Same instructions as Mission 1. Thanks!

Problem 81 Your signature below indicates you have:

(a.) I read Sections 7.1 – 7.5, & 7.8 of Cook's lecture notes: ________________.

Problem 82 Let $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 5 & 6 & 11 \\ 2 & 0 & 2 & 2 \\ 2 & -1 & 1 & 0 \\ 2 & -2 & 0 & -2 \end{bmatrix}$. Find a basis for Col($A$) and Null($A$). Also, calculate the rank and nullity of A and check that the rank-nullity theorem holds true for your calculations.

Problem 83 Let $W = \{f(x) \in P_4 \mid P(1) = 0 \ \& \ P(2) = 0\}$. Find a basis β for W and write the coordinate map Φ_β for W.

Problem 84 Consider the set of quadratic forms in three variables x, y, z. Let $\gamma = \{x^2, y^2, z^2, xy, xz, yz\}$ and define the set of trivariate homogeneous polynomials of order two by

$$W = \text{span}\{x^2, y^2, z^2, xy, xz, yz\}.$$

Observe W is a function space and as it is a span we find $W \leq \mathcal{F}(\mathbb{R}^3, \mathbb{R})$.
If $v = x^2 + (x+y)(y+z)$ then calculate $[v]_\gamma$.

Problem 85 Let V be a finite-dimensional vector space over $\mathbb{R}$ and β its basis. Show that

$$[v+w]_\beta = [v]_\beta + [w]_\beta.$$

Problem 86 Show $L : \mathbb{R}^{m \times n} \to \mathbb{R}^{n \times m}$ be defined by $L(A) = A^T$ is linear transformation.

Problem 87 Let B be a 2×2 matrix and define $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ by $T(A) = AB+BA$ for all $A \in \mathbb{R}^{2\times2}$. Show T is linear.

Problem 88 Suppose $S, T \in \mathcal{L}(V, W)$ and suppose $\beta = \{f_1, \ldots, f_n\}$ is a basis for V and $\gamma = \{g_1, \ldots, g_m\}$ is a basis for W. Prove that $[S+T]_{\beta,\gamma} = [S]_{\beta,\gamma} + [T]_{\beta,\gamma}$

Problem 89 Suppose $T : P_2 \to T(P_2) \leq \mathbb{R}[t]$ be defined by $T(f(x)) = \int_0^t f(x)\,dx$. Let P_2 have basis $\beta = \{3x^2, 2x, 1\}$ and find a basis γ for Range(T). Finally, calculate $[T]_{\beta,\gamma}$

Problem 90 Let $C = \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix}$ and define $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ by $T(X) = CX$.
Suppose $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Calculate $[T]_{\beta,\beta}$.

Problem 91 Let $\beta = \{(1,1,1,1), (0,1,1,0)\}$ and $\gamma = \{(1,2,3), (0,-1,2)\}$. Let $V = \text{span}(\beta)$ and $W = \text{span}(\gamma)$. Suppose $T(1,1,1,1) = 4(1,2,3) + 3(0,-1,2)$ and $T(0,1,1,0) = 0$. Find $[T]_{\beta,\gamma}$ and determine the rank and nullity of T.

Problem 92 Let $\beta = \{(t-2)^2, (t-2), 1\}$ form the basis for $P_2 \leq \mathbb{R}[t]$. Suppose that $f(t) = at^2 + bt + c$. Calculate the coordinates of $f(t)$ with respect to β; that is, find $[f(t)]_\beta$.

Problem 93 Let $v = (a, b)$ and find the coordinates of v in the $\beta = \{(1,1),(1,-1)\}$ basis.

Problem 94 Let β be a basis for P_2 such that $[x^2+1]_\beta = (1,0,0)$ and $[2x-1]_\beta = (0,1,0)$ and $[3]_\beta = (0,0,1)$. Find $[ax^2+bx+c]_\beta$ and determine β.

Problem 95 Define $T(f(x)) = f(x) + f'(x) + f''(x)$ for each $f(x) \in P_2$. If possible, find a basis β for $P_2 = \text{span}\{1, x, x^2\}$ for which $[T]_{\beta,\beta} = I$.

Problem 96 Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $T(x,y,z) = (x+2y-z,\ 2x+3y+z,\ x+y+3z,\ x-z)$ for all $(x,y,z) \in \mathbb{R}^3$. Find bases β, γ such that $[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$ for an appropriately sized identity matrix I_p. What can you read by inspection from $[T]_{\beta,\gamma}$?

Problem 97 Suppose P_2 has basis β and $f(x) \in P_2$ has $[f(x)]_\beta = (1,2,3)$. If $\gamma = \{g_1, g_2, g_3\}$ is a basis for which $[g_1]_\beta = (1,1,0)$ and $[g_2]_\beta = (1,-1,0)$ and $[g_3]_\beta = (0,0,2)$ then find $[f(x)]_\gamma$.

