Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 7 of the notes: ____________________.

Problem 73 Suppose V and W are vector spaces over $\mathbb{F}$ and $S, T : V \to W$ are linear transformations. Show $S + cT$ is a linear transformation for any $c \in \mathbb{F}$.

Problem 74 Let $T(x, y, z) = (x + z, y + z, x + z)$ for all $(x, y, z) \in \mathbb{R}^3$. Find the standard matrix of T. Calculate $\text{Ker}(T)$ and $\text{Range}(T)$. If possible, find T^{-1}.

Problem 75 Let $T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2, x_3 - 4x_4)$ for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Find the standard matrix of T. Find bases for $\text{Ker}(T)$ and $\text{Range}(T)$.

Problem 76 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T(1, 1, 1) = (8, 6, 7)$ and $T(1, 2, 2) = (5, 3, 0)$ and $T(1, 2, 1) = (9, 0, 0)$. Find the standard matrix of T.

Problem 77 Suppose S is a linear transformation on $P_2(\mathbb{R})$ for which $S(t^2 - t) = 1$ and $S(t^2 + t) = 3t + 2$ and $S(1) = 0$. Find the formula for $S(at^2 + bt + c)$.

Problem 78 Let $\beta = \{v_1, \dots, v_n\}$ form a basis for V over a field $\mathbb{F}$. Recall, for each $x \in V$, we defined $[x]_\beta$ to be the unique vector $(c_1, \dots, c_n) \in \mathbb{F}^n$ for which $x = c_1 v_1 + \cdots + c_n v_n$. Show $\Phi_\beta : V \to \mathbb{F}^n$ defined by $\Phi_\beta(x) = [x]_\beta$ for each $x \in V$ defines a linear transformation.

Problem 79 Suppose $T(x, y) = (x + y, x - y, 3y)$ and define $S(u, v, w) = (2u + 3v, u - w)$.

(a.) Calculate the formula for $(S \circ T)(x, y)$ from the definition of function composition,

(b.) Find $[T]$ and $[S]$ and $[S \circ T]$,

(c.) Verify that $[S \circ T] = [S][T]$.

Problem 80 Let $T : \mathbb{C} \to \mathbb{C}$ be the function defined by $T(x + iy) = x - iy$. Find the matrix of T with respect to the basis $\beta = \{1, i\}$. (here we view $\mathbb{C}$ as a real vector space with basis β)

Problem 81 Let $L_w(z) = wz$ where $w, z \in \mathbb{C}$. If $w = a + ib$ then find the matrix if L_w with respect to the basis $\{1, i\}$ for $\mathbb{C}$ (viewed as a two-dimensional real vector space). Is it possible to choose some w such that $L_w = T$ where $T(x + iy) = x - iy$?

Problem 82 Let $T(A) = AE_{12}$ where $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is a linear transformation. Also, find $[T]_{\beta,\beta}$ where $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.

Problem 83 Let $T(f(x)) = f(x) - f'(x)$ for each $f(x) \in P_2(\mathbb{R})$.

(a.) Find a basis β for the $\text{Ker}(T)$,

(b.) extend β to a basis γ for $P_2(\mathbb{R})$ by adjoining appropriate vectors from $\{1, x, x^2\}$,

(c.) Calculate $[T]_{\gamma,\gamma}$.

Problem 84 Suppose $T : P_4(\mathbb{R}) \to P_2(\mathbb{R})$ is defined by $T(f(x)) = f''(x)$.
Prove T is surjective, however T is not injective.

Problem 85 If $W \leq V$ and $T : V \to V$ is a linear transformation such that $T(W) \subset W$ then we define $T_W : W \to W$ to be the restriction of T to W with codomain modified to W; $T_W : W \to W$ where $T_W(x) = T(x)$ for each $x \in W$. If such W exists then it is known as an **invariant subspace** of T. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $T(A) = A - A^T$ for each $A \in \mathbb{R}^{2\times 2}$.

(a.) Show $W_1 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ forms and invariant subspace of T. Calculate T_{W_1} and $[T_{W_1}]_{\gamma_1,\gamma_1}$ where $\gamma_1 = \{E_{11}, E_{22}, E_{12} + E_{21}\}$

(b.) Show $W_2 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = -A\}$ forms and invariant subspace of T. Calculate T_{W_2} and $[T_{W_2}]_{\gamma_2,\gamma_2}$ where $\gamma_2 = \{E_{12} - E_{21}\}$

(c.) Consider $\beta = \gamma_1 \cup \gamma_2 = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$ calculate $[T]_{\beta,\beta}$.

Problem 86 Find an explicit isomorphism from the subspace of 3×3 real symmetric matrices and $\mathbb{C}^3$. Notice, I must mean $\mathbb{C}^3$ as a vector space over $\mathbb{R}$ for this question to be reasonable.

