Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 7 of the notes: ____________.

Problem 73 Suppose V and W are vector spaces over $\mathbb{F}$ and $S, T : V \to W$ are linear transformations. Show $S + cT$ is a linear transformation for any $c \in \mathbb{F}$.

Problem 74 Let $T(x, y, z) = (x + z, y + z, x + z)$ for all $(x, y, z) \in \mathbb{R}^3$. Find the standard matrix of T. Calculate $\text{Ker}(T)$ and $\text{Range}(T)$. If possible, find T^{-1}.

Problem 75 Let $T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2, x_3 - 4x_4)$ for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Find the standard matrix of T. Find bases for $\text{Ker}(T)$ and $\text{Range}(T)$.

Problem 76 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T(1, 1, 1) = (8, 6, 7)$ and $T(1, 2, 2) = (5, 3, 0)$ and $T(1, 2, 1) = (9, 0, 0)$. Find the standard matrix of T.

Problem 77 Suppose S is a linear transformation on $P_2(\mathbb{R})$ for which $S(t^2 - t) = 1$ and $S(t^2 + t) = 3t + 2$ and $S(1) = 0$. Find the formula for $S(at^2 + bt + c)$.

Problem 78 Let $\beta = \{v_1, \dots, v_n\}$ form a basis for V over a field $\mathbb{F}$. Recall, for each $x \in V$, we defined $[x]_\beta$ to be the unique vector $(c_1, \dots, c_n) \in \mathbb{F}^n$ for which $x = c_1 v_1 + \cdots + c_n v_n$. Show $\Phi_\beta : V \to \mathbb{F}^n$ defined by $\Phi_\beta(x) = [x]_\beta$ for each $x \in V$ defines a linear transformation.

Problem 79 Suppose $T(x, y) = (x + y, x - y, 3y)$ and define $S(u, v, w) = (2u + 3v, u - w)$.

(a.) Calculate the formula for $(S \circ T)(x, y)$ from the definition of function composition,

(b.) Find $[T]$ and $[S]$ and $[S \circ T]$,

(c.) Verify that $[S \circ T] = [S][T]$.

Problem 80 Let $T : \mathbb{C} \to \mathbb{C}$ be the function defined by $T(x + iy) = x - iy$. Find the matrix of T with respect to the basis $\beta = \{1, i\}$. (here we view $\mathbb{C}$ as a real vector space with basis β)

Problem 81 Let $L_w(z) = wz$ where $w, z \in \mathbb{C}$. If $w = a + ib$ then find the matrix if L_w with respect to the basis $\{1, i\}$ for $\mathbb{C}$ (viewed as a two-dimensional real vector space). Is it possible to choose some w such that $L_w = T$ where $T(x + iy) = x - iy$?

Problem 82 Let $T(A) = AE_{12}$ where $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is a linear transformation. Also, find $[T]_{\beta,\beta}$ where $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.

Problem 83 Let $T(f(x)) = f(x) - f'(x)$ for each $f(x) \in P_2(\mathbb{R})$.

(a.) Find a basis β for the $\text{Ker}(T)$,

(b.) extend β to a basis γ for $P_2(\mathbb{R})$ by adjoining appropriate vectors from $\{1, x, x^2\}$,

(c.) Calculate $[T]_{\gamma,\gamma}$.

Problem 84 Suppose $T : P_4(\mathbb{R}) \to P_2(\mathbb{R})$ is defined by $T(f(x)) = f''(x)$. Prove T is surjective, however T is not injective.

Problem 85 If $W \leq V$ and $T : V \to V$ is a linear transformation such that $T(W) \subset W$ then we define $T_W : W \to W$ to be the restriction of T to W with codomain modified to W; $T_W : W \to W$ where $T_W(x) = T(x)$ for each $x \in W$. If such W exists then it is known as an **invariant subspace** of T. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $T(A) = A - A^T$ for each $A \in \mathbb{R}^{2\times 2}$.

(a.) Show $W_1 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = A\}$ forms and invariant subspace of T. Calculate T_{W_1} and $[T_{W_1}]_{\gamma_1,\gamma_1}$ where $\gamma_1 = \{E_{11}, E_{22}, E_{12} + E_{21}\}$

(b.) Show $W_2 = \{A \in \mathbb{R}^{2\times 2} \mid A^T = -A\}$ forms and invariant subspace of T. Calculate T_{W_2} and $[T_{W_2}]_{\gamma_2,\gamma_2}$ where $\gamma_2 = \{E_{12} - E_{21}\}$

(c.) Consider $\beta = \gamma_1 \cup \gamma_2 = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$ calculate $[T]_{\beta,\beta}$.

Problem 86 Find an explicit isomorphism from the subspace of 3×3 real symmetric matrices and $\mathbb{C}^3$.

Problem 87 We define **hyperbolic numbers** $\mathcal{H} = \{x + jy \mid x, y \in \mathbb{R}\}$ where $j^2 = 1$. In particular, we add and multiply in the natural fashion:

$$(a + bj) + (x + jy) = a + x + j(b + y)$$

$$(a + bj)(x + jy) = ax + by + j(ay + bx)$$

for all $a+jb, x+jy \in \mathcal{H}$. Scalar multiplication is a special case of the vector multiplication in $\mathcal{H}$; $c(a+bj) = ca + j(cb)$. It is straightforward to verify $\mathcal{H}$ forms a vector space over $\mathbb{R}$. Prove $\Psi(x + jy) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ is a vector space isomorphism from $\mathcal{H}$ to $V = \Psi(\mathcal{H})$. Also, show:

$$\Psi\left((x + jy)(a + jb)\right) = \Psi(a + jy)\Psi(a + jb)$$

Remark: *a vector space V paired with a multiplication is called an* **algebra**. *For example, $\mathbb{C}$ is an algebra since $\mathbb{C}$ is a vector space where we also have a natural method to multiply vectors. Likewise, $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ form algebras with respect to the usual matrix multiplication. When given two algebras it is interesting to ask if they are isomorphic as algebras. This requires they have the same linear structure (which is the sense of isomorphism this course focuses primarily upon)* **and** *the multiplication is preserved in the natural fashion. To be precise, if $\mathcal{A}$ has multiplication $\star$ and $\mathcal{B}$ has multiplication $\circ$ then $\Psi : \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism if*

$$\Psi(x + cy) = \Psi(x) + c\Psi(y), \qquad \Psi(x \star y) = \Psi(x) \circ \Psi(y)$$

for all $x, y \in \mathcal{A}$ and $c \in \mathbb{F}$. We also insist that when $\mathcal{A}$ has a multiplicative identity $1_{\mathcal{A}}$ and likewise $1_{\mathcal{B}}$ for $\mathcal{B}$ then $\Psi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. In the above problem, we see $\Psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ hence Ψ is an algebra isomorphism of hyperbolic numbers and the subspace of 2×2 matrices of the special form $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$. This discussion echos Example 7.3.17 in my notes.

