Same rules as Homework 1.

Problem 71 Your signature below indicates you have:

(a.) I read what Cook has posted of Chapters 3 and 4 of the Lecture Notes: ____________.

(b.) I read parts of Chapter 3 of Curtis: ____________________.

Problem 72 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by $T(x, y, z) = (x + y, x - y + z, x, x + y - 2z)$ for all $(x, y, z) \in \mathbb{R}^3$. Find $[T]$ and discuss if T is injective, surjective or neither. Also, find the rank and nullity of T.

Problem 73 Let $T : \mathbb{R}^n \to \mathbb{R}$ be defined by $T(x) = x_1 + 2x_2 + \cdots + nx_n$ for all $x \in \mathbb{R}^n$. Find $[T]$ and discuss if T is injective, surjective or neither. Also, find the rank and nullity of T.

Problem 74 Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be defined by its values on the standard basis; $T(e_j) = e_j + e_{j+1}$ for $j = 1, \dots, n-1$ and $T(e_n) = e_n$. Find $[T]$ (use the dot-dot-dot type notation) and calculate the rank and nullity of T. Is T injective, surjective or neither? Find the formula for T^{-1} if possible.

Problem 75 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = f''(x) + 2f(x)$. Find a basis for $\text{Ker}(T)$ and find a basis for $\text{Range}(T)$. Finally, find a bases β, γ for $P_2(\mathbb{R})$ for which $[T]_{\beta\gamma}$ is formed by concantenating the standard basis and columns of zero(s). (this illustrates Theorem 4.6.2). Note: the empty set $\emptyset$ is the basis for the zero space.

Problem 76 Consider $W = \text{span}\{e^x, \cos(x), \sin(x)\}$ and define $T, S \in L(W)$ as follows $T = D - 1$ and $S = D^2 + 1$ where $D = d/dx$ the the product denoted the composition of operators. For example, $(D^2 + 1)[y] = D^2[y] + y = D[D[y]] + y = y'' + y$. Let $\beta = \{e^x, \cos(x), \sin(x)\}$.

(a) find $[T]_{\beta\beta}$

(b) find $[S]_{\beta\beta}$

(c) Calculate $T \circ S$ (set $f(x) = ae^x + b\cos x + c\sin x$ and simplify $(T \circ S)(f(x))$)

(d) Calculate $[T]_{\beta\beta}[S]_{\beta\beta}$

Problem 77 Let $W = \{P(x) \in P_5(\mathbb{R}) \mid P(1) = 0,\ P'(1) = 0,\ \&\ P(2) = 0\}$. Find a basis β for W and write the coordinate map Φ_β for W. *Hint: I'll try to use Taylor's Theorem and the factor theorem, but, it might just be ugly.*

Problem 78 Consider the set of quadratic forms in three variables x, y, z. Let $\gamma = \{x^2, y^2, z^2, xy, xz, yz\}$ and define the set of trivariate homogeneous polynomials of order two by

$$W = \text{span}\{x^2, y^2, z^2, xy, xz, yz\}.$$

Observe W is a function space and as it is a span we find $W \leq \mathcal{F}(\mathbb{R}^3, \mathbb{R})$.
If $v = 3x^2 + (x - y)(y + z)$ then calculate $[v]_\gamma$.

Problem 79 Suppose $T : P_2 \to T(P_2) \leq \mathbb{R}[t]$ be defined by $T(f(x)) = \int_0^t f(x)\, dx$. Let P_2 have basis $\beta = \{3x^2, 2x, 1\}$ and find a basis γ for $\text{Range}(T)$. Finally, calculate $[T]_{\beta,\gamma}$

Problem 80 Let $C = \begin{bmatrix} 8 & 6 \\ 7 & -5 \end{bmatrix}$ and define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(X) = CX$. Suppose $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Calculate $[T]_{\beta,\beta}$.

Problem 81 Suppose $P_2(\mathbb{R})$ has basis β and $f(x) \in P_2$ has $[f(x)]_\beta = (3, 3, 12)$. If $\gamma = \{g_1, g_2, g_3\}$ is a basis for which $[g_1]_\beta = (1, 1, 1)$ and $[g_2]_\beta = (0, 1, 1)$ and $[g_3]_\beta = (0, 0, 1)$ then find $[f(x)]_\gamma$.

Problem 82 Let $\beta = \{(t-2)^2, (t-2), 1\}$ form the basis for $P_2 \leq \mathbb{R}[t]$. Suppose that $f(t) = at^2 + bt + c$. Calculate the coordinates of $f(t)$ with respect to β; that is, find $[f(t)]_\beta$.

Problem 83 Let $v = (a, b)$ and find the coordinates of v in the $\beta = \{(1, 2), (3, -1)\}$ basis.

Problem 84 Define $T(f(x)) = f(x) + f'(x) + f''(x)$ for each $f(x) \in P_2$. If possible, find a basis β for $P_2 = \text{span}\{1, x, x^2\}$ for which $[T]_{\beta,\beta} = I$. (or show why it can't be done)

Problem 85 Let S be a set of objects then $S^{m\times n}$ is the set of $m \times n$ matrices of objects in S. For example, if $S = P_2$ then $S^{2\times 2}$ is the set of 2×2 matrices with quadratic polynomial components. Let $V = \{A \in (P_2)^{2\times 2} \mid \text{trace}(A) = 0 \;\&\; A = A^T\}$ find an isomorphism from V to $W = \{X \in \mathbb{R}^{3\times 3} \mid X^T = X\}$.

Problem 86 Find an isomorphism from the set V of 3×3 antisymmetric matrices to the set W of 4×4 traceless diagonal matrices.

