Your PRINTED NAME indicates you read Chapter 5 of the notes: ____________________.

Problem 41 Explain why the following formulas do **not** define an inner product:

(a.) $\langle (a,b),(c,d)\rangle = ac - bd$ on $\mathbb{R}^2$

(b.) $\langle A, B\rangle = tr(A+B)$ on $\mathbb{R}^{2\times 2}$

(c.) $\langle (z_1,z_2),(w_1,w_2)\rangle = z_1w_1 + z_2w_2$ on $\mathbb{C}^2(\mathbb{C})$

(d.) $\langle (z_1,z_2),(w_1,w_2)\rangle = z_1w_1 + z_2w_2$ on $\mathbb{C}^2(\mathbb{R})$

Problem 42 Let β be a basis for an inner product space $(V, \langle\ ,\ \rangle)$. Prove:

(a.) If $x \in V$ and $\langle x, z\rangle = 0$ for all $z \in \beta$ then $x = 0$.

(b.) If $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$ then $x = y$.

Problem 43 Suppose $\{v_1, \dots, v_k\}$ is an orthogonal subset of the inner product space $(V, \langle\ ,\ \rangle)$ and suppose $a_1, \dots, a_k$ are scalars. Show $\| \sum_{i=1}^k a_i v_i \|^2 = \sum_{i=1}^k |a_i|^2 \|v_i\|^2$.

Problem 44 Let W_1 and W_2 be subspaces of finite dimensional vector space V. Prove the following:

(a.) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

(b.) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

Problem 45 Let V be the vector space of continuous real-valued functions on $[-1,1]$. Let W_e be the subspace of even functions in V and let W_o be the subspace of odd functions in V. Prove $W_e^\perp = W_o$ with respect to $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)\,dt$.

Problem 46 Let $W = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ for a subspace in $\mathbb{R}^{2\times 2}$ with the standard Frobenius inner product $\langle A, B\rangle = \text{trace}(AB^T)$.

(a.) Find an orthonormal basis for $W^\perp$

(b.) Calculate $Proj_{W^\perp} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(c.) Calculate $Proj_{W} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(d.) Find the matrix in W which is closest to $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

Problem 47 Let T be a linear transformation on a finite dimensional real inner product space $(V, \langle\ ,\ \rangle)$. Suppose T has an orthornormal eigenbasis. Prove T is self-adjoint. That is, prove $T = T^*$.

Problem 48 Let $\text{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^TR = I, det(R) = 1\}$. Show that: If $R \in \text{SO}(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 49 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3\times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the distinct eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 u u^T, \qquad E_2 = \lambda_2 v v^T, \qquad E_3 = \lambda_3 w w^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ *and the fact you are given* $Au = \lambda_1 u$ *etc.*

Problem 50 Let $T : V \to V$ be a linear transformation on the finite dimensional inner product space $(V, \langle\ ,\ \rangle)$. Show the following are equivalent:

(a.) $T^*T = Id$,

(b.) $TT^* = Id$,

(c.) $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$

(d.) If β is an orthnormal basis for V then $T(\beta)$ is an orthonormal basis for V

(e.) $\|T(x)\| = \|x\|$ for all $x \in V$