Problem 98 Suppose $T : V \to V$ is a linear transformation where $\dim(V) < \infty$. Explain why it is reasonable to define $\text{tr}(T) = \text{tr}([T]_{\beta,\beta})$.

Problem 99 Let S be a set of objects then $S^{m\times n}$ is the set of $m \times n$ matrices of objects in S. For example, if $S = P_2$ then $S^{2\times 2}$ is the set of 2×2 matrices with quadratic polynomial components. Let $V = \{A \in (P_2)^{2\times 2} \mid \text{trace}(A) = 0 \ \&\ A = A^T\}$ find an isomorphism from V to $W = \{X \in \mathbb{R}^{3\times 3} \mid X^T = X\}$.

Problem 100 Find an isomorphism from the set V of 3×3 antisymmetric matrices to the set W of 4×4 traceless diagonal matrices.

MISSION 5 SOLUTION: LINEAR ALGEBRA

$\boxed{P82}$ $\text{rref}\begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 5 & 6 & 11 \\ 2 & 0 & 0 & 2 \\ 2 & -1 & -1 & 0 \\ 2 & -2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.) Hence $\boxed{\beta = \{(1,1,2,2,2),\ (2,5,0,-1,-2)\}}$ serves as basis for $Col(A)$ and we see $\#\beta = \boxed{2 = \text{rank}(A)}$ (used CCP to make claim here)

2.) $Ax = 0 \implies \begin{array}{l} x_1 = -x_3 - x_4 \\ x_2 = -x_3 - 2x_4 \end{array}$ $\therefore\ x = (-x_3 - x_4,\ -x_3 - 2x_4,\ x_3,\ x_4)$
$= (-1,-1,1,0)x_3 + (-1,-2,0,1)x_4$

Thus $\boxed{\gamma = \{(-1,-1,1,0),\ (-1,-2,0,1)\}}$ gives basis for $Null(A)$ moreover $\#\gamma = \boxed{2 = \text{nullity}(A)}$

<u>Remark</u>: $4 = \#$ of columns of $A = rk(A) + \nu(A)$. Also, while $\exists\ \infty$ly many choices for β & γ, the dimensions of $Col(A)$ and $Null(A)$ are same, no matter the choice...

$\boxed{P83}$ $W = \{P(x) \mid P(1) = 0,\ P(2) = 0\} \subset P_4 = \{A_4x^4 + A_3x^3 + \cdots + A_0 \mid A_4, \dots, A_0 \in \mathbb{R}\}$
Find basis β for W and write formula for Φ_β

I'll let highschool algebra help out here. By <u>FACTOR Th^m</u>, $P(x) \in W$,

$$P(x) = (Ax^2 + Bx + C)(x-1)(x-2)$$
$$= (Ax^2 + Bx + C)(x^2 - 3x + 2)$$
$$= A(x^4 - 3x^3 + 2x^2) + B(x^3 - 3x^2 + 2x) + C(x^2 - 3x + 2)$$

Thus $\boxed{\beta = \{x^4 - 3x^3 + 2x^2,\ x^3 - 3x^2 + 2x,\ x^2 - 3x + 2\}}$ serves as basis for W and

$$\Phi_\beta(Ax^4 + Bx^3 + Cx^2 + Dx + E) = (A,\ B + 3A,\ C - 2A + 3B + 9A)$$

Let me check my conjecture above, Nope, wrong $\Phi_\beta^{-1}(e_1) = \Phi_\beta^{-1}(1,0,0)$
well $A = 1$, $B + 3A = 0 \therefore B = -3$, $C + 7A + 3B = 0 \to C = 9 - 7 = 2$
$\Phi_\beta^{-1}(e_1) = (x^4 - 3x^3 + 2x^2) - 3(x^3 - 3x^2 + 2x) + 2(x^2 - 3x + 2)$
$= x^4 - 6x^3 + 11x^2 - 6x + 2x^2 - 6x + 4$

I leave this to show you the flaws in my process ☺

$\boxed{P83}$ $P(x) \in W \Rightarrow P(x) = (Ax^2+Bx+C)(x-1)(x-2)$

then it is simple to write

$$\boxed{\Phi_\beta\left((Ax^2+Bx+C)(x-1)(x-2)\right) = (A, B, C)}$$

However, if we wish a formula for $\Phi_\beta(C_4 x^4 + C_3x^3 + C_2x^2 + C_1x + C_0)$ I need to think a bit more,

$$\Phi_\beta\left(Ax^4 + (B-3A)x^3 + (C-3B+2A)x^2 + (2B-3C)x + 2C\right) = (A, B, C)$$

$$\Phi_\beta(C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0) =$$

Where I observed the substitutions,

$C_4 = A \quad \therefore \quad \underline{A = C_4}$.

$C_3 = B - 3A \Rightarrow B = C_3 + 3A = \underline{C_3 + 3C_4 = B}$.