Problem 87 Curtis §13 #2 on page 107 (this is likely easier than it looks)

Problem 88 Curtis §13 #7 on page 108 (definitions, logic, math glorious math)

Problem 89 Curtis §13 #9 on page 108 (definitions, logic, math glorious math)

Problem 90 We say $\text{tr}(M)$ is the **trace** of M and define $\text{tr}(M) = \sum_{i=1}^{n} M_{ii}$ for each $M \in \mathbb{F}^{n\times n}$. Show that $\text{tr} \in L(\mathbb{F}^{n\times n}, \mathbb{F})$ and prove the fascinating identity $\text{tr}(AB) = \text{tr}(BA)$ for multipliable matrices $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}$.

$\boxed{P72}$ $T(x,y,z) = \begin{bmatrix} x+y \\ x-y+z \\ x \\ x+y-2z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\therefore$ $\boxed{[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}}$.

Observe, $[T] \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}[T]$

We find $\text{rank}([T]) = 3 \Rightarrow \text{nullity}([T]) = 0$

Thus, $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ is <u>injective</u>, but not <u>surjective</u> $(T(\mathbb{R}^3) \neq \mathbb{R}^4)$

Recall, T is 1-1 $\iff \text{Ker}\, T = \{0\} \iff \text{Null}[T] = \{0\} \iff \text{nullity}[T] = 0$.

$\boxed{P73}$ $T(x) = x_1 + 2x_2 + \cdots + nx_n \quad \forall x \in \mathbb{R}^n$ thus $T : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Note, $T(e_j) = j \quad \therefore \quad \boxed{[T] = [1, 2, 3, \dots, n]}$. We find

$\text{Range}(T) = \text{span}\{1\} = \mathbb{R}$ thus <u>T is surjective</u>; $T(\mathbb{R}^n) = \mathbb{R}$.

However, $n = \text{rank}(T) + \text{nullity}(T) \Rightarrow \dim(\text{Ker}(T)) = n-1 \neq 0$

$\therefore$ <u>T is <u>NOT</u> injective</u>.

To recap, $\boxed{\text{rank}(T) = 1}$, $\boxed{\text{nullity}(T) = n-1}$

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I didn't ask for it, but, it's fun to find basis for Null(T)

$$T(x) = 0 \iff x_1 = -2x_2 - 3x_3 - \cdots - nx_n$$

$$\iff x = (-2x_2 - 3x_3, \; x_2, x_3, \dots, x_n)$$

$$\iff x = x_2(-2e_1 + e_2) + x_3(-3e_1 + e_3) + \cdots + x_n(-ne_1 + e_n)$$

Thus <u>$\text{Null}([T]) = \text{Ker}\, T = \text{span}\{-je_1 + e_j\}_{j=2}^{n}$</u>.

P74

$$T(e_j) = e_j + e_{j+1} \quad \text{for } j = 1, 2, \dots, n-1$$
$$T(e_n) = e_n$$

We then extend to $T : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ by the values given above.

$$[T] = [T(e_1) \mid T(e_2) \cdots \mid T(e_n)] = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \leftarrow \begin{array}{l} 1\text{'s down diagonal} \\ 1\text{'s down sub-diagonal} \end{array}$$

Notice $x \in \operatorname{Ker}(T) \Rightarrow T(x) = 0$

$$\Rightarrow (x_1, x_1 + x_2, x_2 + x_3, \dots, x_{n-2} + x_{n-1}, x_{n-1} + x_n) = 0$$
$$\Rightarrow x_1 = 0, \ x_1 + x_2 = 0, \ x_2 + x_3 = 0, \dots, x_{n-1} + x_n = 0$$
$$\Rightarrow x_1 = 0, \ x_2 = 0, \ x_3 = 0, \dots, x_n = 0$$
$$\Rightarrow x = 0$$
$$\Rightarrow \operatorname{Ker}(T) = \{0\}$$
$$\Rightarrow \operatorname{Range}(T) = \mathbb{F}^n \ \& \ T \text{ is linear bijection}$$
$$\Rightarrow \boxed{\operatorname{rank}(T) = n} \text{ and } \boxed{\operatorname{nullity}(T) = 0}$$

We find T is <u>injective</u> and <u>surjective</u>. We can find a formula for $T^{-1}(y)$ by solving $T(x) = y$ for x.

$$\begin{array}{lll} x_1 = y_1 & \Rightarrow & x_1 = y_1 \\ x_1 + x_2 = y_2 & \Rightarrow & x_2 = y_2 - x_1 = y_2 - y_1 \\ x_2 + x_3 = y_3 & \Rightarrow & x_3 = y_3 - x_2 = y_3 - (y_2 - y_1) = y_3 - y_2 + y_1 \\ \vdots & & \vdots \\ x_{n-1} + x_n = y_n & \Rightarrow & x_n = y_n - x_{n-1} = y_n - (y_{n-1} + y_{n-2} + y_{n-3} - \cdots) \end{array}$$

Thus,

$$\boxed{T^{-1}(y) = (y_1, \ y_2 - y_1, \ y_3 - y_2 + y_1, \ y_4 - y_3 - y_2 + y_1, \dots, y_n - y_{n-1} + \cdots \pm y_1)}$$

+ if n odd
− if n even

Or, we could say $T^{-1}(y) = [T]^{-1} y$ where,

$$[T]^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \pm 1 & \cdots & 1 & -1 & 1 \end{bmatrix}$$

$\boxed{P75}$ $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by $T(f(x)) = f''(x) + 2f(x)$.

$$\begin{aligned} T(ax^2+bx+c) &= (ax^2+bx+c)'' + 2(ax^2+bx+c) \\ &= 2a + 2ax^2 + 2bx + 2c \\ &= 2a+2c + 2bx + 2ax^2 \\ &= 2ax^2 + 2bx + 2a+2c \\ &= a(2x^2+2) + b(2x) + c(2) \end{aligned}$$

(keeping with my ordering $x^2, x, 1$ from outset)

Thus, $\text{Range}(T) = \text{span}\{2x^2+2, 2x, 2\}$ ...

note $\underline{\gamma = \{2x^2+2, 2x, 2\} \text{ serves as basis for } \text{Range}(T)}$.