Problem 88 Let V be a finite dimensional vector space over $\mathbb{F}$ of dimension n. Prove $\mathcal{L}(V)$ is isomorphic to $\mathbb{F}^{n\times n}$ as algebras over $\mathbb{F}$. In particular, find an isomorphism which preserves addition, scalar multiplication, and has $\Psi(T \circ S) = \Psi(T)\Psi(S)$ for all $T, S : V \to V$ and has $\Psi(Id_V) = I$. Here the product $\Psi(T)\Psi(S)$ is that of matrix multiplication.

Hint: think about how to create maps from $V \to V$ which naturally correspond to the matrix units E_{ij}. Remember, we know $\mathbb{F}^{n\times n}$ has basis formed by E_{ij} for $1 \leq i, j \leq n$. Probably picking a basis for V is a good starting point. If you're lost, try $n = 1$ or $n = 2$ to get started

Problem 89 Let $\text{SL}(3, \mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} \mid \det(A) = 1\}$. Suppose $T(x) = Ax$. Show T preserves the volume of a parallel-piped.

Problem 90 Consider $\beta = \{1, e_1, e_2, e_1 \wedge e_2\}$ serve to generate $V = \text{span}(\beta)$ as a real vector space of dimension 4. I'll arrange the $\wedge$ products in a table:

$\wedge$	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$-e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

Linear transformations on V naturally correspond to 4×4 matrices. Notice, we can define the left multiplication maps with respect to the wedge product: $L_x(y) = x \wedge y$. Then, the set of $[L_x]_{\beta,\beta} \in \mathbb{R}^{4\times 4}$. I'll set it up:

$$\begin{aligned}[L_x]_{\beta,\beta} &= [[L_x(1)]_\beta | [L_x(e_1)]_\beta | [L_x(e_2)]_\beta | [L_x(e_1 \wedge e_2)]_\beta] \\ &= [[x]_\beta | [x \wedge e_1]_\beta | [x \wedge e_2]_\beta | [x \wedge e_1 \wedge e_2]_\beta]\end{aligned}$$

For example,

$\beta = \{1, e_1, e_2, e_1 \wedge e_2\}$

$$[L_{e_1}]_{\beta,\beta} = [[e_1]_\beta | [e_1 \wedge e_1]_\beta | [e_1 \wedge e_2]_\beta | [e_2 \wedge e_1 \wedge e_2]_\beta] = [[e_1]_\beta | 0 | [e_1 \wedge e_2]_\beta | 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a.) Complete my work by finding $[L_{e_2}]_{\beta,\beta}$, $[L_{e_1 \wedge e_2}]_{\beta,\beta}$ and the far more relaxing $[L_1]_{\beta,\beta}$.

(b.) Check that $([L_{e_1}]_{\beta,\beta})^2 = 0$, $([L_{e_2}]_{\beta,\beta})^2 = 0$ and $[L_{e_1}]_{\beta,\beta}[L_{e_2}]_{\beta,\beta} = [L_{e_1 \wedge e_2}]_{\beta,\beta}$ whereas $[L_{e_2}]_{\beta,\beta}[L_{e_1}]_{\beta,\beta} = -[L_{e_1 \wedge e_2}]_{\beta,\beta}$

Remark: *I know that part (b.) will work out since we can easily calculate in general that $L_x \circ L_y(z) = L_x(y \wedge z) = x \wedge y \wedge z = L_{x\wedge y}(z)$. The neat thing here is that if you forget about the abstract $e_1 \wedge e_2$ and just think about these 4×4 matrices then you can see that the algebra of the wedge product on $\mathbb{R}^2$ is easily implemented with 4×4 matrices. Generally, we can follow much the same construction to build the wedge product algebra on $\mathbb{R}^n$ by representing it on $2^n \times 2^n$ matrices. That said, that's not really how I think about the wedge product. Ask me in office hours sometime if you'd like to know more.*

Mission 5 Solution

P73 Suppose V, W are vector spaces over $\mathbb{F}$ and $S, T \in \mathcal{L}(V, W)$ and $c \in \mathbb{F}$. Let $x, y \in V$ and $b \in \mathbb{F}$. Consider,

$$
\begin{aligned}
(S + cT)(bx+y) &= S(bx+y) + c\,T(bx+y) && : \text{def}^{\underline{n}} \text{ of } S+cT. \\
&= b\,S(x) + S(y) + c\big(b\,T(x) + T(y)\big) && : \text{linearity of } S \,\&\, T \\
&= b\big(S(x) + c\,T(x)\big) + S(y) + c\,T(y) && : bc = cb \text{ as } b, c \in \mathbb{F} \\
&= b\,(S+cT)(x) + (S+cT)(y) && : S + cT \text{ defined by pointwise evaluation.}
\end{aligned}
$$

Thus $S + cT \in \mathcal{L}(V, W)$.

<u>Remark</u>: since $0 : V \to W$ defined by $0(x) = 0 \in W \ \forall x \in V$ defines zero map $0 \in \mathcal{L}(V, W)$ (you can easily check, $0(bx+y) = b\,0(x) + 0(y)$) we have $\mathcal{L}(V, W) \neq \emptyset$ and as $\mathcal{L}(V, W) \subseteq \mathcal{F}(V, W)$ we have $\underbrace{\mathcal{L}(V, W)}_{} \leq \underbrace{\mathcal{F}(V, W)}_{\infty}$.

