Same instructions as Mission 1. Thanks!

Problem 101 Your signature below indicates you have:

(a.) I read Section 7.6 – 7.7 of Cook's lecture notes: ________________.

Problem 102 Let V be a finite dimensional real vector space and suppose $T : V \to V$ has $T^2 = Id_V$. Define $M = \{x \in V \mid T(x) = x\}$ and $N = \{x \in V \mid T(x) = -x\}$. Show that $V = M \oplus N$.

Problem 103 Let $S, T \in \mathcal{L}(V, W)$ and define $M = \{x \in V \mid S(x) = T(x)\}$. Show $M \leq V$.

Problem 104 Let $T : V \to W$ be a bijective linear mapping. For each $S \in \mathcal{L}(V)$ define $\phi(S) = T \circ S \circ T^{-1}$. Prove that $\phi : \mathcal{L}(V) \to \mathcal{L}(W)$ is a bijective linear mapping such that $\phi(R \circ S) = \phi(R) \circ \phi(S)$.

Problem 105 Let V be a vector space and $M, N \leq V$ and $x, y \in V$. Prove:

$$x + M \subseteq y + N \qquad \text{if and only if} \qquad M \subseteq N \text{ and } x - y \in N.$$

Problem 106 Suppose V is a vector space over $\mathbb{R}$. Furthermore, suppose $V_1 \leq V$ and $V_2 \leq V$. If $V_1 + V_2 = \{x + y \mid x \in V_1, y \in V_2\}$ then is $V_1 + V_2 \leq V$? Prove or disprove.

Problem 107 Show set of antisymmetric $n \times n$ matrices over a field $\mathbb{F}$ forms a subspace of $\mathbb{F}^{n \times n}$.

Problem 108 Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, 2x + y)$ find the matrix of T with respect to the basis $\beta = \{(1, 1), (1, -1)\}$. Also, find the matrix of $T^2 = T \circ T$ with respect to the β basis.

Problem 109 Let V be a finite dimensional vector space and $T \in \mathcal{L}(V)$. Prove:
$V = \text{Ker}(T) + \text{Im}(T)$ iff $\text{Ker}(T) \cap \text{Im}(T) = \{0\}$.
there is an interesting continuation to this problem when Ker(T) ∩ Im(T) ≠ {0}, perhaps we will return to it later in the course

Problem 110 Let N be the set of vectors in $\mathbb{R}^5$ whose last two coordinates are zero:

$$N = \{(x_1, x_2, x_3, 0, 0) \mid x_1, x_2, x_3 \in \mathbb{R}\}$$

Prove that $N \leq \mathbb{R}^5$ and show $\mathbb{R}^3 \approx N$ and $\mathbb{R}^5/N \approx \mathbb{R}^2$

Problem 111 Let V be a finite dimensional vector space and M, N subspaces of V. Prove that

$$\dim(M) + \dim(N) = \dim(M + N) + \dim(M \cap N)$$

Problem 112 If $V = W_1 \oplus W_2$ then show $\dim(V) = \dim(W_1) + \dim(W_2)$. Assume V is finite-dimensional.

Problem 113 Show that $\dim(V/W) = \dim(V) - \dim(W)$.

Problem 114 Consider the quotient $V = \mathbb{R}^{n \times n}/W$ where $W = \{A \in \mathbb{R}^{n \times n} \mid A^T = A\}$. Show V is isomorphic to the antisymmetric matrices.

Problem 115 Let V, W be finite dimensional vector spaces and suppose $S \subseteq V$. Further, you are given $T : V \to W$ is an isomorphism. Prove S is LI if and only if $T(S)$ is LI.

Problem 116 Suppose $T : V \to W$ is a linear transformation with $\ker(T) \neq \{0\}$. Let $U \leq V$ and consider the quotient vector space V/U. Suppose we define $\overline{T}(x+U) = T(x)$ for each $x + U \in V/U$. Is $\overline{T} : V/U \to W$ a well-defined function ? If needed, add a condition on U to make $\overline{T}$ a function.

Problem 117 Let $F \leq \mathcal{L}(V, W)$ be a subspace of non-injective linear transformations. Given that $\dim(V) = n$ and $\dim(W) = m$, What are the possible dimensions of F ?

Problem 118 Let $V^* = \mathcal{L}(V, \mathbb{R})$ for a finite dimensional vector space V over $\mathbb{R}$. Thus $\alpha \in V^*$ iff $\alpha : V \to \mathbb{R}$ is a linear transformation. Suppose $V = \mathbb{R}^n$ and define $e^i(v) = v^i$ for $v = \sum_{i=1}^n v^i e_i$ (this notation does not make v^i a power, rather it is just another notation for the i-th component). Show that $\beta^* = \{e^1, e^2, \dots, e^n\}$ forms a basis for V^* and show $e^i(e_j) = \delta_{ij}$.

Problem 119 If $\alpha(v_1, v_2, v_3) = v_1 + 3v_3$ then find $[\alpha]_{\beta^*}$ in view of the notation of the previous problem.