As $\dim(\text{Ker}(T)) + \dim(\text{Range}(T)) = \dim(\text{dom}(T)) = \dim(P_2(\mathbb{R})) = 3$

we find $\underline{\text{Ker}(T) = \{0\} \text{ thus } \emptyset \text{ is its basis}}$.

Sorry, kind of boring, $\beta = \{x^2, x, 1\}$ and thus,

$$\begin{aligned} [T]_{\beta\gamma} &= \left[[T(x^2)]_\gamma \,\middle|\, [T(x)]_\gamma \,\middle|\, [T(1)]_\gamma \right] \\ &= \left[[2+2x^2]_\gamma \,\middle|\, [2x]_\gamma \,\middle|\, [2]_\gamma \right] \\ &= \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}. \end{aligned}$$

See Remark below for a different possible outcome to this sort of problem

Remark: $\tilde{T}(f(x)) = f''(x)$ then $\tilde{T}(ax^2+bx+c) = 2a$

$\therefore$ $\text{Range}(\tilde{T}) = \text{span}\{2\}$ and $\text{Ker}(\tilde{T}) = \text{span}\{x, 1\}$

Let $\beta = \{x^2, x, 1\}$ and $\gamma = \{2, x, x^2\}$ $(x, x^2 \notin \text{Range}(\tilde{T}))$

then
$$\begin{aligned} [\tilde{T}]_{\beta\gamma} &= \left[[\tilde{T}(x^2)]_\gamma \,\middle|\, [\tilde{T}(x)]_\gamma \,\middle|\, [\tilde{T}(1)]_\gamma \right] \\ &= \left[[2]_\gamma \,\middle|\, [0]_\gamma \,\middle|\, [0]_\gamma \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$\boxed{P76}$ $\beta = \{e^x, \cos x, \sin x\}$ and $W = \operatorname{span}\beta$.

Let $T, S : W \longrightarrow W$ be defined in terms of $D = d/dx \in L(W)$,

$T = D - 1$, $S = D^2 + 1$

(a.) $$[T]_{\beta\beta} = \left[[T(e^x)]_\beta \,\middle|\, [T(\cos(x))]_\beta \,\middle|\, [T(\sin(x))]_\beta \right]$$

$$= \left[[0]_\beta \,\middle|\, [-\sin(x) - \cos x]_\beta \,\middle|\, [\cos x - \sin x]_\beta \right]$$

$$= \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = [T]_{\beta\beta}}$$

(b.) $$S(\underbrace{ae^x + b\cos x + c\sin(x)}_{f(x)}) = \frac{d^2 f}{dx^2} + f(x)$$

$$= ae^x - b\cos x - c\sin x + ae^x + b\cos x + c\sin x$$

$$= 2ae^x$$

$$\begin{tikzcd}
f(x) \arrow[r, "S"] \arrow[d, "\Phi_\beta"'] & S(f(x)) = 2ae^x \arrow[d] \\
{[f(x)]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \arrow[r, "L_{[S]_{\beta\beta}}"] & {[S(f(x))]_\beta = \begin{bmatrix} 2a \\ 0 \\ 0 \end{bmatrix}}
\end{tikzcd}$$

$$\boxed{[S]_{\beta\beta} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

(c.) $$(T \circ S)(ae^x + b\cos x + c\sin x) = T(S(ae^x + b\cos x + c\sin x))$$

$$= T(2ae^x)$$

$$= D(2ae^x) - 2ae^x$$

$$= 2ae^x - 2ae^x$$

$$= 0$$

$\therefore \boxed{T \circ S = 0}$

(d.) $$[T]_{\beta\beta}[S]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = [T \circ S]_{\beta\beta}$$

P77 $W = \{ P(x) \in P_5(\mathbb{R}) \mid P(1) = P'(1) = 0 \ \& \ P(2) = 0 \}$

Find basis β for W and formula for Φ_β

Let $P(x) \in W$.

Observe, $P(2) = 0 \implies P(x) = (x-2)\, g_1(x)$

Observe, $P(1) = P'(1) = 0 \implies P(x) = (x-1)^2\, g_2(x)$

Combining these, $P(x) = (x-1)^2 (x-2)\left(Ax^2 + Bx + C\right)$

then, multiplying,

$$P(x) = Ax^2 (x-1)^2 (x-2) + Bx(x-1)^2(x-2) + C(x-1)^2(x-2)$$

$$\beta = \{ x^2(x-1)^2(x-2),\ x(x-1)^2(x-2),\ (x-1)^2(x-2) \}$$

Or, in more standard notation,
$$\begin{aligned}(x-1)^2(x-2) &= (x^2-2x+1)(x-2) \\ &= x^3 - 2x^2 + x - 2x^2 + 4x - 2 \\ &= \underline{x^3 - 4x^2 + 5x - 2}.\end{aligned}$$

$$\boxed{\beta = \{ x^5 - 4x^4 + 5x^3 - 2x^2,\ x^4 - 4x^3 + 5x^2 - 2x,\ x^3 - 4x^2 + 5x - 2 \}}$$

this server as a basis for W. Consider,

$$\begin{aligned} V = ax^5 + bx^4 + cx^3 + dx^2 + ex + f = \ & c_1(x^5 - 4x^4 + 5x^3 - 2x^2) \\ &+ c_2(x^4 - 4x^3 + 5x^2 - 2x) \\ &+ c_3(x^3 - 4x^2 + 5x - 2) \end{aligned}$$

$$= c_1 x^5 + (-4c_1 + c_2)x^4 + (5c_1 - 4c_2 + c_3)x^3 + (-2c_1 + 5c_2 - 4c_3)x^2 + (-2c_2 + 5c_3)x + (-2c_3)\cdot 1$$