$C_2 = C - 3B + 2A \Rightarrow C = C_2 + 3B - 2A = C_2 + 3(C_3 + 3C_4) - 2C_4$

$\underline{C = C_2 + 3C_3 + 7C_4}$

Thus,

$$\boxed{\Phi_\beta(C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0) = (C_4,\ C_3 + 3C_4,\ C_2 + 3C_3 + 7C_4)}$$

Alternatively,

$C_0 = 2C \quad \therefore \quad C = C_0/2 \longrightarrow \cancel{\Phi_\beta(C_4x^4 + \cdots + C_0) =}$

$$\boxed{\Phi_\beta(C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0) = (C_4,\ C_3 + 3C_4,\ C_0/2)} \quad -(*)$$

there are many, many other ways to write this formula, the tricky thing is that the coefficients which don't appear in the RHS of $\Phi_\beta(\ ) = (\ )$ are implicitly fncts of those which do by the nature of W.

Let's check $*$, $\Phi_\beta^{-1}(e_1) = x^4 - 3x^3 + 2x^2$, $\Phi_\beta^{-1}(e_2) = x^3 - 3x^2 + 2x$ and $\Phi_\beta^{-1}(e_3) = x^2 - 3x + 2$ if my f-la is correct.

$$\underbrace{\Phi_\beta^{-1}(1,0,0)} = x^4 - 3x^3 + \underbrace{C_2x^2 + C_1x} + 0$$

$C_4 = 1$

$C_3 + 3C_4 = 0 \quad \therefore \quad C_3 = -3$

$C_0/2 = 0$

turns out $C_2 = 2$, $C_1 = 0$ by structure of W and assumption $C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 \in W$.

Remark : THIS PAGE shows you why we should insist on working with basis.

$\boxed{P84}$ $\gamma = \{x^2, y^2, z^2, xy, xz, yz\}$ and $W = \text{span}\,\gamma$
If $v = x^2 + (x+y)(y+z)$ then ~~to~~ calculate $[v]_\gamma$

$$v = x^2 + xy + xz + y^2 + yz$$
$$[v]_\gamma = [x^2 + y^2 + 0 \cdot z^2 + xy + xz + yz]_\gamma$$
$$\therefore \boxed{[v]_\gamma = (1, 1, 0, 1, 1, 1)}$$

Remark: it's funny how $\boxed{83}$ looks easier than $\boxed{84}$.

$\boxed{P85}$ Prove $[v+w]_\beta = [v]_\beta + [w]_\beta$ for finite-dim'l vector space V with basis β

Let $\beta = \{g_1, g_2, \dots, g_n\}$ and consider $v, w \in V$. As $\text{span}\,\beta = V$ we have $\exists x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ for which

$$v = x_1 g_1 + x_2 g_2 + \cdots + x_n g_n = \sum_{i=1}^{n} x_i g_i$$
$$w = y_1 g_1 + \cdots + y_n g_n = \sum_{i=1}^{n} y_i g_i$$

By definition,

$$\left.\begin{aligned} [v]_\beta &= (x_1, x_2, \dots, x_n) \\ [w]_\beta &= (y_1, y_2, \dots, y_n) \end{aligned}\right\} (*)$$

However,

$$v + w = \sum_{i=1}^{n} x_i g_i + \sum_{i=1}^{n} y_i g_i = \sum_{i=1}^{n} x_i g_i + y_i g_i = \sum_{i=1}^{n} (x_i + y_i) g_i$$

Hence, by def^n, $[v+w]_\beta = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$. $(**)$
Consequently, by vector add. in $\mathbb{R}^n$, we find comparing $*$ & $**$

$$\underline{[v+w]_\beta = [v]_\beta + [w]_\beta}.$$

Remark: this problem proves additivity of $\Phi_\beta : V \longrightarrow \mathbb{R}^n$ as $\Phi_\beta(v+w) = [v+w]_\beta = [v]_\beta + [w]_\beta = \Phi_\beta(v) + \Phi_\beta(w)$.

P86 Show $L : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{n\times m}$ def$^{\underline{z}}$ by $L(A) = A^T$ is lin. trans.

1.) $L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$

2.) $L(cA) = (cA)^T = c\,A^T = c\,L(A)$

By properties of the transpose we conclude $L \in \mathcal{L}(\mathbb{R}^{m\times n}, \mathbb{R}^{n\times m})$ //

P87 Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ for some $B \in \mathbb{R}^{2\times 2}$ by the f-la $T(A) = AB + BA$. Show T linear

$$\left.\begin{aligned} T(X+Y) &= (X+Y)B + B(X+Y) \\ &= XB + YB + BX + BY \\ &= XB + BX + YB + BY \\ &= T(X) + T(Y) \end{aligned}\right\}$$

T is additive is supported by properties of matrix multiplication and addition.

Likewise $T(cX) = (cX)B + B(cX) = c\,(XB + BX) = c\,T(X)$.