Problem 120 Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ with bases β and γ respective. Also, define dual spaces $V^* = \mathcal{L}(V, \mathbb{R})$ and $W^* = \mathcal{L}(W, \mathbb{R})$. If $T : V \to W$ is a linear transformation and $S : W^* \to V^*$ is defined by

$$(S(\alpha))(v) = \alpha(T(v))$$

for all $\alpha \in W^*$ and $v \in V$. **Then show S is a linear transformation and find $[S]_{\gamma^*, \beta^*}$.** Here, we define dual bases β^* and γ^* as follows: if $\beta = \{f_1, \dots, f_n\}$ and $\gamma = \{g_1, \dots, g_m\}$ then $f^j : V \to \mathbb{R}$ and $g^j : W \to \mathbb{R}$ are defined by linearly extending the formulas below:

$$f^j(f_i) = \delta_{ij} \qquad \& \qquad g^j(g_i) = \delta_{ij}.$$

Note, we set-aside the usual notation for exponents in this context; c^i is not the number c raised to the i-th power. A useful lemma is given by the following observation, if $x = \sum_{i=1}^n c^i f_i$ then $f^i(x) = c^i$. In other words, the dual vector f^i gives the i-coordinate of x upon evaluation. (your answer should relate the matrix for S to the matrix $[T]_{\beta,\gamma}$)

MISSION 6 SOLUTION : LINEAR ALGEBRA

P102 Let V be $\mathbb{R}$-vector space with $\dim V < \infty$ and a linear transformation $T: V \to V$ s.t. $T^2 = Id_V$.

Def$^{\underline{n}}$ $M = \{x \in V \mid T(x) = x\}$ & $N = \{x \in V \mid T(x) = -x\}$

Show $V = M \oplus N$

Let $x \in V$ then observe $x = \underbrace{\tfrac{1}{2}(x + T(x))}_{y} + \underbrace{\tfrac{1}{2}(-T(x) + x)}_{z}$

Consider, with y & z defined as above,

$$\begin{aligned} T(y) &= T\left[\tfrac{1}{2}(x + T(x))\right] \\ &= \tfrac{1}{2}T(x) + \tfrac{1}{2}T(T(x)) \\ &= \tfrac{1}{2}T(x) + \tfrac{1}{2}x \\ &= y \quad \therefore \ y \in M. \end{aligned}$$

$$\begin{cases} Id_V(x) = x \\ T^2(x) = T(T(x)) = x. \end{cases}$$

Likewise,

$$\begin{aligned} T(z) &= T\left(\tfrac{1}{2}(x - T(x))\right) \\ &= \tfrac{1}{2}T(x) - \tfrac{1}{2}T(T(x)) \\ &= \tfrac{1}{2}(T(x) - x) \\ &= -\tfrac{1}{2}(x - T(x)) \\ &= -z \quad \therefore \ z \in N \end{aligned}$$

But, $x = y + z$ was arbitrary pt. in $V \Rightarrow V = M + N$.
Next, consider, $x \in M \cap N$ thus $T(x) = x$ and $T(x) = -x$ thus $x = -x \Rightarrow \underline{x = 0}$. $\therefore M \cap N = \{0\}$. We have shown $\underline{V = M \oplus N}$.

(for two subspaces it suffices to show the sum of the subspaces gives the total space <u>and</u> the intersection of the subspaces is trivial.
For 3 or more it's a bit more subtle.

Btw, $M = \{\lambda = 1$ eigenspace of $T\}$ $N = \{\lambda = -1$ eigenspace of $T\}$

P103 Let $S, T \in \mathcal{L}(V, W)$ and $M = \{x \in V \mid S(x) = T(x)\}$.
Show $M \leq V$.

Observe,
$$\begin{aligned} M &= \{x \in V \mid S(x) = T(x)\} \\ &= \{x \in V \mid (S-T)(x) = 0\} \\ &= \mathrm{Ker}(S-T) \leq V. \end{aligned}$$
That is, M is a subspace of V as it is the Kernel of the linear transformation $S-T$. //

P104 Let $T : V \to W$ be a linear bijection. For each $S \in \mathcal{L}(V) = \mathcal{L}(V, V)$ define $\phi(S) = T \circ S \circ T^{-1}$. Prove $\phi : \mathcal{L}(V) \to \mathcal{L}(W)$ is a bijective linear mapping such that $\underline{\phi(R \circ S) = \phi(R) \circ \phi(S)}$ $*$

Consider, by properties of linear maps and the Lemma from Mission 6,
$$\begin{aligned} \phi(cS_1 + S_2) &= T \circ (cS_1 + S_2) \circ T^{-1} \\ &= c\, T \circ S_1 \circ T^{-1} + T \circ S_2 \circ T^{-1} \\ &= c\,\phi(S_1) + \phi(S_2) \end{aligned}$$
Thus ϕ is linear. Moreover
$$\phi(S) = T \circ S \circ T^{-1} : W \xrightarrow{T^{-1}} V \xrightarrow{S} V \xrightarrow{T} W$$
thus $\phi(S) : W \to W$ and is linear as it is formed by the composition of linear maps. Thus $\phi : \mathcal{L}(V) \to \mathcal{L}(W)$ is linear transformation. We could say $\phi \in \mathcal{L}(\mathcal{L}(V), \mathcal{L}(W))$.

- It remains to show ϕ is a bijection and $*$ holds.