We must solve,

$a = c_1$, $b = -4c_1 + c_2$, $c = 5c_1 - 4c_2 + c_3$, $d = -2c_1 + 5c_2 - 4c_3$
$e = -2c_2 + 5c_3$
$f = -2c_3$

There are many ways to solve these. I prefer,

$\underline{c_1 = a,\ c_2 = 4a + b,\ c_3 = -f/2}$ $\quad\therefore\ \underline{[V]_\beta = (a,\ 4a+b,\ -f/2)}$

PROBLEM 77 continued

Given my 1^{st} choice of β,

$$\Phi_\beta(\underbrace{ax^5+bx^4+cx^3+dx^2+ex+f}_{v}) = \begin{bmatrix} a \\ 4a+b \\ -f/2 \end{bmatrix}.$$

But, even so, we could also write

$$c_3 = -f/2, \quad c_2 = \tfrac{1}{2}(-e+5c_3) = \tfrac{1}{2}\left(-e+5\left(\tfrac{-f}{2}\right)\right) = \tfrac{-e}{2} - \tfrac{5f}{4}$$

$$c_1 = a.$$

Hence $$[v]_\beta = \begin{bmatrix} a \\ -\frac{1}{2}e - \frac{5}{4}f \\ -f/2 \end{bmatrix}.$$

Sorry to grader: $\exists$ ∞ly many correct answers!

The issue is a, b, c, d, e, f are not independent. We are given $P(x) = ax^5+bx^4+cx^3+dx^2+ex+f \in W$ has

$$P(1) = a+b+c+d+e+f = 0$$
$$P'(1) = 5a+4b+3c+2d+e = 0$$
$$P(2) = 32a+16b+8c+4d+2e+f = 0$$

So we can rewrite some of $a, b, c, \ldots, f$ in terms of the other variables in $a, b, c, \ldots, f$.

$$\text{rref}\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 & 0 \\ 32 & 16 & 8 & 4 & 2 & 1 & 0 \end{array}\right] = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1/2 & 5/4 & 17/8 \\ 0 & 1 & 0 & -2 & -9/2 & -29/4 \\ 0 & 0 & 1 & 5/2 & 17/4 & 49/8 \end{bmatrix} \end{array} \quad (*)$$

Thus,
$$a = -d/2 - (5/4)e - \left(\tfrac{17}{8}\right)f$$
$$b = 2d + (9/2)e + (29/4)f$$
$$c = \left(-\tfrac{5}{2}\right)d + \left(\tfrac{17}{4}\right)e + \left(\tfrac{49}{8}\right)f$$

$\boxed{P77}$ nicest answer, we've shown

$$W = \text{span}\{x^5 - 4x^4 + 5x^3 - 2x^2,\ x^4 - 4x^3 + 5x^2 - 2x,\ x^3 - 4x^2 + 5x - 2\}$$

Hence $f(x) \in W$ has form:

$$f(x) = c_1(x^5 - 4x^4 + 5x^3 - 2x^2) + c_2(x^4 - 4x^3 + 5x^2 - 2x) + c_3(x^3 - 4x^2 + 5x - 2)$$

Thus $\underline{[f(x)]_\beta = (c_1, c_2, c_3)}$.

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Finally, many of you likely solved as in (*) of last page then

$$\begin{aligned} ax^5 + bx^4 + cx^3 + dx^2 + ex + f = &\left(\tfrac{-d}{2} - \tfrac{5}{4}e - \tfrac{17}{8}f\right)x^5 \\ &+ \left(2d + \tfrac{9}{2}e + \tfrac{29}{4}f\right)x^4 \\ &+ \left(-\tfrac{5}{2}d + \tfrac{17}{4}e + \tfrac{49}{8}f\right)x^3 \\ &+ dx^2 + ex + f \\ \hline = &\ d\left(x^2 - \tfrac{1}{2}x^5 + 2x^4 - \tfrac{5}{2}x^3\right) \\ &+ e\left(x - \tfrac{5}{4}x^5 + \tfrac{9}{2}x^4 + \tfrac{17}{4}x^3\right) \\ &+ f\left(1 - \tfrac{17}{8}x^5 + \tfrac{29}{4}x^4 + \tfrac{49}{8}x^3\right) \end{aligned}$$

Thus $\tilde{\beta} = \{x^2 - \frac{1}{2}x^5 + 2x^4 - \frac{5}{2}x^3,\ x - \frac{5}{4}x^5 + \frac{9}{2}x^4 + \frac{17}{4}x^3,\ 1 - \frac{17}{8}x^5 + \frac{29}{4}x^4 + \frac{49}{8}x^3\}$

Then

$$\underline{[ax^5 + bx^4 + cx^3 + dx^2 + ex + f]_{\tilde{\beta}} = (d, e, f)}.$$

$\boxed{P78}$ $W = \text{span}\,\gamma$, $\gamma = \{x^2, y^2, z^2, xy, xz, yz\}$

$$V = 3x^2 + (x-y)(y+z) = 3x^2 + xy - y^2 + xz - yz$$

$$V = 3x^2 - y^2 + (0)z^2 + xy + xz - yz$$

$$\therefore \boxed{[V]_\gamma = (3, -1, 0, 1, 1, -1)}$$

$\boxed{P79}$ $T : P_2(\mathbb{R}) \longrightarrow T(P_2(\mathbb{R})) \leq \mathbb{R}[t]$

defined by $T(f(x)) = \int_0^t f(x)\,dx$. Let $\beta = \{3x^2, 2x, 1\}$ be basis for $P_2(\mathbb{R})$ and find basis γ for Range(T). Calc. $[T]_{\beta,\gamma}$

$$T(ax^2+bx+c) = \int_0^t (ax^2 + bx + c)\,dx$$

$$= \left(\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx\right)\Big|_0^t$$

$$= \frac{1}{3}at^3 + \frac{1}{2}bt^2 + ct$$

We find Range$(T) = \text{span}\{t^3, t^2, t\}$. Let $\gamma = \{t^3, t^2, t\}$.