Thus $T \in \mathcal{L}(\mathbb{R}^{2\times2}, \mathbb{R}^{2\times2})$; that is T is a linear map on $\mathbb{R}^{2\times 2}$.

[in yet more notation news] $\longrightarrow$ [we could say $T \in gl(\mathbb{R}^{2\times2})$] –

P88 Suppose $S, T \in \mathcal{L}(V, W)$ and suppose $\beta = \{f_1, \dots, f_n\}$ a basis for V and $\gamma = \{g_1, \dots, g_m\}$ a basis for W. Prove that $[S+T]_{\beta,\gamma} = [S]_{\beta,\gamma} + [T]_{\beta,\gamma}$

$$\begin{tikzcd} V \arrow[r, shift left, "S"] \arrow[r, shift right, "T"'] \arrow[d, "\Phi_\beta"'] & W \arrow[d, "\Phi_\gamma"] \\ \mathbb{R}^n \arrow[r, shift left, "L_{[S]_{\beta,\gamma}}"] \arrow[r, shift right, "L_{[T]_{\beta,\gamma}}"'] & \mathbb{R}^m \end{tikzcd}$$

$[S]_{\beta,\gamma} = [L_{[S]_{\beta,\gamma}}]$ *

$[T]_{\beta,\gamma} = [L_{[T]_{\beta,\gamma}}]$

(notice $L_A(x) = Ax$ has $[L_A] = A$ for $L_A : \mathbb{R}^n \to \mathbb{R}^m$)

By def$^{\underline{n}}$ $L_{[S]_{\beta,\gamma}} = \Phi_\gamma \circ S \circ \Phi_\beta^{-1}$ & $L_{[T]_{\beta,\gamma}} = \Phi_\gamma \circ T \circ \Phi_\beta^{-1}$

Using * consider,

I'll explain

$$\begin{aligned} [S]_{\beta,\gamma} + [T]_{\beta,\gamma} &= [\Phi_\gamma \circ S \circ \Phi_\beta^{-1}] + [\Phi_\gamma \circ T \circ \Phi_\beta^{-1}] \\ &= [\Phi_\gamma \circ S \circ \Phi_\beta^{-1} + \Phi_\gamma \circ T \circ \Phi_\beta^{-1}] = [\Phi_\gamma \circ (S+T) \circ \Phi_\beta^{-1}] \\ &= [S+T]_{\beta,\gamma}. // \end{aligned}$$

P88 continued

$$[S]_{\beta,\gamma} + [T]_{\beta,\gamma} = [\Phi_\gamma \circ S \circ \Phi_\beta^{-1}] + [\Phi_\gamma \circ T \circ \Phi_\beta^{-1}]$$
$$= [\Phi_\gamma \circ S \circ \Phi_\beta^{-1} + \Phi_\gamma \circ T \circ \Phi_\beta^{-1}]$$
$$= [\Phi_\gamma \circ (S+T) \circ \Phi_\beta^{-1}]$$
$$= [S \circ T]_{\beta,\gamma}$$

property of standard matrices from PART I.

prop. of linear maps. (Lemma 2))

using $*$ of last page again.

Oh, maybe this proof is not what you want, try the next then,
~~$C(\ldots[S]_{\beta,\gamma}) = \ldots[S]_{\beta,\gamma}\ldots$~~ perhaps I should give more detail

Lemma: Let ψ, ϕ, S, T be linear transformations on vector spaces then if $\psi \circ (S+T) \circ \phi$ is reasonable then
$$\psi \circ (S+T) \circ \phi = \psi \circ S \circ \phi + \psi \circ T \circ \phi$$

Proof: Suppose $\psi : W \to U_1$ and $\phi : U_2 \to V$ and $S, T : V \to W$ so $\psi \circ S \circ \phi : U_2 \to V \to W \to U_1$ and $\psi \circ T \circ \phi : U_2 \to V \to W \to U_1$ are well-defined then, assuming all maps above are linear, if $x \in U_2$,

$$\begin{aligned}
(\psi \circ (S+T) \circ \phi)(x) &= \psi((S+T)(\phi(x))) && : \text{def}^{\underline{n}} \text{ of composition} \\
&= \psi(S(\phi(x)) + T(\phi(x))) && : \text{def}^{\underline{n}} \text{ of } S+T \\
&= \psi(S(\phi(x))) + \psi(T(\phi(x))) && : \text{linearity of } \psi \\
&= (\psi \circ S \circ \phi)(x) + (\psi \circ T \circ \phi)(x) && : \text{def}^{\underline{n}} \text{ of composite} \\
&= [(\psi \circ S \circ \phi) + (\psi \circ T \circ \phi)](x) && : \text{def}^{\underline{n}} \text{ of fnct. addition}
\end{aligned}$$

But, the above holds for all $x \in U_2$ hence

$$\boxed{\psi \circ (S+T) \circ \phi = \psi \circ S \circ \phi + \psi \circ T \circ \phi}$$