$\mathcal{F}(V, W)$: all functions from V to W with vector space structure given by pointwise add and scalar mult. of functions.

If $\dim(V), \dim(W) < \infty$ then $\mathcal{L}(V, W)$ also finite dim'l over $\mathbb{F}$

∞ dim'l. for $\mathbb{F}$ char. zero.

P74 $T(x, y, z) = \begin{bmatrix} x+z \\ y+z \\ x+z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\therefore$ $\boxed{[T] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$

$\mathrm{Ker}(T) = \mathrm{Null}[T]$ in this context, so calculate as usual,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \ x \in \mathrm{Ker}(T) \text{ has } \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \end{aligned}$$

$\Rightarrow \boxed{\mathrm{Ker}(T) = \mathrm{span}\{(-1, -1, 1)\}}$

By rank-nullity $\text{Th}^{\underline{m}}$ we know $\dim(\mathrm{Range}(T)) = 3 - 1 = 2$

Thus $\boxed{\mathrm{Range}(T) = \mathrm{Col}[T] = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.}$

P74 continued

Remark: since $3 = \dim(\cancel{\text{Ker } T}^{\,1}) + \dim(\text{Range}(T)) \Rightarrow \text{Range}(T)$ is a 2-dim'l space. But, $\text{Col}[T] = \text{Range}(T)$ for $T: \mathbb{R}^2 \to \mathbb{R}^3$ hence we need only select two LI vectors from columns of $[T]$ to create a basis for Range(T).

Hence $\beta_{12} = \{(1,0,1), (0,1,0)\}$ or $\beta_{13} = \{(1,0,1), (1,1,1)\}$ both do nicely. (although technically, I did not ask for basis here ☺)

Continuing, can we calculate T^{-1}? **NO!**

If a function is not 1-1 then it has no inverse and $\text{Ker}(T) \neq 0 \Rightarrow T$ not 1-1 $\therefore T^{-1}$ d.n.e.

P75 Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $T(x) = (x_1 - 2x_2, x_3 - 4x_4)$

$$[T] = [T(e_1)|T(e_2)|T(e_3)|T(e_4)] = \underbrace{\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix}}_{\text{standard matrix of } T}$$

Again, since T is map on column-vectors we have $\text{Ker } T = \text{Null}[T]$ and $\text{Range}(T) = \text{Col}[T]$ so calculate

$$x \in \text{Ker}(T) \Rightarrow \begin{array}{l} x_1 = 2x_2 \\ x_3 = 4x_4 \end{array} \quad \therefore \quad x = (2x_2, x_2, 4x_4, x_4) = x_2(2,1,0,0) + x_4(0,0,4,1)$$

$$\therefore \boxed{\beta_1 = \{(2,1,0,0), (0,0,4,1)\}}$$

BASIS FOR Ker(T)

By rank-nullity Th$^{\text{m}}$ we note $4 = \underbrace{\dim(\text{Range}(T))}_{\substack{rk(T) \\ (\text{rank})}} + \underbrace{\dim(\text{Ker } T)}_{\substack{\nu(T) \\ (\text{nullity})}}$ since $\nu(T) = 2 \Rightarrow rk(T) = 2$.

Anyway, we know $\boxed{\beta_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$ serves as basis for $\text{Col}[T]$ hence β_2 is basis for Range(T).

$\boxed{P76}$ $T: \mathbb{R}^3 \to \mathbb{R}^3$ is linear transformation s.t.

$$\left.\begin{aligned} T(1,1,1) &= (8,6,7) = A(1,1,1) \\ T(1,2,2) &= (5,3,0) = A(1,2,2) \\ T(1,2,1) &= (9,0,0) = A(1,2,1) \end{aligned}\right\} \text{ setting } [T] = A$$

Notice $(1,2,2) - (1,2,1) = (0,0,1)$ hence

$$Ae_3 = A[(1,2,2)-(1,2,1)] = A(1,2,2) - A(1,2,1) = (5,3,0) - (9,0,0)$$

Thus $\underline{\text{col}_3(A) = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}}$. One down, two to go.

Ah, $2(1,1,1) - (1,2,2) = (1,0,0) = e_1$ thus,

$$\begin{aligned} Ae_1 &= A(2(1,1,1) - (1,2,2)) \\ &= 2A(1,1,1) - A(1,2,2) \\ &= 2(8,6,7) - (5,3,0) \\ &= (11,9,14) \quad \therefore \quad \underline{\text{Col}_2[A] = \begin{bmatrix} 11 \\ 9 \\ 14 \end{bmatrix}}. \end{aligned}$$

But $A(1,1,1) = Ae_1 + Ae_2 + Ae_3 = (8,6,7)$

$$\Rightarrow Ae_1 = (8,6,7) - (11,9,14) - (-4,3,0)$$

$$\therefore \text{Col}_1(A) = \begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix} \hookrightarrow \boxed{A = \begin{bmatrix} 1 & 11 & -4 \\ -6 & 9 & 3 \\ -7 & 14 & 0 \end{bmatrix} = [T]}$$

Remark: I solved this by a rather ad hoc method. If you wish a more formulaic approach, simply note

$$A\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 9 \\ 6 & 3 & 0 \\ 7 & 0 & 0 \end{bmatrix} \Rightarrow \underline{A = \begin{bmatrix} 8 & 5 & 9 \\ 6 & 3 & 0 \\ 7 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}^{-1}}$$

(I don't like to calculate inverse for 3x3 if I can help it)

$\boxed{P77}$ Suppose $S \in \mathcal{L}(P_2(\mathbb{R}))$ where $P_2(\mathbb{R}) = \mathbb{R}[t]_{n \leq 2}$
and $S(t^2 - t) = 1$, $S(t^2+t) = 3t+2$, $S(1) = 0$
Find formula for $S(at^2+bt+c)$.