$$[T]_{\beta,\gamma} = \Big[[T(3x^2)]_\gamma \;\Big|\; [T(2x)]_\gamma \;\Big|\; [T(1)]_\gamma \Big]$$

$$= \Big[[t^3]_\gamma \;\Big|\; [t^2]_\gamma \;\Big|\; [t]_\gamma \Big]$$

$$= \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [T]_{\beta,\gamma}}$$

(different choice of γ makes for different $[T]_{\beta,\gamma}$

For examples $\tilde{\gamma} = \{t, t^2, t^3\} \hookrightarrow [T]_{\beta,\tilde{\gamma}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

P80 $C = \begin{bmatrix} 8 & 6 \\ 7 & -5 \end{bmatrix}$ and let $T : \mathbb{R}^{2\times2} \longrightarrow \mathbb{R}^{2\times 2}$ be defined by $T(X) = CX$. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Calc. $[T]_{\beta\beta}$

$$T\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} = \begin{bmatrix} 8 & 6 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[\begin{array}{c|c} 8a+6c & 8b+6d \\ \hline 7a-5c & 7b-5d \end{array}\right]$$

$$\begin{tikzcd}
\boxed{\begin{array}{c}\mathbb{R}^{2\times2} \\ A\end{array}} \arrow[r, "T"] \arrow[d] & \boxed{T(A)} \arrow[d] \\
\boxed{[A]_\beta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}} \arrow[r, "\text{multiply by}", "\begin{bmatrix} 8 & 0 & 6 & 0 \\ 0 & 8 & 0 & 6 \\ 7 & 0 & -5 & 0 \\ 0 & 7 & 0 & -5 \end{bmatrix}"'] & \boxed{[T(A)]_\beta = \begin{bmatrix} 8a + 6c \\ 8b + 6d \\ 7a - 5c \\ 7b - 5d \end{bmatrix}}
\end{tikzcd}$$

$$\therefore\ \underbrace{\boxed{[T]_{\beta\beta} = \left[\begin{array}{cc|cc} 8 & 0 & 6 & 0 \\ 0 & 8 & 0 & 6 \\ \hline 7 & 0 & -5 & 0 \\ 0 & 7 & 0 & -5 \end{array}\right]}}_{\text{interesting pattern.}}$$

P81 $P_2(\mathbb{R})$ has $\beta = \{V_1, V_2, V_3\}$ and $f(x) \in P_2(\mathbb{R})$ has $[f(x)]_\beta = (3, 3, 12)$. If $\gamma = \{g_1, g_2, g_3\}$ has $[g_1]_\beta = (1,1,1)$ & $[g_2]_\beta = (0,1,1)$ & $[g_3]_\beta = (0,0,1)$ Find $[f(x)]_\gamma$

We have $f(x) = 3V_1 + 3V_2 + 12V_3$ from $[f(x)]_\beta = (3, 3, 12)$.

Also,
$$[g_1]_\beta = (1,1,1) \implies g_1 = V_1 + V_2 + V_3$$
$$[g_2]_\beta = (0,1,1) \implies g_2 = V_2 + V_3$$
$$[g_3]_\beta = (0,0,1) \implies g_3 = V_3$$

We wish to express $f(x)$ in terms of g_1, g_2, g_3,
$$V_3 = g_3$$
$$V_2 = g_2 - V_3 = g_2 - g_3$$
$$V_1 = g_1 - V_2 - V_3 = g_1 - (g_2 - \cancel{g_3}) - \cancel{g_3} = g_1 - g_2$$

Thus
$$f(x) = 3V_1 + 3V_2 + 12V_3$$
$$= 3(g_1 - \cancel{g_2}) + 3(\cancel{g_2} - g_3) + 12g_3$$
$$= 3g_1 + 9g_3$$

$$\therefore \boxed{[f(x)]_\gamma = (3, 0, 9)}$$

P81 Alternatively, once we've found

$$\boxed{\begin{aligned} v_1 &= g_1 - g_2 \\ v_2 &= g_2 - g_3 \\ v_3 &= g_3 \end{aligned}}$$

$$P_{\beta,\gamma} = \left[[v_1]_\gamma \,\middle|\, [v_2]_\gamma \,\middle|\, [v_3]_\gamma \right]$$

$$= \left[[g_1 - g_2]_\gamma \,\middle|\, [g_2 - g_3]_\gamma \,\middle|\, [g_3]_\gamma \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Thus $[f(x)]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}$.

P82 $\beta = \{(t-2)^2, (t-2), 1\}$ form the basis $P_2(\mathbb{R}) \leq \mathbb{R}[t]$
find $[f(t)]_\beta$ for $f(t) = at^2 + bt + c$

$$f(t) = f(2) + f'(t)(t-2) + \tfrac{1}{2} f''(2)(t-2)^2 \quad \text{by Taylor's Th}^{\underline{m}}!$$

$$= 4a + 2b + c + (4a + b)(t-2) + a(t-2)^2$$

$f'(t) = 2at + b$
$f''(t) = 2a$

$$\therefore \boxed{[f(t)]_\beta = (a, 4a+b, 4a+2b+c)}$$

P83 $v = (a, b)$ find $[v]_\beta$ for $\beta = \{(1,2), (3,-1)\}$ $\quad [\beta] = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

$$[v]_\beta = [\beta]^{-1} v = \frac{1}{-7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7}(a+3b) \\ \frac{1}{7}(2a-b) \end{bmatrix}$$

$$= \boxed{\left(\tfrac{1}{7}(a+3b), \tfrac{1}{7}(2a-b) \right)}$$

We can check answer,

$$\tfrac{1}{7}(a+3b)(1,2) + \tfrac{1}{7}(2a-b)(3,-1) \stackrel{\checkmark}{=} (a, b).$$