<u>Remark:</u> I was too greedy perhaps, we could do with linearity of just ψ, I didn't need the linearity of S, T or ϕ for this Lemma However, as they appear $\psi = \Phi_\gamma$ and $\phi = \Phi_\beta^{-1}$ the linearity is essential to define $[S]_{\beta,\gamma}$, $[T]_{\beta,\gamma}$ etc. (But, this Lemma is beyond the problem)

P89 $T: P_2 \to T(P_2) \leq \mathbb{R}[t]$ where $T(f(x)) = \int_0^t f(x)\,dx$
Let $\beta = \{3x^2, 2x, 1\}$ serve as basis for P_2 and find a basis γ for Range $(T) = T(P_2)$. Also calculate $[T]_{\beta,\gamma}$

$$\int_0^t (3ax^2 + 2bx + c)\,dx = \left(ax^3 + 2bx^2 + cx\right)\Big|_0^t$$

$$\underline{T(3ax^2+2bx+c) = at^3 + 2bt^2 + ct} \hookrightarrow \boxed{\gamma = \{t^3, 2t^2, t\}}$$

a basis for Range (T)

$$T(3x^2) = t^3 \longrightarrow [T(3x^2)]_\gamma = (1,0,0)$$
$$T(2x) = 2t^2 \longrightarrow [T(2x)]_\gamma = (0,1,0)$$
$$T(1) = t \longrightarrow [T(1)]_\gamma = (0,0,1)$$

$$\boxed{[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

– (of course other choices for γ will produce different $[T]_{\beta,\gamma}$) –

P90 Let $C = \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix}$ and define $T(X) = CX$.
Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ serve as basis for $\mathbb{R}^{2\times 2}$. Calculate $[T]_{\beta,\beta}$.

$$T(X) = T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[\begin{array}{c|c} -a+4c & -b+4d \\ \hline 2a+6c & 2b+6d \end{array}\right]$$

$$[T(X)]_\beta = (-a+4c, -b+4d, 2a+6c, 2b+6d)$$
$$[X]_\beta = (a,b,c,d)$$

$$[T]_{\beta,\beta}[X]_\beta = [T(X)]_\beta \implies \boxed{[T]_{\beta,\beta} = \left[\begin{array}{cc|cc} -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 \\ \hline 2 & 0 & 6 & 0 \\ 0 & 2 & 0 & 6 \end{array}\right]}$$

hmm... neat.

[later we discuss, this is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix}$]

$\boxed{P91}$ $\beta = \{(1,1,1,1), (0,1,1,0)\}$, $V = \text{span}\,\beta$
$\gamma = \{(1,2,3), (0,-1,2)\}$, $W = \text{span}\,\gamma$

Let $T(1,1,1,1) = 4(1,2,3) + 3(0,-1,2)$
$T(0,1,1,0) = 0$

Find $[T]_{\beta,\gamma}$ and determine the rank (T) and nullity (T).

$$[T(V_1)]_\gamma = [T(1,1,1,1)]_\gamma = [4(1,2,3)+3(0,-1,2)]_\gamma = (4,3)$$

$$[T(V_2)]_\gamma = [T(0,1,1,0)]_\gamma = [0]_\gamma = (0,0)$$

But $[T]_{\beta,\gamma} = \left[\, [T(V_1)]_\gamma \mid [T(V_2)]_\gamma \,\right]$ where $\beta = \{V_1, V_2\}$ thus we find $\boxed{[T]_{\beta,\gamma} = \begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix}} \Rightarrow \boxed{\text{rank}(T) = 1}$

Since $T: V \longrightarrow W$ has $\dim V = 2$ we know by rank/nullity theorem for linear transformations on finite-dim'l v-spaces that $2 = \text{rank}(T) + \text{nullity}(T) \Longrightarrow \boxed{\text{nullity}(T) = 1}$

$\boxed{P92}$ Let $\beta = \{(t-2)^2, (t-2), 1\}$ form basis for $P_2 \leq \mathbb{R}[t]$. Suppose $f(t) = at^2 + bt + c$. Find $[f(t)]_\beta$.

I will use... ALGEBRA... (ADDED ZERO WITH INTENT)

$$\begin{aligned} f(t) &= at^2 + bt + c \\ &= a(t-2+2)^2 + b(t-2+2) + c \\ &= a\left[(t-2)^2 + 4(t-2) + 4\right] + b\left[(t-2) + 2\right] + c \\ &= a(t-2)^2 + (4a+b)(t-2) + 4a + 2b + c \end{aligned}$$

$$\therefore \boxed{[f(t)]_\beta = (a,\ 4a+b,\ 4a+2b+c)}$$

- Alternatively, use $f(t) = f(2) + f'(2)(t-2) + \frac{f''(2)}{2}(t-2)^2$ to select coefficients for 1, $(t-2)$ and $(t-2)^2$.