☆ Notice, $t^2 - t + t^2 + t = 2t^2$ hence using linearity of S,

$$S(2t^2) = S(t^2-t) + S(t^2+t) = 1 + 3t + 2 = 3(t+1)$$

Thus $\underbrace{S(t^2) = \frac{3}{2}(t+1)}_{*}$ But, $S(t^2-t) = S(t^2) - S(t) = 1$

thus $S(t) = S(t^2) - 1 = \frac{3}{2}(t+1) - 1 = \underbrace{\frac{3}{2}t + \frac{1}{2} = S(t)}_{**}$

Consequently, using $*$ and $**$,

$$\begin{aligned} S(at^2 + bt + c) &= a\,S(t^2) + b\,S(t) + c\,S(1) \\ &= a\left(\frac{3}{2}(t+1)\right) + b\left(\frac{3}{2}t + \frac{1}{2}\right) + c(0) \\ &= \frac{1}{2}(3a+3b)t + \frac{1}{2}(3a+b) \\ &= \boxed{\frac{3}{2}(a+b)t + \frac{1}{2}(3a+b)} \end{aligned}$$

Remark: What if ☆ does not occur to us in the calculation? How would we approach this problem?
I give a rather different approach ⤵

$\boxed{P77}$ A coordinated approach, w.r.t. $\beta = \{t^2, t, 1\}$

$$S(t^2 - t) = 1 \quad \Rightarrow \quad [S]_{\beta\beta} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S(t^2 + t) = 3t + 2 \quad \Rightarrow \quad [S]_{\beta\beta} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$S(1) = 0 \quad \Rightarrow \quad [S]_{\beta\beta} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$[S]_{\beta\beta} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow [S]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Block-diagonal inverse nice!

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3/2 & 3/2 & 0 \\ 3/2 & 1/2 & 0 \end{bmatrix} = \Big[[S(t^2)]_\beta \,\Big|\, [S(t)]_\beta \,\Big|\, [S(1)]_\beta \Big]$$

and hence I can assemble the formula for S,

$$\boxed{S(at^2 + bt + c) = \frac{3}{2}(a+b)t + \frac{1}{2}(3a+b)}$$

$$\Big(\; [S(at^2+bt+c)]_\beta = [S]_{\beta\beta} [at^2+bt+c]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 3/2 & 3/2 & 0 \\ 3/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2}(a+b) \\ \frac{1}{2}(3a+b) \end{bmatrix}$$

$$[S(at^2+bt+c)]_\beta = \left(0, \tfrac{3}{2}(a+b), \tfrac{1}{2}(3a+b)\right) \Rightarrow \underline{S(at^2+bt+c) = \tfrac{3}{2}(a+b)t + \tfrac{1}{2}(3a+b)}$$

$\boxed{P78}$ Let $\beta = \{v_1, v_2, \dots, v_n\}$ be basis for $V(\mathbb{F})$.

Let $x = \sum_{i=1}^{n} c_i v_i$ and $y = \sum_{i=1}^{n} b_i v_i$ hence

$[x]_\beta = (c_1, \dots, c_n) = c$ and $[y]_\beta = (b_1, \dots, b_n) = b$

by defn of coordinates w.r.t. basis β.

Define $\Phi_\beta(\tilde{x}) = [\tilde{x}]_\beta \quad \forall \tilde{x} \in V$. Consider for x, y as described above and $\alpha \in \mathbb{F}$,

$$\begin{aligned}
\Phi_\beta(\alpha x + y) &= \Phi_\beta\left(\alpha \sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{n} b_i v_i\right) \\
&= \Phi_\beta\left(\sum_{i=1}^{n} (\alpha c_i + b_i) v_i\right) \\
&= (\alpha c_1 + b_1, \alpha c_2 + b_2, \dots, \alpha c_n + b_n) \\
&= \alpha (c_1, \dots, c_n) + (b_1, \dots, b_n) \\
&= \alpha [x]_\beta + [y]_\beta \\
&= \alpha \Phi_\beta(x) + \Phi_\beta(y) \quad \therefore \quad \Phi_\beta \in \mathcal{L}(V, \mathbb{F}^n) //
\end{aligned}$$

$\boxed{P79}$ $T(x,y) = (x+y,\ x-y,\ 3y)$, $T : \mathbb{R}^2 \to \mathbb{R}^3$

$S(u,v,w) = (2u+3v,\ u-w)$, $S : \mathbb{R}^3 \to \mathbb{R}^2$

(a.)
$$
\begin{aligned}
(S\circ T)(x,y) &= S(x+y,\ x-y,\ 3y) \\
&= \big(2(x+y)+3(x-y),\ x+y-3y\big) \\
&= \underline{(5x-y,\ x-2y)} \quad *
\end{aligned}
$$

(b.) We read from formulas of given T & S that

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad [S] = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and from $*$ we find $[S\circ T] = \begin{bmatrix} 5 & -1 \\ 1 & -2 \end{bmatrix}$.

(c.)
$$[S][T] = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & -2 \end{bmatrix} \overset{\checkmark}{=} [S\circ T].$$

$\boxed{P80}$ Let $T : \mathbb{C} \to \mathbb{C}$ be given by $T(x+iy) = x-iy$. (it is assumed that $x,y \in \mathbb{R}$). Let $\beta = \{1, i\}$.

$$
\begin{aligned}
[T]_{\beta\beta} &= \big[\, [T(1)]_\beta \,\big|\, [T(i)]_\beta \,\big] \\
&= \big[\, [1]_\beta \,\big|\, [-i]_\beta \,\big] \\
&= \boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}
\end{aligned}
$$

$$
\begin{tikzcd}
z = x+iy \arrow[r, "T"] \arrow[d, "\Phi_\beta"'] & x-iy = T(z) \arrow[d, "\Phi_\beta"] \\
{[z]_\beta = \begin{bmatrix} x \\ y \end{bmatrix}} \arrow[r, "{[T]_{\beta\beta}}"] & {[T(z)]_\beta = \begin{bmatrix} x \\ -y \end{bmatrix}}
\end{tikzcd}
$$

Hence $[T]_{\beta\beta} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to get $\begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

<u>Remark</u>: now I realize why you guys were snickering as I found $[T]_{\beta\beta}$ for <u>P81</u> you $\dddot{x}$'s.