Let $X \in \mathcal{L}(W)$ and consider, $T^{-1} \circ X \circ T : V \to V$ with $\phi(T^{-1} \circ X \circ T) = T \circ (T^{-1} \circ X \circ T) \circ T^{-1} = X$ thus ϕ is surjective. Now to prove ϕ is injective, notice $\phi^{-1}(Y) = T^{-1} \circ Y \circ T$ is an inverse mapping for ϕ;
$$\begin{aligned} \phi(\phi^{-1}(Y)) &= \phi(T^{-1} \circ Y \circ T) = T \circ T^{-1} \circ Y \circ T \circ T^{-1} = Y \\ \phi^{-1}(\phi(\tilde{Y})) &= \phi^{-1}(T \circ \tilde{Y} \circ T^{-1}) = T^{-1} \circ T \circ \tilde{Y} \circ T^{-1} \circ T = \tilde{Y} \end{aligned}$$
for all $Y \in \mathcal{L}(W)$ and $\tilde{Y} \in \mathcal{L}(V)$. continued

$\boxed{P104}$ it is easy to prove 1-1 from inverse existing,

$$\phi(X) = \phi(Y) \Rightarrow \phi^{-1}(\phi(X)) = \phi^{-1}(\phi(Y)) \Rightarrow X = Y.$$

Finally turn to $*$,

$$\begin{aligned} \phi(R \circ S) &= T \circ (R \circ S) \circ T^{-1} \\ &= T \circ R \circ T^{-1} \circ T \circ S \circ T^{-1} \\ &= (T \circ R \circ T^{-1}) \circ (T \circ S \circ T^{-1}) \\ &= \phi(R) \circ \phi(S). \end{aligned}$$

Notice $\mathcal{L}(V)$ and $\mathcal{L}(W)$ are <u>algebras</u> w.r.t. composition. The map ϕ is an <u>algebra</u> <u>isomorphism</u> it preserves both the vector space structure of $\mathcal{L}(V)$ and the composition operation as it transfers them to $\mathcal{L}(W)$.

$\boxed{P105}$ Let V be a vector space and $M, N \underset{\text{subspace}}{\leq} V$ and $x, y \in V$

Prove $x + M \underset{\text{subset}}{\subseteq} y + N \iff M \subseteq N$ and $x - y \in N$.

$\Rightarrow$ | Assume $x + M \subseteq y + N$. Let $m \in M$ then $x + m \in x + M \subseteq y + N \Rightarrow \exists n \in N$ s.t. $x + m = y + n$.

Also, $x + 0 \in x + M \subseteq y + N \Rightarrow x \in y + N \Rightarrow \exists n_2 \in N$ s.t. $x = y + n_2$

Put these facts together, $x = y + n - m = y + n_2 \Rightarrow n - m = n_2$

thus $m = n - n_2$ and as $n, n_2 \in N \Rightarrow m \in N \therefore M \subseteq N$.

Moreover, $x - y = n - m \in N$ as $m \in M \subseteq N \Rightarrow m \in N$ and $n \in N$ so $n - m \in N$ as N is subspace of V.

$\Leftarrow$ | Assume $M \subseteq N$ and $x - y \in N$. Let $z \in x + M$ then $\exists m \in M$ s.t. $z = x + m$. Now, $x - y \in N \Rightarrow x - y = n$ for some $n \in N$. Furthermore, note $m \in N$ as we have $M \subseteq N$.

Observe, $z = x + m = x - y + y + m = n + y + m = y + \underbrace{m + n}_{\in N} \in y + N$

thus $x + M \subseteq y + N$. //

P106 V a vector space over $\mathbb{R}$, $V_1 \leq V$ and $V_2 \leq V$.
If $V_1 + V_2 = \{x+y \mid x \in V_1,\ y \in V_2\}$ then is $V_1 + V_2 \leq V$?

Claim: $V_1 + V_2 \leq V$

Proof: By subspace theorem. Since V_1 & V_2 are subspaces of V they both contain $0 \in V$ hence $0+0 = 0 \in V_1 + V_2 \neq \emptyset$.
Let $z_1, z_2 \in V_1 + V_2$ then by definition of the sum $V_1 + V_2$ we have $\exists\, x_1, x_2 \in V_1$ and $y_1, y_2 \in V_2$ such that $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$. Let $c \in \mathbb{R}$ and consider,

$$\begin{aligned} c z_1 + z_2 &= c(x_1 + y_1) + x_2 + y_2 \\ &= \underbrace{(c x_1 + x_2)}_{\substack{\text{element of} \\ V_1 \text{ as } V_1 \leq V}} + \underbrace{(c y_1 + y_2)}_{\substack{\text{element of} \\ V_2 \text{ as} \\ V_2 \leq V}} \in V_1 + V_2 \end{aligned}$$

But, c, z_1, z_2 were arbitrary and it follows that $c z_1 + z_2 \in V_1 + V_2$ whenever $z_1, z_2 \in V_1 + V_2$ thus by subspace theorem we conclude $V_1 + V_2 \leq V$.

P107 Show $M = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\} \leq \mathbb{F}^{n \times n}$

Proof ①/ Observe $0^T = 0 = -0 \therefore 0 \in M \neq \emptyset$. If $A, B \in M$ then $(A+B)^T = A^T + B^T = -A - B = -(A+B)$ thus $A + B \in M$. Likewise, $A \in M$, $c \in \mathbb{R}$ we have $(cA)^T = c A^T = -(cA) \Rightarrow cA \in M$. Hence, by subspace th^{m} we conclude $M \leq \mathbb{F}^{n\times n}$. //

Proof ② Let $\psi(A) = (A + A^T)(\frac{1}{2})$ observe $\mathrm{Ker}(\psi) = M$ and ψ is clearly linear as

$$\begin{aligned} \psi(cA+B) &= \tfrac{1}{2}(cA + B + (cA+B)^T) \\ &= \tfrac{c}{2}(A + A^T) + \tfrac{1}{2}(B + B^T) \\ &= c\psi(A) + \psi(B). \end{aligned}$$