P93 Let $v = (a,b)$ and find $[v]_\beta$ for $\beta = \{(1,1), (1,-1)\}$

In $\mathbb{R}^2$ we have <u>standard</u> technique based on matrix algebra.

<u>Step 1</u>: formulate $[\beta] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

<u>Step 2</u>: find $[\beta]^{-1} = \frac{1}{-1-1}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (2x2 inverse f-la thw.)

<u>Step 3</u>: $[v]_\beta = [\beta]^{-1}v$ (as derived in my notes)

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} a/2 + b/2 \\ a/2 - b/2 \end{bmatrix} \quad \therefore \quad \boxed{[v]_\beta = \left(\tfrac{1}{2}(a+b), \tfrac{1}{2}(a-b)\right)}$$

P94 Let β be basis for P_2 such that $[x^2+1]_\beta = (1,0,0)$ and $[2x-1]_\beta = (0,1,0)$ and $[3]_\beta = (0,0,1)$. Find $[ax^2+bx+c]_\beta$ and determine β.

Recall, if $\beta = \{v_1, v_2, v_3\}$ then $\Phi_\beta(v_1) = e_1 = (1,0,0)$ and $\Phi_\beta(v_2) = (0,1,0)$ and $\Phi_\beta(v_3) = (0,0,1)$. We find by the given data that

$$v_1 = x^2+1, \quad v_2 = 2x-1, \quad v_3 = 3$$

That is $\boxed{\beta = \{x^2+1, 2x-1, 3\}}$. Furthermore,

$$[ax^2+bx+c]_\beta = \left[a(x^2+1) - a + b\left(\tfrac{2x-1}{2} + \tfrac{1}{2}\right) + c\left(\tfrac{3}{3}\right)\right]_\beta$$

$$= \left[a(x^2+1) + \tfrac{b}{2}(2x-1) + \left(\tfrac{c}{3} - \tfrac{a}{3} + \tfrac{b}{6}\right)3\right]_\beta$$

$$= \boxed{\left(a, \tfrac{b}{2}, \tfrac{-a}{3} + \tfrac{b}{6} + \tfrac{c}{3}\right)}$$

Check answer,

$$a(x^2+1) + \tfrac{b}{2}(2x-1) + \left(\tfrac{-a}{3} + \tfrac{b}{6} + \tfrac{c}{3}\right)3 =$$

$$= ax^2 + \cancel{a} + bx - \cancel{b/2} - \cancel{a} + \cancel{b/2} + c$$

$$= ax^2 + bx + c. \checkmark$$

$\boxed{P95}$ Let $T(f(x)) = f(x) + f'(x) + f''(x)$ for each $f(x) \in P_2$.
If possible, find a basis β for P_2 for which $[T]_{\beta,\beta} = I$

Since $T : P_2 \to P_2$, it is possible $[T]_{\beta,\beta} = I$ as $[T]_{\beta,\beta}$ is 3×3.
However, we also need $\mathrm{Ker}(T) = \{0\}$ or $\mathrm{Range}(T) = P_2$.

$$\mathrm{Ker}(T) = \{f(x) \mid f(x) + f'(x) + f''(x) = 0\}$$

Let $f(x) = ax^2+bx+c \in \mathrm{Ker}(T)$, we have

$$ax^2+bx+c + 2ax+b+2a = 0$$
$$ax^2 + (b+2a)x + c+b+2a = 0$$
$$\Rightarrow a=0,\ b+2a=0,\ c+b+2a=0$$
$$\Rightarrow a=0,\ b=0,\ c=0 \quad \therefore \mathrm{Ker}(T)=\{0\} \Rightarrow T \text{ is 1-1}$$

and by rank, nullity theorem $\mathrm{Range}(T) = P_2$, that is, $\mathrm{rank}(T)=3$.
Consequently $\exists\beta$ for which $[T]_{\beta\beta} = I$. How to find it?
FALSE
$[T]_{\beta\beta}[v]_\beta = [T(v)]_\beta$ if $\beta = \{v_1, v_2, v_3\}$ we have to solve
$[v_j]_\beta = e_j$, more to the point; $T(v_j) = v_j$ for $j = 1,2,3$.
Let $v_j = ax^2+bx+c$,

$$T(ax^2+bx+c) = ax^2+bx+c$$
$$\underline{ax^2} + \underline{\underline{(b+2a)}}x + \underline{\underline{c+b+2a}} = \underline{ax^2} + \underline{\underline{b}}x + \underline{\underline{\underline{c}}}$$

Hence $b+2a = b \Rightarrow 2a = 0 \therefore a = 0$.
$c+b+2a = c \Rightarrow b = 0$ and c is free

Let $v_1 = 1$ clearly $T(1) = 1$. That's it. We can't find v_2 & v_3 with $\{v_1,v_2,v_3\}$ LI and $T(v_2) = v_2$, $T(v_3) = v_3$.
We just worked out that only $v_1 = 1$ is available for this T. In conclusion, it is $\boxed{\text{not possible}}$ to find β s.t. $[T]_{\beta\beta} = I$.