$\boxed{P81}$ $L_w(z) = wz \quad \forall w, z \in \mathbb{C}$.

Let $a, b \in \mathbb{R}$ and hence $a + ib = w \in \mathbb{C}$.

Let $x, y \in \mathbb{R}$ and hence $x + iy = z \in \mathbb{C}$ and calculate,

$$
\begin{aligned}
L_w(z) &= (a+ib)(x+iy) \\
&= ax - by + i(bx + ay)
\end{aligned}
$$

$\beta = \{1, i\}$ again.

$$
\begin{tikzcd}
\mathbb{C} \ni \boxed{z = x+iy} \arrow[r, "L_w"] \arrow[d, "\Phi_\beta"'] & \boxed{wz} = L_w(z) = ax - by + i(bx+ay) \in \mathbb{C} \arrow[d, "\Phi_\beta"] \\
\mathbb{R}^2 \ni \boxed{[z]_\beta = \begin{bmatrix} x \\ y \end{bmatrix}} \arrow[r, "{[L_w]_{\beta\beta}}", "\text{multiplying}"'] & \boxed{[L_w(z)]_\beta = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}} \in \mathbb{R}^2
\end{tikzcd}
$$

Thus, $\boxed{[L_w]_{\beta\beta} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}}$ $\left(\begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$

<u>Alternatively,</u>

$$
\begin{aligned}
[L_w]_{\beta\beta} &= \big[[L_w(1)]_\beta \,\big|\, [L_w(i)]_\beta \big] \\
&= \big[[w]_\beta \,\big|\, [iw]_\beta \big] \\
&= \big[[a+ib]_\beta \,\big|\, [ia - b]_\beta \big] \\
&= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \quad \text{yep.}
\end{aligned}
$$

In any event, certainly $T \neq L_w$ for any choice of w since $[T]_{\beta\beta} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for any choice of $a, b \in \mathbb{R}$ $(1 \neq -1)$.

$\boxed{P82}$ $T(A) = A E_{12}$ where $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} T(cA_1 + A_2) &= (cA_1 + A_2)E_{12} \\ &= c\, A_1 E_{12} + A_2 E_{12} \\ &= c\, T(A_1) + T(A_2) \quad \forall A_1, A_2 \in \mathbb{R}^{2\times 2} \text{ and } c \in \mathbb{R} \end{aligned}$$

Thus $T \in \mathcal{L}(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2})$ aka $T \in \mathcal{L}(\mathbb{R}^{2\times 2})$.

$$T\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}}_{T(A)}$$

$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$\begin{tikzcd}[column sep=huge] \mathbb{R}^{2\times 2}: \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[r, "T"] \arrow[d, "\Phi_\beta"'] & \mathbb{R}^{2\times 2}: \; T(A) = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \arrow[d, "\Phi_\beta"] \\ [A]_\beta = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \arrow[r, "\text{multiply by}", "{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}"'] & \begin{bmatrix} 0 \\ a \\ 0 \\ c \end{bmatrix} \end{tikzcd}$$

Thus $\boxed{[T]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$

__Alternatively__, use $E_{ij} E_{kl} = \delta_{jk} E_{il}$ to save lots of writing,

$$\begin{aligned} [T]_{\beta\beta} &= \Big[[E_{11}E_{12}]_\beta \,\Big|\, [E_{12}E_{12}]_\beta \,\Big|\, [E_{21}E_{12}]_\beta \,\Big|\, [E_{22}E_{12}]_\beta \Big] \\ &= \Big[[E_{12}]_\beta \,\Big|\, [0]_\beta \,\Big|\, [E_{22}]_\beta \,\Big|\, [0]_\beta \Big] \qquad \beta = \{E_{11}, E_{12}, E_{21}, E_{22}\} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [e_2 | 0 | e_4 | 0]. \end{aligned}$$

$\boxed{P83}$ $T(f(x)) = f(x) - f'(x)$ for $f(x) \in P_2(\mathbb{R})$.

(a.) $T(a+bx+cx^2) = a+bx+cx^2 - (b+2cx) \underset{\uparrow}{=} 0$

$f(x) = a+bx+cx^2 \in \text{Ker}(T)$.

Thus

$f(x) = a+bx+cx^2 \in \text{Ker}(T)$

must satisfy,

$a+bx+cx^2 - b - 2cx = 0$

$(a-b) + (b-2c)x + cx^2 = 0$

by LI of $\{1, x, x^2\}$

$\xrightarrow{1} a-b=0$

$\xrightarrow{x} b-2c=0$

$\xrightarrow{x^2} c=0$

Thus $a=b$, $b=2c$ and $c=0$. Well, apparently $\text{Ker}(T) = \{0\}$ hence $\underline{\beta = \emptyset \text{ is basis}}$.

(b.) CURSES. Well guess you all got lucky here!

$\gamma = \{1, x, x^2\}$ is nice basis for $P_2(\mathbb{R})$.

(c.) Since $T(a+bx+cx^2) = (a-b) + (b-2c)x + cx^2$

$$[T(a+bx+cx^2)]_\gamma = (a-b,\ b-2c,\ c)$$

$$= [T]_{\gamma\gamma} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \leftarrow [a+bx+cx^2]_\gamma = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Hence, $\boxed{[T]_{\gamma\gamma} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}$

Alternatively,

$$[T]_{\gamma\gamma} = \Big[[T(1)]_\gamma \Big| [T(x)]_\gamma \Big| [T(x^2)]_\gamma \Big] = \Big[[1]_\gamma \Big| [x-1]_\gamma \Big| [x^2-2x]_\gamma \Big]$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark: $\boxed{P83}$ is more exciting when T is chosen s.t. $\ker(T) \neq \{0\}$. For example,

$$T(f(x)) = f'(x) - f''(x).$$

$$T(a+bx+cx^2) = b+2cx - 2c$$

$b - 2c = 0$, $2c = 0$ $\Big\}$ $b = c = 0$, a free

Hence $\ker T = \{a+bx+cx^2 \mid b-2c+2cx = 0\} = \text{span}\{1\}$.