$\boxed{P\,84}$ $T(f(x)) = f(x) + f'(x) + f''(x)$ for each $f(x) \in P_2(\mathbb{R})$

If possible, find basis for which $[T]_{\beta\beta} = I$ (or show not possible)

If $[T]_{\beta\beta} = I$ and $\beta = \{V_1, V_2, V_3\}$ then $T(V_1) = V_1$

and $T(V_2) = V_2$ and $T(V_3) = V_3$ by def^{n} of $[T]_{\beta\beta}$.

Can we solve $T(f(x)) = f(x)$? Consider,

$$T(f(x)) = f(x)$$

$$f(x) + f'(x) + f''(x) = f(x)$$

$$\Rightarrow f'(x) + f''(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(f(x) + \frac{df}{dx}\right) = 0$$

$$\Rightarrow f(x) + f'(x) = C_0$$

$$\Rightarrow ax^2 + bx + c + 2ax + b = C_0$$

$$\Rightarrow ax^2 + (b+2a)x + c + b - C_0 = 0$$

$$\Rightarrow a = 0, \quad b + 2a = 0, \quad c + b - C_0 = 0$$

$$\Rightarrow a = 0, \quad b = 0, \quad c = C_0.$$

Hence $f(x) = C_0$ is the only sol^{n} to $T(f(x)) = f(x)$

$\therefore \nexists\ \beta$ a basis for $P_2(\mathbb{R})$ such that $[T]_{\beta\beta} = I$.

(this was not expected, this part of the sol$^{\underline{n}}$ is foreshadowing)

[P84 continued]

$$(T - I)V_2 = V_1 = 1$$

I can solve $T(V_2) - V_2 = V_1$ aka $\underline{T(V_2) = V_2 + V_1}$ $*$

$$T(f(x)) - f(x) = 1$$

$$\cancel{f(x)} + f'(x) + f''(x) - \cancel{f(x)} = 1$$

$$f(x) + f'(x) = x + c_1$$

$$ax^2 + bx + c + 2ax + b = x + c_1 \quad \hookrightarrow a = 0$$

$$(b + 2a)x + c + b = x + c_1$$

$$b + 2a = 1 \longrightarrow b = 1$$

$$c + b = c_1 \longrightarrow c + 1 = c_1 \quad \therefore \quad c = c_1 - 1$$

$\underline{\text{choose } c = 0}$.

Thus $V_2 = x$. This has

$$(T - I)(x) = x + 1 - x = 1.$$

Next, I'll solve,

$$(T - I)(V_3) = V_2 \quad \text{aka} \quad \underline{T(V_3) = V_3 + V_2} \; **$$

$$T(V_3) - V_3 = x \quad , V_3 = f(x)$$

$$f'(x) + f''(x) = x$$

$$f(x) + f'(x) = \tfrac{1}{2}x^2 + c_1$$

$$ax^2 + bx + c + 2ax + b = \tfrac{1}{2}x^2 + c_1 \quad \Rightarrow \underline{a = 1/2}.$$

$$b + 2a = 0 \quad \Rightarrow \underline{b = -1}.$$

$$c + b = c_1 \quad \Rightarrow c = c_1 - b = c_1 + 1$$

Thus $V_3 = \frac{1}{2}x^2 - x$ will do. With $\beta = \{1, x, \frac{1}{2}x^2 - x\}$ we have

$$T(1) = 1 \qquad [T(1)]_\beta = (1, 0, 0)$$

$$(*) \quad T(x) = x + 1 \qquad [T(x)]_\beta = (1, 1, 0)$$

$$(**) \quad T(\tfrac{1}{2}x^2 - x) = \tfrac{1}{2}x^2 - x + x \qquad [T(\tfrac{1}{2}x^2 - x)]_\beta = (0, 1, 1)$$

$$\underbrace{[T]_{\beta\beta} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

this is arguably the next best thing to I we can reach for $[T]_{\beta\beta}$

$\boxed{P85}$

$$V = \{ A \in (P_2(\mathbb{R}))^{2\times 2} \mid tr(A) = 0,\ A = A^T \}$$

$$W = \{ X \in \mathbb{R}^{3\times 3} \mid X^T = X \}$$

Find an isomorphism from V to W

<u>Remark</u>: "find" is different than "find and prove"

$$A \in V \Rightarrow A = \begin{bmatrix} f(x) & g(x) \\ h(x) & i(x) \end{bmatrix} = \begin{bmatrix} f(x) & h(x) \\ g(x) & i(x) \end{bmatrix} = A^T$$

$$tr(A) = f(x) + i(x) = 0$$

$$\Rightarrow A = \begin{bmatrix} f(x) & g(x) \\ g(x) & -f(x) \end{bmatrix} \quad \text{for } f(x), g(x) \in P_2(\mathbb{R})$$

$$\Rightarrow A = \left[\begin{array}{c|c} a_0 + a_1 x + a_2 x^2 & b_0 + b_1 x + b_2 x^2 \\ \hline b_0 + b_1 x + b_2 x^2 & -a_0 - a_1 x - a_2 x^2 \end{array}\right]$$

But, $W = \{ X \in \mathbb{R}^{3\times 3} \mid X^T = X \}$ likewise has $X = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ by similar examination of $X^T = X$. We construct,

$$\boxed{\Psi \left(\left[\begin{array}{c|c} a_0 + a_1 x + a_2 x^2 & b_0 + b_1 x + b_2 x^2 \\ \hline b_0 + b_1 x + b_2 x^2 & -a_0 - a_1 x - a_2 x^2 \end{array}\right] \right) = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & b_0 & b_1 \\ a_2 & b_1 & b_2 \end{bmatrix}}$$

To prove Ψ is isomorphism we'd need to give some supporting comments/arguments that Ψ is a linear bijection.