Therefore $M = \mathrm{Ker}\,\psi \leq \mathbb{F}^{n\times n}$ //

P108 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+2y, 2x+y)$ find $[T]_{\beta\beta}$ for $\beta = \{(1,1), (1,-1)\}$ and find matrix of T^2 w.r.t. β.

$$[\beta] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \hookrightarrow [\beta]^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Observe $[T] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is standard matrix of T thus, (see pg. 172 of my notes)

$$\begin{aligned} [T]_{\beta\beta} &= [\beta]^{-1}[T][\beta] \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = [T]_{\beta\beta}} \end{aligned}$$

Likewise, $[T^2]_{\beta\beta} = [T]_{\beta\beta}[T]_{\beta\beta} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} = [T^2]_{\beta\beta}}$

(what follows is bonus remark, you may ignore if you like)

Remark: $\mathbb{R}^2 = \mathbb{R} \oplus j\mathbb{R}$ with $j^2 = 1$ gives

$T(x+jy) = (1+2j)(x+jy)$

$\beta = \{1+j, 1-j\}$

$T(1+j) = (1+2j)(1+j) = 1+3j+2j^2 = 3(1+j)$

$T(1-j) = (1+2j)(1-j) = 1+2j-j-2j^2 = -1+j = -(1-j)$

That is $T(1+j) = 3(1+j)$

$T(1-j) = -1(1-j)$

$\beta = \{1+j, 1-j\}$ gives $[T]_{\beta\beta} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

the calculation above uses hyperbolic numbers, I happened to recognizize $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ as the matrix representative of $1+2j$. Multiplication by such matrices is interchangable with multiplication by hyperbolic # $1+2j \in \mathbb{R} \oplus j\mathbb{R}$.

$\boxed{P109}$ Prove for $T: V \to V$ a linear transformation,
$V = \text{Ker}(T) + T(V) \iff \text{Ker}(T) \cap T(V) = \{0\}$.

Observe $T: V \to V$ has both $\text{Ker}(T)$ and $T(V)$ as subspaces of V. Therefore we may apply the result of $\boxed{P111}$ to nicely solve this: $M = \text{Ker}\, T$, $N = T(V)$

$$\dim(\text{Ker}(T)) + \dim(T(V)) = \dim(\text{Ker}\, T + T(V)) + \dim(\text{Ker}\, T \cap T(V))$$

Notice, by rank-nullity theorem we have

$$\dim V = \dim \text{Ker}\, T + \dim T(V)$$

Thus,

$$\underline{\dim V = \dim(\text{Ker}\, T + T(V)) + \dim(\text{Ker}\, T \cap T(V))} \; *$$

Suppose $V = \text{Ker}\, T + \text{Im}(T)$ then from $*$ we deduce $\dim(\text{Ker}\, T \cap T(V)) = 0$ $\therefore$ $\underline{\text{Ker}\, T \cap T(V) = \{0\}}$.

Conversely, if $\text{Ker}\, T \cap T(V) = \{0\}$ then $\dim(\text{Ker}\, T \cap T(V)) = 0$ and we find $\dim V = \dim(\text{Ker}\, T + T(V))$. But, $\text{Ker}\, T + T(V) \leq V \Rightarrow \underline{\text{Ker}\, T + T(V) = V}$.

<u>Remark</u>: there is probably some direct argument to give here, I used <u>many</u> theorems to cipher the result desired.

P110 Let $N = \{(x_1, x_2, x_3, 0, 0) \mid x_1, x_2, x_3 \in \mathbb{R}\} \subseteq \mathbb{R}^5$
Prove $N \leq \mathbb{R}^5$ and show $\mathbb{R}^3 \approx N$ and $\mathbb{R}^5/N \approx \mathbb{R}^2$

Let $\psi : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ be defined by

$$\psi(x_1, x_2, x_3, x_4, x_5) = (x_4, x_5)$$

then observe

$$\begin{aligned} Ker(\psi) &= \{(x_1, \dots, x_5) \in \mathbb{R}^5 \mid (x_4, x_5) = (0,0)\} \\ &= \{(x_1, x_2, x_3, 0, 0) \mid x_1, x_2, x_3 \in \mathbb{R}\} \\ &= N \quad \therefore \quad \boxed{N = Ker(\psi) \leq \mathbb{R}^5} \end{aligned}$$

as ψ is clearly linear (~~product~~ observe $[\psi] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ if you prefer to think in those terms)

Also, notice $\psi(\mathbb{R}^5) = \mathbb{R}^2$

thus $\underbrace{\mathbb{R}^5/Ker(\psi) \approx Im(\psi)}_{1^{st} \text{ isomorphism theorem}} \implies \boxed{\mathbb{R}^5/N \approx \mathbb{R}^2}$

Finally, to see $\mathbb{R}^3 \approx N$ let $F : \mathbb{R}^3 \longrightarrow N$ be defined by $F(x, y, z) = (x, y, z, 0, 0)$. I invite the reader to verify F is a linear bijection. Hence $\mathbb{R}^3 \underset{F}{\approx} N$. //

P111 Let $M, N \leq V$ with $\dim V < \infty$. Prove that $\dim M + \dim N = \dim(M+N) + \dim(M \cap N)$

Let $\beta_1 = \{v_1, \dots, v_r\}$ be basis for M.
Let $\beta_2 = \{w_1, \dots, w_s\}$ be basis for N.