Thus $\beta = \{1\}$ so $\gamma = \{1, x, x^2\}$ and

$$[T]_{\gamma\gamma} = \left[\, [T(1)]_\gamma \,\middle|\, [T(x)]_\gamma \,\middle|\, [T(x^2)]_\gamma \right] = \left[\, [0]_\gamma \,\middle|\, [1]_\gamma \,\middle|\, [2x-2]_\gamma \right]$$
$$= \underline{\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}.$$

Remark 2:

Oh, maybe $T(f(x)) = f(0) + f(1)x$ will be more interesting.

Yes, $T(a+bx+cx^2) = a + (a+b+c)x$ thus $f(x) = a+bx+cx^2$ in $\ker(T)$ has $a = 0$ and $a+b+c = 0 \Rightarrow b = -c$

thus $f(x) = b(x - x^2) \Rightarrow \underline{\ker T = \text{span}\{x - x^2\}}$.

$\beta = \{x - x^2\}$ so

$\gamma = \{x - x^2, 1, x\}$
$\gamma = \{x - x^2, 1, x^2\}$ $\Big\}$ L I
$\gamma = \{x - x^2, x, x^2\}$ ← No GOOD.

$$[T]_{\gamma\gamma} = \left[\, [T(x-x^2)]_\gamma \,\middle|\, [T(1)]_\gamma \,\middle|\, [T(x)]_\gamma \right]$$
$$= \left[\, [0]_\gamma \,\middle|\, [1+x]_\gamma \,\middle|\, [x]_\gamma \right] = \underline{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

$\boxed{P\,84}$ $T : P_4(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ given by $T(f(x)) = f''(x)$.
Prove T is surjective, however T not 1-1.

$$\ker(T) = \{ f(x) \in P_4(\mathbb{R}) \mid f''(x) = 0 \} = \text{span}\{1, x\}.$$

Clearly $1 \in \ker(T)$ since $T(1) = 1'' = 0 \therefore \ker(T) \neq \{0\}$ and we find T is not injective.

- (I don't need to prove $\ker(T) = \text{span}\{1,x\}$ to see fail of injectivity, even one nonzero element in the kernel spoils injectivity for T) -

---

TURNING TO SHOWING T is SURJECTION. Let $a + bx + cx^2 \in P_2(\mathbb{R})$. Consider,

$$T\left(\tfrac{1}{2}ax^2 + \tfrac{1}{6}bx^3 + \tfrac{1}{12}cx^4\right) = \left(\tfrac{1}{2}ax^2 + \tfrac{1}{6}bx^3 + \tfrac{1}{12}cx^4\right)''$$

$$= \left(ax + \tfrac{1}{2}bx^2 + \tfrac{1}{3}cx^3\right)'$$

$$= a + bx + cx^2$$

Thus T is a surjection as we have shown the arbitrary pt. $a+bx+cx^2$ in the codomain of T is mapped to under T by $\underline{\tfrac{1}{2}ax^2 + \tfrac{1}{6}bx^3 + \tfrac{1}{12}cx^4 \in P_4(\mathbb{R})}$. $*$

Remark: CALCULUS is a prerequisite. Sometimes we even use it here ☺. I simply $\iint$ to find $*$.

$\boxed{P85}$ Let $T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ where $T(A) = A - A^T$ for each $A \in \mathbb{R}^{2\times 2}$

(a.) Let $W_1 = \{ A \in \mathbb{R}^{2\times 2} \mid A^T = A \}$. If $A \in W_1$ then $T(A) = A - A^T = A - A = 0$ and as $0^T = 0$ we find $T(A) \in W_1$ and hence $T(W_1) \subseteq W_1$

Thus W_1 is an <u>invariant</u> subspace of T.

$T_{W_1} : W_1 \longrightarrow W_1$ is defined by $T_{W_1}(A) = T(A) \ \forall A \in W_1$.

Thus $T_{W_1}(A) = T(A) = 0 \ \forall A \in W_1 \Rightarrow \boxed{[T_{W_1}]_{\gamma_1\gamma_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

where $\gamma_1 = \{ E_{11}, E_{22}, E_{12} + E_{21} \}$ serves as basis for W_1.

(b.) $W_2 = \{ A \in \mathbb{R}^{2\times 2} \mid A^T = -A \}$. Let $A \in W_2$ and consider $T(A) = A - A^T = 2A$ thus $(T(A))^T = (2A)^T = 2A^T$ which yields $(T(A))^T = 2(-A) = -(2A) = -(T(A))$ hence $T(A) \in W_2$ and we've shown $T(W_2) \subseteq W_2$.

We find $T_{W_2} : W_2 \longrightarrow W_2$ given by $T_{W_2}(A) = T(A)$ $\forall A \in W_2$ has $T_{W_2}(A) = T(A) = 2A \ \forall A \in W_2$ which means $\boxed{T_{W_2} = 2 Id_{W_2}}$ thus $[T_{W_2}]_{\gamma_2\gamma_2} = [2]$

where $\gamma_2 = \{ E_{12} - E_{21} \}$

(c.) $\beta = \gamma_1 \cup \gamma_2 = \{ E_{11}, E_{12}, E_{12} + E_{21}, E_{12} - E_{21} \}$

$$[T]_{\beta\beta} = \Big[[T(E_{11})]_\beta \Big| [T(E_{11})]_\beta \Big| [T(E_{12}+E_{21})]_\beta \Big| [\underbrace{T(E_{12}-E_{21})}_{2(E_{12}-E_{21})}]_\beta \Big]$$

$$\boxed{[T]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}} = \left[\begin{array}{c|c} [T_{W_1}]_{\gamma_1\gamma_1} & 0 \\ \hline 0 & [T_{W_2}]_{\gamma_2\gamma_2} \end{array} \right] \text{ btw.}$$

P86 $V = \{ A \in \mathbb{R}^{3\times 3} \mid A^T = A \}$ vs. $\mathbb{C}^3$ over $\mathbb{R}$

$$A^T = A \Rightarrow A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

Hence, $\boxed{\psi\left(\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}\right) = (a+ib,\ c+id,\ e+if)}$

gives isomorphism from V to $\mathbb{C}^3$ as real vector spaces.

