{\underline{n}}$ of coordinate map we have

$$A = E_{11} + 2iE_{11} + 3E_{12} + 4iE_{12} + 5E_{21} + 6iE_{21} + 7E_{22} + 8iE_{22}$$

which yields

$$\boxed{A = \left[\begin{array}{c|c} 1+2i & 3+4i \\ \hline 5+6i & 7+8i \end{array}\right]}$$

P66 Suppose $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 3 & 4 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix}$

a.) find basis for $Col(A)$ using columns of A
b.) find basis for $Row(A)$ using rows of A
c.) find basis for $Null(A)$

(a.)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 3 & 4 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix} \xrightarrow[r_4 - 3r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4/3 & 5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $rref(A) = \begin{bmatrix} 1 & 0 & -1/3 & -2/3 \\ 0 & 1 & 4/3 & 5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and by CCP

we find $col_3(A) = -\frac{1}{3}col_1(A) + \frac{4}{3}col_2(A)$ & $col_4(A) = -\frac{2}{3}col_1(A) + \frac{5}{3}col_2(A)$

it follows $\{col_1(A), col_2(A)\}$ form basis for $Col(A)$ since

$\boxed{\{(1,2,0,3), (1,2,3,6)\} \text{ is } \underbrace{\text{LI spanning set}}_{\text{that is, a \underline{basis}}} \text{ for } Col(A)}$.

(b.) Notice $3row_1(A) + row_3(A) = row_4(A)$ and $2row_1(A) = row_2(A)$

thus $Row(A) = span\{row_1(A), row_3(A)\}$ and as

$\boxed{\{[1,1,1,1], [0,3,4,5]\} \text{ is LI we find it serves as \underline{basis} for } Row(A)}$

(c.) $x \in Null(A)$ means $Ax = 0$ hence by calculation in (a.) find

$$x_1 = \frac{1}{3}x_3 + \frac{2}{3}x_4 \quad \& \quad x_2 = -\frac{4}{3}x_3 - \frac{5}{3}x_4$$

Thus $x = x_3(1/3, -4/3, 1, 0) + x_4(2/3, -5/3, 0, 1)$ and

hence $\boxed{\{(1,-4,3,0), (2,-5,0,3)\} \text{ is basis for } Null(A).}$

$\boxed{P67}$ Suppose $V(\mathbb{F})$ is vector space with $U \leq V$ and $W \leq V$

Since U and W are subspaces we know $0 \in U$ and $0 \in W$ thus $0 \in U \cap W$.

Hence $U \cap W \neq \phi$. Clearly $U \cap W \subseteq V$. Suppose $x, y \in U \cap W$

and $c \in \mathbb{F}$. Since $U \leq V$ we have $cx + y \in U$ and

since $W \leq V$ we have $cx + y \in W$ $\therefore$ $cx + y \in U \cap W$

and hence $cx,\ x+y \in U \cap W$. Thus $U \cap W \leq V$ by subspace test Th$^=$. //

$\boxed{P68}$ Let $V(\mathbb{F})$ be vector space and $S = \{v_1, \dots, v_n\} \subseteq V$.

$\Rightarrow]$ Suppose S is LI and suppose $\exists\, a_i, b_i \in \mathbb{F}$ for $i = 1, 2, \dots, n$ such that $a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$. Then

$$a_1 v_1 - b_1 v_1 + \cdots + a_n v_n - b_n v_n = 0$$

and hence

$$(a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n = 0$$

and by LI of $S = \{v_1, \dots, v_n\}$ we find $a_1 - b_1 = 0, \dots, a_n - b_n = 0$

Thus $a_1 = b_1, \dots, a_n = b_n$.

$\Leftarrow]$ Suppose $\exists\, a_i, b_i \in \mathbb{F}$ such that $a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n \Rightarrow a_1 = b_1, \dots, a_n = b_n$.

Consider $\underline{c_1 v_1 + \cdots + c_n v_n = 0}_{*}$ for some $c_1, \dots, c_n \in \mathbb{F}$. Notice $\underline{0 \cdot v_1 + \cdots + 0 \cdot v_n = 0}_{**}$. Consequently, as $0 = 0$ comparing $*$ and $**$ and using our assumption in $\Leftarrow]$ part of proof we find $c_1 = 0, \dots, c_n = 0$ which provides that S is LI.