Observe $M+N = \text{span}(\beta_1 \cup \beta_2)$ as $x + y \in M+N$ has $x \in M$ and $y \in N$ hence $x = \sum' c_i v_i$ and $y \in \sum' b_j w_j$

so $x+y = \sum_i c_i v_i + \sum_j b_j w_j \in \text{span}\{v_1, \dots, v_r, w_1, \dots, w_s\} = \text{span}(\beta_1 \cup \beta_2)$

However, $\beta_1 \cup \beta_2$ might not be LI. If $\exists$ a linear dependence in $\beta_1 \cup \beta_2$ then it must arise between β_1 & β_2 as a lin. dep. in β_1 or β_2 alone is forbidden by LI of β_1 & β_2 $\rightarrow$

P111 continued

If $v_i = \sum_{j=1}^{s} b_j w_j$ then $v_i \in M$ and $v_i \in N$

thus $v_i \in M \cap N$. Modify $\beta_1 \cup \beta_2$ by removing each $v_i \in M \cap N$ until we are left with a LI set. We know this process is finite as $\dim(V) < \infty$.

$$\gamma = \{ v_i \in \beta_1 \mid v_i \in M \cap N \} \quad (\text{define } \# \gamma = p \leq r)$$

$$\beta_1 \cup \beta_2 - \gamma = \text{LI set which spans } M + N$$

By the $\pm$ theorem which stated we may remove vectors from a spanning set w/o altering the span provided each vector removed was itself in the span of the remaining vectors.

Finally, we need to explain why γ forms a basis for $M \cap N$. Notice $M \cap N$ is a subspace as is easily checked by the subspace theorem: $c \in \mathbb{R}$, $x, y \in M \cap N \Rightarrow x, y \in M$ and $x, y \in N \Rightarrow cx + y \in M$ and $cx + y \in N \Rightarrow cx + y \in M \cap N$.

Also, $\gamma \subseteq \beta_1$ thus γ is LI. Let $z \in M \cap N$ then $z \in M$ and $z \in N$ if $z \notin \text{span}\, \gamma$ then $\exists\, v_{i_1}, \dots, v_{i_n} \in \beta_1 - \gamma$ s.t. $z = c_{i_1} v_{i_1} + \cdots + c_{i_n} v_{i_n} \in N = \text{span}\, \beta_2$ thus $\exists$ a linear dependence between $\beta_1 - \gamma$ and β_2. However $\beta_1 \cup \beta_2 - \gamma$ was constructed to be LI hence this shows $z \notin \text{span}\, \gamma$ is impossible (except $z = 0$). Thus $\text{span}\, \gamma = M \cap N$ and we count,

$$\begin{aligned}
\dim M &= r = \#(\beta_1) \\
\dim N &= s = \#(\beta_2) \\
\dim M \cap N &= p = \#(\gamma) \\
\dim(M+N) &= r + s - p = \#(\beta_1 \cup \beta_2 - \gamma)
\end{aligned}$$

$$\therefore \boxed{\dim M + \dim N = \dim(M+N) + \dim(M \cap N)} \Leftrightarrow r + s = (r+s-p) + (p)$$

P112 Suppose $V = W_1 \oplus W_2$. Prove $\dim V = \dim W_1 + \dim W_2$

Observe, by assumption, if $V = W_1 \oplus W_2$ then $V = W_1 + W_2$ for $W_1, W_2 \leq V$ and $W_1 \cap W_2 = \{0\}$. Let β_1 be basis for W_1 and β_2 a basis for W_2. We argue $\beta = \beta_1 \cup \beta_2$ is a basis for V. First, if $z \in V$ then $\exists\, x, y \in V$ with $x \in W_1$ and $y \in W_2$ s.t. $z = x + y$. But $x \in \operatorname{span}\beta_1$ and $y \in \operatorname{span}\beta_2 \Rightarrow x = c_1 v_1 + \cdots + c_r v_r$ where $\beta_1 = \{v_1, \dots, v_r\}$ and $y = b_1 w_1 + \cdots + b_s w_s$ where $\beta_2 = \{w_1, \dots, w_s\}$. Hence

$$z = x + y = c_1 v_1 + \cdots + c_r v_r + b_1 w_1 + \cdots + b_s w_s \in \operatorname{span}(\beta_1 \cup \beta_2).$$

Thus $\beta_1 \cup \beta_2$ spans V. Next, consider

$$0 = \underbrace{c_1 v_1 + \cdots + c_r v_r}_{x} + \underbrace{b_1 w_1 + \cdots + b_s w_s}_{y} = 0 + 0$$

Hence, by uniqueness of expansion w.r.t. W_1 & W_2 respective we find $x = 0$ and $y = 0$. But,

$$x = 0 = c_1 v_1 + \cdots + c_r v_r \Rightarrow c_1 = 0, \dots, c_r = 0.$$
$$y = 0 = b_1 w_1 + \cdots + b_s w_s \Rightarrow b_1 = 0, \dots, b_s = 0.$$

Thus $\beta_1 \cup \beta_2$ is LI. Consequently $\beta = \beta_1 \cup \beta_2$ is basis for V

Note,
$$\#(\beta) = \#(\beta_1) + \#(\beta_2)$$
$$\underline{\dim V = \dim W_1 + \dim W_2} \quad //$$

Remark: probably I could have short-cut much of this by making use of my Lecture Notes. Well not quite, I actually refer to Hefferon for Th^{m} 7.7.11 part (iii) which is what we really need. In fact,

$$V = W_1 \oplus W_2 \iff \beta = \beta_1 \cup \beta_2 \text{ is basis for } V$$

given β_1 & β_2 bases for W_1, W_2 respective.