- Unless you invent a weird $\mathbb{C}$-scalar mult. for V as a point-set it is certainly not the case V and $\mathbb{C}^3$ are isomorphic as complex vector spaces.

-(I leave it to the reader to check ψ is a bijective linear transformation) -

Remark:
$$\psi(E_{11}) = (1,0,0)$$
$$\psi(E_{12}+E_{21}) = (i,0,0)$$
$$\psi(E_{13}+E_{31}) = (0,1,0)$$
$$\psi(E_{22}) = (0,i,0)$$
$$\psi(E_{23}+E_{32}) = (0,0,1)$$
$$\psi(E_{33}) = (0,0,i)$$

extended linearly suffices to define ψ hence linearity of ψ is clear. Moreover, ψ maps basis of V to basis of $\mathbb{C}^3$, both are 6-dim'l over $\mathbb{R}$ thus ψ is a bijection. (can you prove this?) I state it as lemma ?)

Lemma: If $\tilde{\Psi} : \beta_V \to \beta_W$ is extended linearly to define $\Psi : V \longrightarrow W$ where β_V & β_W are bases for V & W respectively then Ψ is an isomorphism. -(we assume $\#\beta_V = \#\beta_W$ over some $\mathbb{F}$ from the out set to be clear)-

$\boxed{P87}$ $\mathcal{H} = \{ x + jy \mid x, y \in \mathbb{R} \}$ where $j^2 = 1$.

Prove $\Psi(x+jy) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ is vector space isomorphism from $\mathcal{H}$ to $V = \Psi(\mathcal{H}) = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} \,\middle|\, x+jy \in \mathcal{H} \right\}$

It is clear $\Psi : \mathcal{H} \longrightarrow \Psi(\mathcal{H})$ is a surjection!

If $x + jy \in \text{Ker}\,\Psi$ then $\Psi(x+jy) = \begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

thus $x = 0$ and $y = 0$ $\Rightarrow$ $\underline{\text{Ker}\,\Psi = \{0\}}$.

Let's check linearity, if $z_1, z_2 \in \mathcal{H}$ and $c \in \mathbb{R}$,

$$\Psi(z_1 + cz_2) = \Psi\big((x_1 + jy_1) + c(x_2 + jy_2)\big)$$

(expanding z_1, z_2 into their real parts $x_1, y_1, x_2, y_2 \in \mathbb{R}$)

$$= \Psi\big(x_1 + cx_2 + j(y_1 + cy_2)\big)$$

$$= \begin{bmatrix} x_1 + cx_2 & y_1 + cy_2 \\ y_1 + cy_2 & x_1 + cx_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & y_1 \\ y_1 & x_1 \end{bmatrix} + c \begin{bmatrix} x_2 & y_2 \\ y_2 & x_2 \end{bmatrix}$$

assuming $\Psi(\mathcal{H})$ is vector space w.r.t. usual $\mathbb{R}^{2\times 2}$ vect. space structure.

$$= \Psi(x_1 + jy_1) + c\,\Psi(x_2 + jy_2)$$

$$= \Psi(z_1) + c\,\Psi(z_2)$$

Thus $\Psi \in \mathcal{L}(\mathcal{H}, \Psi(H))$. Thus Ψ is isomorphism of $\mathcal{H}$ and $\Psi(\mathcal{H}) \leq \mathbb{R}^{2\times 2}$.

P87 continued:

I usually say $\Psi(\mathcal{H}) = M_{\mathcal{H}} \leq \mathbb{R}^{2\times 2}$. In fact $M_{\mathcal{H}}$ forms the <u>regular</u> <u>matrix</u> <u>representation</u> of hyperbolic #s. You found $M_{\mathbb{C}}$ in an early problem, $M_{\mathbb{C}} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \middle| \ a,b \in \mathbb{R} \right\}$. $\mathbb{R}^{2\times 2}$ allows us to represent many interesting objects in terms of matrix multiplication...

Oh, still need to show something,

$$\Psi\big((x+jy)(a+jb)\big) = \Psi\big(xa+yb+j(xb+ya)\big)$$

$$= \left[\begin{array}{c|c} xa+yb & xb+ya \\ \hline xb+ya & xa+yb \end{array}\right] \ast$$

$$\Psi(x+jy)\,\Psi(a+jb) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$= \left[\begin{array}{c|c} xa+yb & xb+ya \\ \hline ya+xb & yb+xa \end{array}\right] \ast\ast$$

Comparing $\ast$ and $\ast\ast$ we find,

$$\Psi\big((x+jy)(a+jb)\big) = \Psi(x+jy)\,\Psi(a+jb)$$

aka $\quad \Psi(z_1 z_2) = \Psi(z_1)\Psi(z_2) \quad \forall z_1, z_2 \in \mathcal{H}.$

P88 $V(\mathbb{F})$ with $\dim(V) = n$.

$\mathcal{L}(V) = \{T : V \longrightarrow V \mid T \text{ linear}\}$.

Prove $\mathcal{L}(V) \cong \mathbb{F}^{n\times n}$ as algebras. In particular, find $\Psi : \mathcal{L}(V) \longrightarrow \mathbb{F}^{n\times n}$ s.t. $\Psi(T \circ S) = \Psi(T)\Psi(S)$ $\forall\, T, S \in \mathcal{L}(V)$ and $\Psi(Id_V) = I \in \mathbb{F}^{n\times n}$.