PROBLEM 86

Find an isomorphism from 3x3 antisymmetric matrices to 4x4 traceless diagonal matrices.

$$V = \{A \in \mathbb{R}^{3\times3} \mid A^T = -A\}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$$

$a = e = i = 0$

$b = -d,\ c = -g,\ h = -f$

$$A = \begin{bmatrix} 0 & d & g \\ -d & 0 & h \\ -g & -h & 0 \end{bmatrix}$$

$$V = \text{span}\underbrace{\left\{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right\}}_{\beta = \text{BASIS for } V}$$

$$W = \Big\{ \underbrace{\begin{bmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & d \end{bmatrix}}_{X} \Big/ \underbrace{a+b+c+d=0}_{d = -a-b-c} \Big\}$$

$$X = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & -a-b-c \end{bmatrix}$$

$$W = \text{span}\underbrace{\left\{\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}, \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\right\}}_{\gamma = \text{BASIS FOR } W}$$

Th$^{\underline{m}}$/ If $\#\beta = \#\gamma = n$ and β, γ bases for V, W respective then $\beta = \{v_i\}$, $\gamma = \{w_i\}$

$\Psi(v_i) = w_i$

for $i = 1, 2, \dots, n$ extended linearly defines isomorphism from V to W

Define $\psi : \beta \to \gamma$ by:

$$\psi\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}$$

$$\psi\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{bmatrix}$$

$$\psi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

$$\boxed{\psi\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -x-y-z \end{bmatrix}}$$

Then extend linearly to obtain isomorphism $\psi : V \to W$.

$\boxed{P87}$ §13 #2 pg. 107. Given that:

$S(u_1) = u_1 + u_2$ $\quad T(u_1) = u_1 - u_2$ $\quad U(u_1) = u_1 + u_2 - u_3$

$S(u_2) = -u_1 - u_2$ $\quad T(u_2) = 2u_2$ $\quad U(u_2) = u_2 - 3u_3$

$U(u_3) = -u_1 - 3u_2 - 2u_3$

$\beta = \{u_1, u_2\}$ basis for V $\quad \beta = \{u_1, u_2\}$ basis of V $\quad \gamma = \{u_1, u_2, u_3\}$ basis for W

$S : V \longrightarrow V$ $\quad T : V \longrightarrow V$ $\quad U : W \longrightarrow W$

<u>Remark</u>: reading ahead I could have saved work by doing (c.) and (d.) first!

(a.) $[S]_{\beta\beta} = \left[[S(u_1)]_\beta \,\middle|\, [S(u_2)]_\beta \right] = \left[[u_1+u_2]_\beta \,\middle|\, [-u_1-u_2]_\beta \right] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

clearly $\text{col}_2 [S]_{\beta\beta} = -\text{col}_1 [S]_{\beta\beta} \Rightarrow \text{rank}[S]_{\beta\beta} = 1$

and $\therefore$ $2 = \text{rank}([S]_{\beta\beta}) + \text{nullity}([S]_{\beta\beta}) \Rightarrow \boxed{\begin{array}{c} \nu(S) = 1 \\ \hline \text{rank}(S) = 1 \end{array}}$

//

or we can calculate directly,

$T(xu_1 + yu_2) = 0$

$\Rightarrow xT(u_1) + yT(u_2) = 0$

$\Rightarrow x(u_1 - u_2) + y(2u_2) = 0$

$\Rightarrow xu_1 + (2y - x)u_2 = 0$

$\Rightarrow x = 0, \; 2y - x = 0$ by LI of $\{u_1, u_2\}$

$\Rightarrow x = 0, \; y = 0 \;\therefore\; \text{Ker}(T) = 0$

Hence $\nu(T) = \boxed{\text{nullity}(T) = 0}$ and $\therefore$ $\nu(T) + \text{rank}(T) = 2$ we find $\boxed{\text{rank}(T) = 2}$

//

$$[U]_{\gamma\gamma} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & -3 & -2 \end{bmatrix} \xrightarrow[r_3 + r_1]{r_2 - r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{r_3 + 3r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & -9 \end{bmatrix}$$

By now it is clear $\text{rank}[U]_{\gamma\gamma} = 3$ thus,

$\boxed{\text{rank}(U) = 3}$ and $\boxed{\text{nullity}(U) = 0}$

P 87 continued

(b.) both T and U are bijections hence T^{-1} and U^{-1} exist. (he did not ask us to find T^{-1} & U^{-1})

(c.) Find bases for ~~null spaces~~ $Ker(S)$, $Ker(T)$, $Ker(U)$

$$S(xu_1 + yu_2) = xS(u_1) + yS(u_2)$$
$$= x(u_1 + u_2) + y(-u_1 - u_2)$$
$$= (x-y)u_1 + (x-y)u_2 \qquad (*)$$

Hence $xu_1 + yu_2 \in Ker(S) \Rightarrow (x-y)u_1 + (x-y)u_2 = 0$ thus $x - y = 0$ by LI of $\{u_1, u_2\}$. Thus, $x = y$ and $Ker(S) = span\{u_1 + u_2\}$. That is, $\boxed{\{u_1 + u_2\} \text{ basis for } Ker(S)}$

In (a.) I showed nullity of T and U were zero it follows $Ker(T) = 0$ and $Ker\, U = 0$ thus $\boxed{\emptyset \text{ is basis for } Ker\, T \text{ and } Ker\, U}$

(d.) Find basis for ranges of S, T, U.