[P113] Show $\dim(V/W) = \dim(V) - \dim(W)$

Let $W \leq V$ and consider $\beta_W = \{w_1, \dots, w_r\}$ a basis for W.
Extend β_W to basis $\beta = \{w_1, \dots, w_r, v_1, \dots, v_p\}$ a basis for V.

Notice $\dim V = \#(\beta) = r + p$. We seek to show

$\gamma = \{v_1 + W, v_2 + W, \dots, v_p + W\}$

serves as a basis for the quotient V/W. I'll begin by examining LI of γ, consider:

$$c_1(v_1 + W) + c_2(v_2 + W) + \dots + c_p(v_p + W) = 0 + W$$

$$\Rightarrow c_1v_1 + c_2v_2 + \dots + c_pv_p + W = 0 + W$$

$$\Rightarrow c_1v_1 + c_2v_2 + \dots + c_pv_p \in W.$$

However, span $\{w_1, \dots, w_r\} = W$ and β is LI thus

$c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$ as $c_1v_1 + \dots + c_pv_p \neq 0$

would imply a linear dependence between $\{v_i\}$ & $\{w_i\}$.

Note then $c_1v_1 + c_2v_2 + \dots + c_pv_p = 0 \Rightarrow c_1 = 0, c_2 = 0, \dots, c_p = 0$
by LI of β thus γ is a LI set in V/W.

Next, consider $x + W \in V/W$ then $\exists c_1, \dots, c_r, b_1, \dots, b_p$
such that $x = c_1v_1 + \dots + c_pv_p + b_1w_1 + \dots + b_pw_p$ and

$$x + W = c_1v_1 + \dots + c_rv_r + W \overset{\text{def}^{\underline{n}}\text{ of addition in } V/W}{=} c_1(v_1 + W) + \dots + c_r(v_r + W)$$

as $x - c_1v_1 - \dots - c_pv_p = b_1w_1 + \dots + b_pw_p \in W$ (using Prop. 7.6.1)

Thus $x + W \in \text{span}\{v_1 + W, \dots, v_p + W\} = \text{span}(\gamma)$. Therefore, we find γ is basis for V/W with $\#(\gamma) = p$.

$$\dim(V/W) = p$$
$$\dim(V) - \dim(W) = (r + p) - r = p$$

$$\therefore \underline{\dim(V/W) = \dim V - \dim(W)} \;//$$

P114 Let $S = \{A \mid A^T = A\} = W$ and $A = \{A \mid A^T = -A\}$ where $S, A \subseteq \mathbb{R}^{n\times n}$. Show $\mathbb{R}^{n\times n}/W \approx A$.

Let $T : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ be defined such that $\mathrm{Ker}(T) = W$. A moments reflection suggests we use

$$T(A) = A - A^T$$

$$\mathrm{Ker}(T) = \{A \in \mathbb{R}^{n\times n} \mid A - A^T = 0\} = \{A \in \mathbb{R}^{n\times n} \mid A = A^T\} = W.$$

To see T is linear, just use properties of transpose,

$$\begin{aligned} T(cA + B) &= cA + B - (cA+B)^T \\ &= c(A - A^T) + (B - B^T) \\ &= c\,T(A) + T(B). \end{aligned}$$

Consider $B \in T(\mathbb{R}^{n\times n}) \Rightarrow B = A - A^T$ for some $A \in \mathbb{R}^{n\times n}$

But, then $B^T = (A - A^T)^T = A^T - A = -(A - A^T) = -B$

hence $T(\mathbb{R}^{n\times n})$ is comprised of antisymmetric matrices.

Moreover, $T(B/2) = \frac{B}{2} - \left(\frac{B}{2}\right)^T = \frac{B}{2} + \frac{B}{2} = B$ for each antisymmetric matrix $\therefore T(\mathbb{R}^{n\times n}) = A$.

The 1st isomorphism theorem applied to $T : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ tells us that

$$\mathbb{R}^{n\times n}/\mathrm{Ker}(T) \approx T(\mathbb{R}^{n\times n}) \quad \therefore \quad \underline{\mathbb{R}^{n\times n}/W \approx A}.$$

P115 Let V, W be v-spaces of finite dimension. Let $S \subseteq V$. Let $T: V \longrightarrow W$ be an isomorphism. Prove S LI $\Leftrightarrow$ $T(S)$ is LI

Let $S = \{ s_1, s_2, \dots, s_r \}$.

$\Rightarrow$) Suppose S is LI. Consider $T(S) = \{ T(s_1), T(s_2), \dots, T(s_r) \}$ if $c_1 T(s_1) + c_2 T(s_2) + \cdots + c_r T(s_r) = 0$ then by linearity $T(c_1 s_1 + \cdots + c_r s_r) = 0$. But T is invertible hence $T^{-1}(0) = 0$ and we find $\underbrace{c_1 s_1 + \cdots + c_r s_r = 0}_{*}$. Note LI of S then reveals from $*$ that $c_1 = 0, \dots, c_r = 0$ thus $T(S)$ is LI.