Let $\beta = \{v_1, v_2, \dots, v_n\}$ form basis over $\mathbb{F}$ for V. If $T : V \longrightarrow V$ is linear then $[T]_{\beta\beta} \in \mathbb{F}^{n\times n}$ and we know for $S : V \longrightarrow V$ linear we also have the property $[T \circ S]_{\beta\beta} = [T]_{\beta\beta}[S]_{\beta\beta}$. Moreover

$$[Id_V]_{\beta\beta} = \left[\,[v_1]_\beta \mid [v_2]_\beta \mid \cdots \mid [v_n]_\beta\,\right] = [e_1 \mid e_2 \mid \cdots \mid e_n] = I.$$

Hence define $\Psi(T) = [T]_{\beta\beta}$ and observe,

$$\begin{aligned}\Psi(cT + S) &= [cT + S]_{\beta\beta} \\ &= c[T]_{\beta\beta} + [S]_{\beta\beta} \\ &= c\Psi(T) + \Psi(S)\end{aligned}$$

$\forall\, c \in \mathbb{F}$ and $T, S \in \mathcal{L}(V)$

Hence $\Psi \in \mathcal{L}(\mathcal{L}(V), \mathbb{F}^{n\times n})$ and

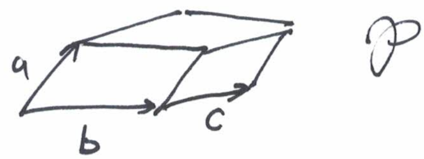
$$\begin{aligned}\Psi(T \circ S) &= [T \circ S]_{\beta\beta} \\ &= [T]_{\beta\beta}[S]_{\beta\beta} \\ &= \Psi(T)\,\Psi(S).\end{aligned}$$

In summary linear transformations on $V(\mathbb{F})$ are isomorphic to $n\times n$ matrices over $\mathbb{F}$ and the isomorphism is simply given by the matrix w.r.t. β map.

$\boxed{P89}$ Let $SL(3,\mathbb{R}) = \{ A \in \mathbb{R}^{3\times 3} \mid \det A = 1 \}$

Suppose $T(x) = Ax$ for some $A \in SL(3,\mathbb{R})$.

Recall parallel-piped $\mathcal{P}$ with sides a, b, c has $\text{vol}(\mathcal{P}) = |\det[a|b|c]|$



$T(\mathcal{P})$ has sides $T(a), T(b), T(c)$

So, calculate,

$$\begin{aligned}
\text{vol}(T(\mathcal{P})) &= |\det[T(a) \mid T(b) \mid T(c)]| \\
&= |\det[Aa \mid Ab \mid Ac]| \\
&= |\det(A[a|b|c])| \\
&= |\underbrace{\det A}_{1} \det[a|b|c]| \\
&= |\det[a|b|c]| \\
&= \text{vol}(\mathcal{P})
\end{aligned}$$

__Remark:__ I'm using a few results from the notes. In particular, the linear image of a line-segment is once more a line-segment or point... To be more self-contained we probably should examine

$$\mathcal{P} = \{ xa + yb + zc \mid 0 \le x,y,z \le 1 \}$$

$$\begin{aligned}
T(\mathcal{P}) &= \{ T(xa+yb+zc) \mid 0 \le x,y,z \le 1 \} \\
&= \underbrace{\{ xT(a) + yT(b) + zT(c) \mid 0 \le x,y,z \le 1 \}}_{\text{//-piped with sides } T(a),\, T(b),\, T(c).}
\end{aligned}$$

$\boxed{P90}$ $\beta = \{1, e_1, e_2, e_1 \wedge e_2\}$ is basis for $V = \operatorname{span}_{\mathbb{R}} \beta$

Suppose we define $\wedge$-product by "extending linearly" the table below

	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$-e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

We define $L_x : V \longrightarrow V$ by $L_x(y) = x \wedge y$. Hence as

$$L_x(y_1 + c y_2) = x \wedge (y_1 + c y_2) = x \wedge y_1 + c\, x \wedge y_2$$
$$= L_x(y_1) + c L_x(y_2)$$

we have $L_x \in \mathcal{L}(V)$ for each $x \in V$. Hence calculating $[L_x]_{\beta\beta} \in \mathbb{R}^{4 \times 4}$ is reasonable,

(a.) $$[L_{e_2}]_{\beta\beta} = \big[\, [e_2 \wedge 1]_\beta \,\big|\, [e_2 \wedge e_1]_\beta \,\big|\, [e_2 \wedge e_2]_\beta \,\big|\, [e_2 \wedge (e_1 \wedge e_2)]_\beta \,\big]$$
$$= \big[\, [e_2]_\beta \,\big|\, [-e_1 \wedge e_2]_\beta \,\big|\, 0 \,\big|\, 0 \,\big]$$
$$= \underline{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}}.$$

$$[L_{e_1 \wedge e_2}]_{\beta\beta} = \big[\, [e_1 \wedge e_2]_\beta \,\big|\, [e_1 \wedge e_2 \wedge e_1]_\beta \,\big|\, [e_1 \wedge e_2 \wedge e_2]_\beta \,\big|\, [e_1 \wedge e_2 \wedge e_1 \wedge e_2]_\beta \,\big]$$
$$= \underline{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}.$$

$$[L_1]_{\beta\beta} = \big[\, [1 \wedge 1]_\beta \,\big|\, [1 \wedge e_1]_\beta \,\big|\, [1 \wedge e_2]_\beta \,\big|\, [1 \wedge e_1 \wedge e_2]_\beta \,\big]$$
$$= \big[\, [1]_\beta \,\big|\, [e_1]_\beta \,\big|\, [e_2]_\beta \,\big|\, [e_1 \wedge e_2]_\beta \,\big]$$
$$= \underline{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

well, $L_1(x) = 1 \wedge x = x \ \forall x \in V$ so, this is not suprising!

P90 continued

$$[L_1]_{\beta\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[L_{e_1}]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow [L_{e_1}]_{\beta\beta}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

$$[L_{e_2}]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \Rightarrow [L_{e_2}]_{\beta\beta}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

$$[L_{e_1\wedge e_2}]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underbrace{[L_{e_1\wedge e_2}]_{\beta\beta}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{} \checkmark$$

well, I guess I didn't ask for this one ☺.

$$[L_{e_1}]_\beta [L_{e_2}]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = [L_{e_1\wedge e_2}]_{\beta\beta} \checkmark$$

$$[L_{e_2}]_\beta [L_{e_1}]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = -[L_{e_1\wedge e_2}]_{\beta\beta} \checkmark$$