Note, by $*$ we have $\boxed{Range(S) = span\{u_1 + u_2\}}$ thus $\boxed{\{u_1 + u_2\} \text{ is basis for } Range(S)}$

Since T, U are surjections to $V = span\{u_1, u_2\}$ and $W = span\{u_1, u_2, u_3\}$ respective we have

Range (T)	has basis	$\{u_1, u_2\}$
Range (U)	has basis	$\{u_1, u_2, u_3\}$

Remark: for Range (T), Range (U) $\exists$ many other choices for basis. In contrast Range (S) and $Ker(S)$ we have not so much freedom (could use $\{k(u_1+u_2)\}$ for $k \neq 0$, but that's it.)

$\boxed{P88}$ §13 # 7 pg. 108 (CURTIS HAS $n(T) = \mathrm{Ker}\, T$)

Let $T \in L(V)$. Prove $T^2 = 0$ iff $T(V) \subseteq \mathrm{Ker}(T)$

$\Rightarrow$] Suppose $\overbrace{T : V \to V}^{\text{linear}}$ and $T^2 = 0$. Let $y \in T(V)$ then $\exists\, x \in V$ s.t. $T(x) = y$. Therefore, $T(y) = T(T(x)) = T^2(x) = 0$. But, this shows $y \in \mathrm{Ker}(T)$ $\therefore$ $T(V) \subseteq \mathrm{Ker}(T)$.

$\Leftarrow$] Suppose $T(V) \subseteq \mathrm{Ker}(T)$ and $T : V \to V$ linear. Let $x \in V$ and consider $T^2(x) = T(T(x))$. Let $T(x) = y$ and observe $y \in T(V)$. But as $T(V) \subseteq \mathrm{Ker}(T)$ it follows $T(y) = 0$. Therefore, $T^2(x) = T(T(x)) = T(y) = 0$. But, $x \in V$ was arbitrary and so we have shown $T^2 = 0$.

This concludes the proof for $T \in L(V)$,

$$T^2 = 0 \iff \underbrace{T(V) \subseteq \mathrm{Ker}(T)}_{\mathrm{Range}(T) \subseteq \mathrm{Ker}(T)}$$

actually, we have in this case,

$$\{0\} \leq \mathrm{Range}(T) \leq \mathrm{Ker}(T) \leq V$$

$\boxed{P89}$ §13#9, p. 108

Let $T \in L(V)$. Prove that $\exists$ a nonzero linear transformation $S \in L(V)$ such that $T \circ S = \phi$ iff $\left\{ \begin{matrix} \exists v \in V,\ v \neq 0 \\ \text{such that } T(v) = 0 \end{matrix} \right\}$

$\Rightarrow$| Assume $T, S \in L(V)$ and $S \neq 0$ and $T \circ S = 0$

Note $S \neq 0 \Rightarrow \exists w \in V,\ w \neq 0$ such that $S(w) \neq 0$ (otherwise we'd have $S(w) = 0\ \forall w \in V \therefore S = 0$ a contradiction)

Now, $T \circ S = 0 \Rightarrow (T \circ S)(w) = T(S(w)) = 0$.

Let $v = S(w)$ and observe $v \neq 0$ yet $T(v) = 0$. //

$\Leftarrow$| Once more suppose $T \in L(V)$. Suppose $\exists v \in V,\ v \neq 0$ such that $T(v) = 0$. We <u>seek</u> <u>to</u> <u>show</u> $\exists S \in L(V)$ for which $T \circ S = 0$ and $S \neq 0$ (S is not the zero transformation).

Let $S(v) = v$ and extend linearly in the sense that $S(x) = 0$ for all $x \in V - \{kv \mid k \in \mathbb{F}\}$. If $\dim V < \infty$ then we could concretely describe this by $\beta = \{v, v_2, \ldots, v_n\}$

$$S(v) = v, \quad S(v_j) = 0$$

for $j = 2, 3, \ldots, n$ the extend linearly off the basis β. That said, I believe my argument works well in the case $\dim(V) = \infty$. $S : V \to V$ is by construction linear and, $S(v) = v \neq 0$ shows $S \neq 0$ yet,

$$\begin{aligned} (T \circ S)(kv) &= T(S(kv)) \\ &= T(k\,S(v)) \\ &= k\,T(v) \\ &= 0 \end{aligned}$$

and $x \notin \text{span}\{v\}$ we have $(T \circ S)(x) = T(S(x)) = T(0) = 0$

Thus, $T \circ S = 0$. //

$\boxed{P\,90}$ $tr(M) = \sum_{i=1}^{n} M_{ii}$ for $M \in \mathbb{F}^{n \times n}$.

Observe $tr(cA+B) = \sum_{i=1}^{n} (cA+B)_{ii}$: for $c \in \mathbb{F}$, $A, B \in \mathbb{F}^{n\times n}$

$= \sum_{i=1}^{n} (cA_{ii} + B_{ii})$: def^{n} of matrix add & scalar mult.

$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$: Σ. properties.

$= c\,tr(A) + tr(B)$

Thus $tr : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}$ is a linear transformation; $tr \in L(\mathbb{F}^{n\times n}, \mathbb{F})$

Let A, B be multipliable ∋ $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{n \times m}$

$tr(AB) = \sum_{i=1}^{m} (AB)_{ii}$: def^{n} of trace

$= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ji}$: def^{n} of AB

$= \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij} B_{ji}$: prop. of Σ

$= \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij}$: $\alpha\beta = \beta\alpha \quad \forall \alpha, \beta \in \mathbb{F}$.

$= \sum_{j=1}^{n} (BA)_{jj}$: def^{n} of matrix mult. BA.

$= tr(BA)$. //