$\Leftarrow$) Suppose $T(S)$ is LI. Consider $c_1 s_1 + \cdots + c_r s_r = 0$ observe $T(c_1 s_1 + \cdots + c_r s_r) = T(0)$ yields

$$c_1 T(s_1) + \cdots + c_r T(s_r) = 0$$

Hence $c_1 = 0, \dots, c_r = 0$ by LI of $T(S)$. Therefore, S is LI //

P116 Suppose $T: V \longrightarrow W$ is lin. transformation with $Ker(T) \neq \{0\}$ Let $U \leq V$ and consider quotient V/U. Suppose we define $\overline{T}(x + U) = T(x)$ for each $x + U \in V/U$. Is $\overline{T}$ so defined a well-defined function? If needed, add a condition on U to make certain $\overline{T}$ is a function

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We need $\overline{T}(x + U) = \overline{T}(x' + U)$ whenever $x + U = x' + U$ this means we need $T(x) = T(x') \Leftrightarrow T(x - x') = 0$. Thus $x - x' \in Ker(T)$ is a needed condition. But, $x + U = x' + U$ just tells us $x - x' \in U$. One obvious fix, require $U \subseteq Ker(T)$ then $x - x' \in U \Rightarrow x - x' \in Ker(T)$ $\Rightarrow T(x - x') = 0 \Rightarrow T(x) = T(x') \Rightarrow \overline{T}(x + U) = \overline{T}(x' + U)$. //

$\therefore$ we require $\boxed{U \leq Ker(T)}$.

~~The~~ If the Kernel of T ~~must~~ contains U ~~for~~ then $\overline{T}$ ~~to be~~ is a function.

-(there are doubtless other ways to modify $\overline{T}$ to get a funct.)-

P117 Let $F \leq \mathcal{L}(V, W)$ be a subspace of non-injective linear transformations. Given that $\dim(V) = n$ and $\dim(W) = m$. What are the possible dimensions of F? (fortunately I made announcement before this befalled you ☺)

We can think about this question in terms of matrices which represent these spaces. Since $\dim V = n$ and $\dim(W) = m$ it follows $T \in \mathcal{L}(V, W)$ can be exchanged for multiplication by $[T]_{\beta,\gamma}$ via coordinate isomorphisms $\Phi_\beta : V \to \mathbb{R}^n$, $\Phi_\gamma : W \to \mathbb{R}^m$

$$T = \Phi_\gamma^{-1} \circ L_{[T]_{\beta,\gamma}} \circ \Phi_\beta$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi_\beta"'] & W \\ \mathbb{R}^n \arrow[r, "L_{[T]_{\beta,\gamma}}"] & \mathbb{R}^m \arrow[u, "\Phi_\gamma^{-1}"'] \end{tikzcd}$$

$$\psi : \mathcal{L}(V, W) \to \mathbb{R}^{m \times n}$$

$$\psi(T) = [T]_{\beta,\gamma}$$

we can show ψ is an isomorphism.

$$F \leq \mathcal{L}(V, W) \implies \psi(F) \leq \mathbb{R}^{m \times n}$$

Notice F non-injective $\Rightarrow \operatorname{Ker}(f) \neq \{0\}$ for each $f \in F$

$\Rightarrow \operatorname{Null}(\psi(f)) \neq \{0\}$ for each $f \in F$

Thus, we seek subspaces of matrices for which the null space of each matrix is non zero. Oh noes! This problem is <u>not reasonable</u>. Consider,

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\substack{\text{non-injective} \\ \text{map} \\ \text{induced}}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{noninjective} \\ \text{map} \\ \text{induced}}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{induces} \\ \text{identity} \\ \text{map}}} \xleftarrow{\ \psi\ } \Pi_1 + \Pi_2 = Id_{\mathbb{R}^2}$$

the sum (and scalar multiple, think about $c = 0$) of non-injective maps need not be non-injective!

P118 Let the dual space of V be $V^* = \mathcal{L}(V, \mathbb{R})$.
Thus $\alpha \in V^*$ iff $\alpha : V \to \mathbb{R}$ is linear map.
Suppose $V = \mathbb{R}^n$ and define $e^i(v) = v^i$ for $v = \sum_{i=1}^{n} v^i e_i$.
Show $\beta^* = \{e^1, e^2, \dots, e^n\}$ forms basis for V^* and show $e^i(e_j) = \delta_{ij}$

Let $v = \sum_{i=1}^{n} v^i e_i$ for $v^1, v^2, \dots, v^n \in \mathbb{R}$. Consider, for $\alpha \in V^*$ we find,

$$\begin{aligned}
\alpha(v) &= \alpha\Big(\sum_{i=1}^{n} v^i e_i\Big) \\
&= \sum_{i=1}^{n} v^i \alpha(e_i) && \text{: by linearity of } \alpha \\
&= \sum_{i=1}^{n} \alpha(e_i)\, e^i(v) && \text{: def}^{\underline{n}} \text{ of } e^i \\
&= \Big(\sum_{i=1}^{n} \alpha(e_i) e^i\Big)(v) && \text{: def}^{\underline{n}} \text{ of } \underline{\text{fnct}} \text{ addition}
\end{aligned}$$

But, as the calculation above holds $\forall v \in \mathbb{R}^n$ we deduce $\alpha = \sum_{i=1}^{n} \alpha(e_i) e^i$ and as $\alpha(e_i) \in \mathbb{R}$ we find that $\alpha \in \text{span}\{e^1, e^2, \dots, e^n\}$. It remains to show β^* is LI, consider,

$$\underline{c_1 e^1 + c_2 e^2 + \dots + c_n e^n = 0}\ *$$

the above is a function equation. We can use the Lemma $e^i(e_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ to evaluate $*$ on e_j

$$c_1 e^1(e_j) + c_2 e^2(e_j) + \dots + c_n e^n(e_j) = 0(e_j) = 0$$

$$\Rightarrow \quad c_j = 0 \quad \text{for arbitrary } j \in \mathbb{N}_n$$

Thus $c_1 = c_2 = \dots = c_n = 0$ and we find β^* is LI and so $V^* = \text{span}\,\beta^*$ for LI β^* hence β^* a basis for V^*.