Problem 87 We define **hyperbolic numbers** $\mathcal{H} = \{x + jy \mid x, y \in \mathbb{R}\}$ where $j^2 = 1$. In particular, we add and multiply in the natural fashion:

$$(a + bj) + (x + jy) = a + x + j(b + y)$$

$$(a + bj)(x + jy) = ax + by + j(ay + bx)$$

for all $a+jb, x+jy \in \mathcal{H}$. Scalar multiplication is a special case of the vector multiplication in $\mathcal{H}$; $c(a + bj) = ca + j(cb)$. It is straightforward to verify $\mathcal{H}$ forms a vector space over $\mathbb{R}$. Prove $\Psi(x + jy) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ is a vector space isomorphism from $\mathcal{H}$ to $V = \Psi(\mathcal{H})$ viewed as a subset of $\mathbb{R}^{2\times 2}$ with respect to the usual addition and scalar multiplication of matrices. Also, show:

$$\Psi\left((x + jy)(a + jb)\right) = \Psi(a + jy)\Psi(a + jb)$$

Remark: *a vector space V paired with a multiplication is called an* **algebra**. *For example, $\mathbb{C}$ is an algebra since $\mathbb{C}$ is a vector space where we also have a natural method to multiply vectors. Likewise, $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ form algebras with respect to the usual matrix multiplication. When given two algebras it is interesting to ask if they are isomorphic as algebras. This requires they have the same linear structure (which is the sense of isomorphism this course focuses primarily upon)* **and** *the multiplication is preserved in the natural fashion. To be precise, if $\mathcal{A}$ has multiplication $\star$ and $\mathcal{B}$ has multiplication $\circ$ then $\Psi : \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism if*

$$\Psi(x + cy) = \Psi(x) + c\Psi(y), \qquad \Psi(x \star y) = \Psi(x) \circ \Psi(y)$$

for all $x, y \in \mathcal{A}$ and $c \in \mathbb{F}$. We also insist that when $\mathcal{A}$ has a multiplicative identity $1_{\mathcal{A}}$ and likewise $1_{\mathcal{B}}$ for $\mathcal{B}$ then $\Psi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. In the above problem, we see $\Psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ hence Ψ is an algebra isomorphism of hyperbolic numbers and the subspace of 2×2 matrices of the special form $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$. This discussion echos Example 7.3.17 in my notes.

Problem 88 Let V be a finite dimensional vector space over $\mathbb{F}$ of dimension n. Prove $\mathcal{L}(V)$ is isomorphic to $\mathbb{F}^{n\times n}$ as algebras over $\mathbb{F}$. In particular, find an isomorphism which preserves addition, scalar multiplication, and has $\Psi(T \circ S) = \Psi(T)\Psi(S)$ for all $T, S : V \to V$ and has $\Psi(Id_V) = I$. Here the product $\Psi(T)\Psi(S)$ is that of matrix multiplication.

Hint: think about how to create maps from $V \to V$ which naturally correspond to the matrix units E_{ij}. Remember, we know $\mathbb{F}^{n\times n}$ has basis formed by E_{ij} for $1 \leq i, j \leq n$. Probably picking a basis for V is a good starting point. If you're lost, try $n = 1$ or $n = 2$ to get started

Problem 89 Let $\mathrm{SL}(3,\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} \mid \det(A) = 1\}$. Suppose $T(x) = Ax$. Show T preserves the volume of a parallel-piped.

Problem 90 Consider $\beta = \{1, e_1, e_2, e_1 \wedge e_2\}$ serve to generate $V = \mathrm{span}(\beta)$ as a real vector space of dimension 4. I'll arrange the $\wedge$ products in a table:

$\wedge$	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$-e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

Linear transformations on V naturally correspond to 4×4 matrices. Notice, we can define the left multiplication maps with respect to the wedge product: $L_x(y) = x \wedge y$. Then, the set of $[L_x]_{\beta,\beta} \in \mathbb{R}^{4\times 4}$. I'll set it up:

$$\begin{aligned}[L_x]_{\beta,\beta} &= [[L_x(1)]_\beta | [L_x(e_1)]_\beta | [L_x(e_2)]_\beta | [L_x(e_1 \wedge e_2)]_\beta] \\ &= [[x]_\beta | [x \wedge e_1]_\beta | [x \wedge e_2]_\beta | [x \wedge e_1 \wedge e_2]_\beta]\end{aligned}$$

For example,

$$[L_{e_1}]_{\beta,\beta} = [[e_1]_\beta | [e_1 \wedge e_1]_\beta | [e_1 \wedge e_2]_\beta | [e_2 \wedge e_1 \wedge e_2]_\beta] = [[e_1]_\beta | 0 | [e_1 \wedge e_2]_\beta | 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a.) Complete my work by finding $[L_{e_2}]_{\beta,\beta}$, $[L_{e_1\wedge e_2}]_{\beta,\beta}$ and the far more relaxing $[L_1]_{\beta,\beta}$.

(b.) Check that $([L_{e_1}]_{\beta,\beta})^2 = 0$, $([L_{e_2}]_{\beta,\beta})^2 = 0$ and $[L_{e_1}]_{\beta,\beta}[L_{e_2}]_{\beta,\beta} = [L_{e_1\wedge e_2}]_{\beta,\beta}$ whereas $[L_{e_2}]_{\beta,\beta}[L_{e_1}]_{\beta,\beta} = -[L_{e_1\wedge e_2}]_{\beta,\beta}$

Remark: *I know that part (b.) will work out since we can easily calculate in general that $L_x \circ L_y(z) = L_x(y \wedge z) = x \wedge y \wedge z = L_{x\wedge y}(z)$. The neat thing here is that if you forget about the abstract $e_1 \wedge e_2$ and just think about these 4×4 matrices then you can see that the algebra of the wedge product on $\mathbb{R}^2$ is easily implemented with 4×4 matrices. Generally, we can follow much the same construction to build the wedge product algebra on $\mathbb{R}^n$ by representing it on $2^n \times 2^n$ matrices. That said, that's not really how I think about the wedge product. Ask me in office hours sometime if you'd like to know more.*