__Remark__: notice Th$^{\underline{m}}$ 7.5.2 allows use to select β, γ possibly distinct for domain & codomain of linear transformation in question. In contrast $\beta = \gamma$ is forced upon us for $\boxed{P95}$.
This is not a minor restriction, this makes it harder to match $[T]_{\beta\beta}$ to I.

P9S continued

(this was not expected, this part of the sol^{n} is foreshadowing for PART III)

$$(T - I)V_2 = V_1 = 1$$

I can solve $T(V_2) - V_2 = V_1$ aka $\underline{T(V_2) = V_2 + V_1}\ *$

$$T(f(x)) - f(x) = 1$$

$$\cancel{f(x)} + f'(x) + f''(x) - \cancel{f(x)} = 1$$

$$f(x) + f'(x) = x + c_1$$

$$ax^2 + bx + c + 2ax + b = x + c_1 \hookrightarrow a = 0$$

$$(b + 2a)x + c + b = x + c_1$$

$$b + 2a = 1 \longrightarrow b = 1$$

$$c + b = c_1 \longrightarrow c + 1 = c_1 \quad \therefore \quad c = c_1 - 1$$

$\underline{\text{choose } c = 0}$.

Thus $V_2 = x$. This has

$$(T - I)(x) = x + 1 - x = 1.$$

Next, I'll solve,

$$(T - I)(V_3) = V_2 \quad \text{aka} \quad \underline{T(V_3) = V_3 + V_2}\ **$$

$$T(V_3) - V_3 = x \quad , V_3 = f(x)$$

$$f'(x) + f''(x) = x$$

$$f(x) + f'(x) = \tfrac{1}{2}x^2 + c_1$$

$$ax^2 + bx + c + 2ax + b = \tfrac{1}{2}x^2 + c_1 \Rightarrow \underline{a = 1/2}.$$

$$b + 2a = 0 \Rightarrow \underline{b = -1}.$$

$$c + b = c_1 \Rightarrow c = c_1 - b = c_1 + 1$$

Thus $V_3 = \frac{1}{2}x^2 - x$ will do. With $\beta = \{1, x, \frac{1}{2}x^2 - x\}$ we have

$$T(1) = 1 \qquad [T(1)]_\beta = (1, 0, 0)$$

$$(*) \quad T(x) = x + 1 \qquad [T(x)]_\beta = (1, 1, 0)$$

$$(**) \quad T(\tfrac{1}{2}x^2 - x) = \tfrac{1}{2}x^2 - x + x \qquad [T(\tfrac{1}{2}x^2 - x)]_\beta = (0, 1, 1)$$

$$\underbrace{[T]_{\beta\beta} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

this is arguably the next best thing to I we can reach for $[T]_{\beta\beta}$

$\boxed{P96}$ $T(x,y,z) = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Null $([T])$ is found from calculation of $\text{rref}([T]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (by technology, (I used www.math.odu.edu/~bogaki/cgi-bin.cgi)

Aww, well, $p = 3$ here as nullity$([T]) = 0$.

One nice choice for β is simply $\beta = \{e_1, e_2, e_3\}$

$$T(e_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \ \& \ T(e_2) = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \ \& \ T(e_3) = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

Let $\gamma = \{ T(e_1), T(e_2), T(e_3), v \}$ where

$v \notin \text{span}\{(1,2,1,1), (2,3,1,0), (-1,1,3,-1)\}$ (*)

there are <u>many</u> choices for v, but, as long as it satisfies * we have,

$$\begin{aligned} [T]_{\beta,\gamma} &= \left[[T(e_1)]_\gamma \,|\, [T(e_2)]_\gamma \,|\, [T(e_3)]_\gamma \right] \\ &= [e_1 | e_2 | e_3] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

For example, $\gamma = \{(1,2,1,1), (2,3,1,0), (-1,1,3,-1), (0,0,0,1)\}$ works nicely and you can check $\text{rref}[\gamma] = [e_1|e_2|e_3|e_4]$ which shows γ is indeed LI.

P97 Suppose P_2 has bases β and $\gamma = \{g_1, g_2, g_3\}$ such that $[g_1]_\beta = (1,1,0)$, $[g_2]_\beta = (1,-1,0)$ and $[g_3]_\beta = (0,0,2)$. If $[f(x)]_\beta = (1,2,3)$ then calculate $[f(x)]_\gamma$.

