Same rules as Homework 1.

Problem 71 Your signature below indicates you have:

(a.) I read what Cook has posted of Chapters 3 and 4 of the Lecture Notes: ____________.

(b.) I read parts of Chapter 3 of Curtis: ____________________.

Problem 72 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $T(x, y, z) = (x + y, x - y + z, x, x + y - 2z)$ for all $(x, y, z) \in \mathbb{R}^3$. Find $[T]$ and discuss if T is injective, surjective or neither. Also, find the rank and nullity of T.

Problem 73 Let $T : \mathbb{R}^n \to \mathbb{R}$ be defined by $T(x) = x_1 + 2x_2 + \cdots + nx_n$ for all $x \in \mathbb{R}^n$. Find $[T]$ and discuss if T is injective, surjective or neither. Also, find the rank and nullity of T.

Problem 74 Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be defined by its values on the standard basis; $T(e_j) = e_j + e_{j+1}$ for $j = 1, \dots, n-1$ and $T(e_n) = e_n$. Find $[T]$ (use the dot-dot-dot type notation) and calculate the rank and nullity of T. Is T injective, surjective or neither? Find the formula for T^{-1} if possible.

Problem 75 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = f''(x) + 2f(x)$. Find a basis for $\text{Ker}(T)$ and find a basis for $\text{Range}(T)$. Finally, find a bases β, γ for $P_2(\mathbb{R})$ for which $[T]_{\beta\gamma}$ is formed by concantenating the standard basis and columns of zero(s). (this illustrates Theorem 4.6.2). Note: the empty set $\emptyset$ is the basis for the zero space.

Problem 76 Consider $W = \text{span}\{e^x, \cos(x), \sin(x)\}$ and define $T, S \in L(W)$ as follows $T = D - 1$ and $S = D^2 + 1$ where $D = d/dx$ the the product denoted the composition of operators. For example, $(D^2 + 1)[y] = D^2[y] + y = D[D[y]] + y = y'' + y$. Let $\beta = \{e^x, \cos(x), \sin(x)\}$.

(a) find $[T]_{\beta\beta}$

(b) find $[S]_{\beta\beta}$

(c) Calculate $T \circ S$ (set $f(x) = ae^x + b\cos x + c\sin x$ and simplify $(T \circ S)(f(x))$)

(d) Calculate $[T]_{\beta\beta}[S]_{\beta\beta}$

Problem 77 Let $W = \{f(x) \in P_5(\mathbb{R}) \mid P(1) = 0,\ P'(1) = 0,\ \&\ P(2) = 0\}$. Find a basis β for W and write the coordinate map Φ_β for W. *Hint: I'll try to use Taylor's Theorem and the factor theorem, but, it might just be ugly.*

Problem 78 Consider the set of quadratic forms in three variables x, y, z. Let $\gamma = \{x^2, y^2, z^2, xy, xz, yz\}$ and define the set of trivariate homogeneous polynomials of order two by

$$W = \text{span}\{x^2, y^2, z^2, xy, xz, yz\}.$$

Observe W is a function space and as it is a span we find $W \le \mathcal{F}(\mathbb{R}^3, \mathbb{R})$.
If $v = 3x^2 + (x - y)(y + z)$ then calculate $[v]_\gamma$.

Problem 79 Suppose $T : P_2 \to T(P_2) \le \mathbb{R}[t]$ be defined by $T(f(x)) = \int_0^t f(x)\,dx$. Let P_2 have basis $\beta = \{3x^2, 2x, 1\}$ and find a basis γ for $\text{Range}(T)$. Finally, calculate $[T]_{\beta,\gamma}$

Problem 80 Let $C = \begin{bmatrix} 8 & 6 \\ 7 & -5 \end{bmatrix}$ and define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(X) = CX$. Suppose $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Calculate $[T]_{\beta,\beta}$.

Problem 81 Suppose $P_2(\mathbb{R})$ has basis β and $f(x) \in P_2$ has $[f(x)]_\beta = (3, 3, 12)$. If $\gamma = \{g_1, g_2, g_3\}$ is a basis for which $[g_1]_\beta = (1, 1, 1)$ and $[g_2]_\beta = (0, 1, 1)$ and $[g_3]_\beta = (0, 0, 1)$ then find $[f(x)]_\gamma$.

Problem 82 Let $\beta = \{(t-2)^2, (t-2), 1\}$ form the basis for $P_2 \leq \mathbb{R}[t]$. Suppose that $f(t) = at^2 + bt + c$. Calculate the coordinates of $f(t)$ with respect to β; that is, find $[f(t)]_\beta$.

Problem 83 Let $v = (a, b)$ and find the coordinates of v in the $\beta = \{(1, 2), (3, -1)\}$ basis.

Problem 84 Define $T(f(x)) = f(x) + f'(x) + f''(x)$ for each $f(x) \in P_2$. If possible, find a basis β for $P_2 = \text{span}\{1, x, x^2\}$ for which $[T]_{\beta,\beta} = I$. (or show why it can't be done)

Problem 85 Let S be a set of objects then $S^{m\times n}$ is the set of $m \times n$ matrices of objects in S. For example, if $S = P_2$ then $S^{2\times 2}$ is the set of 2×2 matrices with quadratic polynomial components. Let $V = \{A \in (P_2)^{2\times 2} \mid \text{trace}(A) = 0 \ \& \ A = A^T\}$ find an isomorphism from V to $W = \{X \in \mathbb{R}^{3\times 3} \mid X^T = X\}$.

Problem 86 Find an isomorphism from the set V of 3×3 antisymmetric matrices to the set W of 4×4 traceless diagonal matrices.

Problem 87 Curtis §13 #2 on page 107 (this is likely easier than it looks)

Problem 88 Curtis §13 #7 on page 108 (definitions, logic, math glorious math)

Problem 89 Curtis §13 #9 on page 108 (definitions, logic, math glorious math)

Problem 90 We say $\text{tr}(M)$ is the **trace** of M and define $\text{tr}(M) = \sum_{i=1}^{n} M_{ii}$ for each $M \in \mathbb{F}^{n\times n}$. Show that $\text{tr} \in L(\mathbb{F}^{n\times n}, \mathbb{F})$ and prove the fascinating identity $\text{tr}(AB) = \text{tr}(BA)$ for multipliable matrices $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}$.