Therefore, S LI $\Leftrightarrow$ S has equating coefficients property.

$\boxed{P69}$ Let $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in R^{n\times n}$ (R any comm. ring with identity)

(a.) $\text{trace}(I_n) = \sum_{i=1}^{n} (I_n)_{ii} = \sum_{i=1}^{n} 1 = n.$ yep.

(b.) $\text{trace}(cA+B) = \sum_{i=1}^{n} (cA+B)_{ii}$ (we assume $A, B \in R^{n\times n}$ and $c \in R$)

$= \sum_{i=1}^{n} (cA_{ii} + B_{ii})$: def^{n} of matrix addition and scalar multiplication.

$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$

$= \underline{c\,\text{trace}(A) + \text{trace}(B)}.$

(c.) Let $A \in R^{p\times n}$, $B \in R^{n\times p}$ and consider $AB \in R^{p\times p}$ whereas $BA \in R^{n\times n}$ and yet:

$\text{trace}(AB) = \sum_{i=1}^{p} (AB)_{ii}$: def^{n} of trace of $p\times p$ matrix

$= \sum_{i=1}^{p}\sum_{j=1}^{n} A_{ij}B_{ji}$: def^{n} of matrix mult.

$= \sum_{j=1}^{n}\sum_{i=1}^{p} A_{ij}B_{ji}$: flipping the order of finite sums. (btw, can you prove this is allowed ?)

$= \sum_{j=1}^{n}\sum_{i=1}^{p} B_{ji}A_{ij}$: A_{ij}, B_{ji} commute as elements in comm. ring R.

$= \sum_{j=1}^{n} (BA)_{jj}$: def^{n} of BA.

$= \text{trace}(BA).$: def^{n} of trace of $n\times n$ matrix.

$\boxed{P70}$ $[A,B] = AB - BA$ for square matrices A, B.

Suppose $[A,B] = kI$ for some $k \neq 0$. Calculate using the results of $\boxed{P69}$,

$$\begin{aligned} \text{trace}([A,B]) &= \text{trace}(AB - BA) \\ &= \text{trace}(AB) + \text{trace}(-BA) \\ &= \text{trace}(AB) - \text{trace}(BA) \\ &= \text{trace}(AB) - \text{trace}(AB) \\ &= 0 \end{aligned}$$

$-BA = (-1)BA$

YET, $\text{trace}(kI) = k\,\text{trace}(I) = kn \neq 0$ $\rightarrow\leftarrow$ $\underbrace{\text{trace}([A,B])}_{0} = \underbrace{\text{trace}(kI)}_{kn}$

Thus $[A,B] \neq kI$ for $k \neq 0$.

$\boxed{P71}$ Consider $W = \{ y \in \underbrace{C^\infty(\mathbb{R})}_{\text{(smooth fncts from } \mathbb{R} \text{ to } \mathbb{R})} \mid y'' - 3y' + 2y = 0 \}$

Observe $0'' - 3(0)' + 2(0) = 0$ $\therefore$ the zero function $0 \in W \neq \emptyset$.

By construction $W \subseteq C^\infty(\mathbb{R})$. Consider $y_1, y_2 \in W$ and $c \in \mathbb{R}$,

$$\begin{aligned} (cy_1 + y_2)'' - 3(cy_1+y_2)' + 2(cy_1+y_2) &= c(y_1'' - 3y_1' + 2y_1) + y_2'' - 3y_2' + 2y_2 && \text{: properties of differentiation} \\ &= c(0) + 0 && \text{: since } y_1, y_2 \in W \\ &= 0 \end{aligned}$$

Thus $cy_1 + y_2 \in W$ and hence $cy_1,\ y_1 + y_2 \in W$ and we conclude $W \leq C^\infty(\mathbb{R})$ by the subspace test Th^{m}. //

$\boxed{P72}$ Consider $y'' = 0$ where $y' = \frac{dy}{dx}$.

$$\frac{d}{dx}(y') = 0 \Rightarrow y' = c_1 \Rightarrow y = c_1 x + c_2 \in \text{span}\{x, 1\}$$

Since $\{x, 1\}$ is LI subset of $\mathcal{F}(\mathbb{R})$ we deduce $\{x,1\}$ serves as basis for $\text{sol}^{\underline{n}}$ space to $y'' = 0$.