Consider $[f(x)]_\gamma = \Phi_\gamma(f(x)) = (\Phi_\gamma \circ \Phi_\beta^{-1} \circ \Phi_\beta)(f(x))$

that is, $[f(x)]_\gamma = \underbrace{[\Phi_\gamma \circ \Phi_\beta^{-1}]}_{P_{\beta,\gamma}}[f(x)]_\beta \qquad (\star)$

However,

$$\begin{aligned}
P_{\gamma,\beta} &= [\Phi_\beta \circ \Phi_\gamma^{-1}] \\
&= [\Phi_\beta(\Phi_\gamma^{-1}(e_1)) \mid \Phi_\beta(\Phi_\gamma^{-1}(e_2)) \mid \Phi_\beta(\Phi_\gamma^{-1}(e_3))] \\
&= [\Phi_\beta(g_1) \mid \Phi_\beta(g_2) \mid \Phi_\beta(g_3)] \\
&= [[g_1]_\beta \mid [g_2]_\beta \mid [g_3]_\beta] \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (P_{\beta,\gamma})^{-1}
\end{aligned}$$

: we need the inverse matrix.

A nice identity exists for block matrices $\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]^{-1} = \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right]$

$B = 2$ has $B^{-1} = 1/2$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has $A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ hence,

$$P_{\beta,\gamma} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Return to $(\star)$,

$$\therefore \quad [f(x)]_\gamma = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 3/2 \end{bmatrix}$$

$$\therefore \boxed{[f(x)]_\gamma = \left(\tfrac{3}{2}, \tfrac{-1}{2}, \tfrac{3}{2}\right)}$$

__Remark:__ to the grader, the students were free to simply quote my notes here, I derive for fun.

P98 Suppose $T: V \to V$ is linear transformation with $\dim V < \infty$. Explain why $tr(T) = tr([T]_{\beta\beta})$ is a reasonable def$^{=}$.

Certainly $[T]_{\beta\beta} \in \mathbb{R}^{n \times n}$ given $\dim V = n$ hence the formula for a particular choice of β is reasonable. However, we should wonder, what if I used $\bar{\beta} \neq \beta$ where $\bar{\beta}$ is a different basis for V. We proved

$$[T]_{\bar{\beta}\bar{\beta}} = P^{-1}[T]_{\beta\beta}P$$

for a particular change of basis matrix P. Notice,

$$\begin{aligned} \text{trace}\left([T]_{\bar{\beta}\bar{\beta}}\right) &= \text{trace}\left(P^{-1}[T]_{\beta\beta}P\right) \\ &= \text{trace}\left(P P^{-1}[T]_{\beta\beta}\right) \\ &= \text{trace}\left(I[T]_{\beta\beta}\right) \\ &= \text{trace}\left([T]_{\beta\beta}\right) \end{aligned}$$

$tr(AB) = tr(BA)$ proved previously

Hence the apparent β-dependence of the def^{n} is illusory. The def$^{=}$ is indeed coordinate-independent.

—(if the def^{n} gave different $tr(T)$ for different β-choices then this would be a bad def$^{=}$ as V has no preferred basis generally)—

P99 $(P_2)^{2\times2}$ is 2×2 matrices of quadratic polynomials. Let $V = \{A \in (P_2)^{2\times2} \mid \text{trace}(A) = 0,\ A = A^T\}$. Find isomorphism to $W = \{X \in \mathbb{R}^{3\times3} \mid X^T = X\}$.

$$A \in V \Rightarrow A = \begin{bmatrix} f(x) & g(x) \\ g(x) & -f(x) \end{bmatrix} = \left[\begin{array}{c|c} ax^2+bx+c & dx^2+ex+f \\ \hline dx^2+ex+f & -ax^2-bx-c \end{array}\right]$$

Natural choice,

$$\boxed{\Psi\left[\begin{array}{c|c} ax^2+bx+c & dx^2+ex+f \\ \hline dx^2+ex+f & -ax^2-bx-c \end{array}\right] = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}}$$

It remains to show Ψ is an isomorphism, but I only asked you <u>find</u> it.

$\boxed{P100}$ Find an isomorphism from 3x3 antisymmetric matrices to 4x4 traceless diagonal matrices.

$$V = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$$

$$a = e = i = 0$$
$$b = -d, \quad c = -g, \quad h = -f$$

$$A = \begin{bmatrix} 0 & d & g \\ -d & 0 & h \\ -g & -h & 0 \end{bmatrix}$$

$$V = \text{span}\underbrace{\left\{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right\}}_{\beta = \text{BASIS for } V}$$

$$W = \{\underbrace{\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{bmatrix}}_{X} \mid \underbrace{a+b+c+d=0}_{d = -a-b-c}\}$$

$$X = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & -a-b-c \end{bmatrix}$$

$$W = \text{span}\underbrace{\left\{\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\right\}}_{\gamma = \text{BASIS FOR } W}$$

Define $\psi : \beta \to \gamma$ by:

$$\psi\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}$$

$$\psi\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}$$

$$\psi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

$$\boxed{\psi\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -x-y-z \end{bmatrix}}$$

Then extend linearly to obtain isomorphism $\psi : V \to W$.