$\boxed{\underline{\text{Lemma}}: \ e^i(e_j) = \delta_{ij}}$

$\underline{\text{Proof}}$: Observe $e_j = \sum_{i=1}^{n} \delta_{ij} e_i \Rightarrow e^i(e_j) = \delta_{ij}$ as $\delta_{ij} = v^i$ for $v = e_j$. //

$\boxed{\text{P119}}$ If $\alpha(V_1, V_2, V_3) = V_1 + 3V_3$ then find $[\alpha]_{\beta^*}$ in terms of $\beta^* = \{e^1, e^2, e^3\}$ as defined in P118

$$\alpha(v) = v^1 + 3v^3 \quad \text{for} \quad v = (v^1, v^2, v^3)$$
$$= e^1(v) + 3e^3(v)$$
$$= (1\cdot e^1 + 0\cdot e^2 + 3\cdot e^3)(v) \hookrightarrow \alpha = e^1 + 0\cdot e^2 + 3\cdot e^3$$
$$\therefore \boxed{[\alpha]_{\beta^*} = (1, 0, 3)}$$

$\boxed{\text{P120}}$ Given $V = \operatorname{span}\beta$, $V^* = \operatorname{span}\beta^*$ & $W = \operatorname{span}\gamma$, $W^* = \operatorname{span}\gamma^*$. If $T : V \longrightarrow W$ a linear tranformation and we define $S^* : W^* \to V^*$ by $(S(\alpha))(v) = \alpha(T(v))$ $\forall v \in V$ and $\alpha \in W^*$. Then show $S \in \mathcal{L}(W^*, V^*)$ and calculate $[S]_{\gamma^*, \beta^*}$

Consider, for $\alpha, \beta \in W^*$ and $c \in \mathbb{R}$,

$$(S(c\alpha + \beta))(v) = (c\alpha + \beta)(T(v))$$
$$= c\,\alpha(T(v)) + \beta(T(v))$$
$$= (c\,S(\alpha) + S(\beta))(v) \quad \forall v \in V \Rightarrow S \in \mathcal{L}(W^*, V^*).$$

Next, consider, $\gamma^* = \{g^1, g^2, \dots, g^m\}$ so,

$$[S]_{\gamma^*, \beta^*} = \left[[S(g^1)]_{\beta^*} \,\middle|\, [S(g^2)]_{\beta^*} \,\middle|\, \cdots \,\middle|\, [S(g^m)]_{\beta^*} \right] \quad *$$

$$= \left[\begin{bmatrix} S(g^1)(f_1) \\ S(g^1)(f_2) \\ \vdots \\ S(g^1)(f_n) \end{bmatrix} \middle| \begin{bmatrix} S(g^2)(f_1) \\ S(g^2)(f_2) \\ \vdots \\ S(g^2)(f_n) \end{bmatrix} \middle| \cdots \middle| \begin{bmatrix} S(g^m)(f_1) \\ S(g^m)(f_2) \\ \vdots \\ S(g^m)(f_n) \end{bmatrix} \right]$$

$$= \begin{bmatrix} g^1(T(f_1)) & g^2(T(f_1)) & \cdots & g^m(T(f_1)) \\ \vdots & & & \\ g^1(T(f_n)) & g^2(T(f_n)) & \cdots & g^m(T(f_n)) \end{bmatrix} \quad **$$

$$= \begin{bmatrix} ([T(f_1)]_\gamma)^T \\ \vdots \\ ([T(f_n)]_\gamma)^T \end{bmatrix}$$

$$= \left[[T(f_1)]_\gamma \,\middle|\, \cdots \, [T(f_n)]_\gamma \right]^T$$

$$= ([T]_{\beta,\gamma})^T$$

I'll expand on * & ** next

P120 continued

$*$: $[S(g^1)]_{\beta^*} = (c_1, c_2, \dots, c_n)$ means that
$S(g^1) = c_1 f^1 + c_2 f^2 + \cdots + c_n f^n$ and we can
select the values of $c_1, c_2, \dots, c_n$ by using $f^i(f_j) = \delta_{ij}$

$$(S(g^1))(f_i) = c_1 \underbrace{f^1(f_i)}_{0} + \cdots + c_i \underbrace{f^i(f_i)}_{1} + \cdots + c_n \underbrace{f^n(f_i)}_{0} = c_i$$

Therefore,

$$[S(g^1)]_{\beta^*} = \big((S(g^1))(f_1), (S(g^1))(f_2), \dots, (S(g^1))(f_n) \big)$$

Of course this calculation holds for $\alpha \in V^*$ just the same

$$[\alpha]_{\beta^*} = (\alpha(f_1), \alpha(f_2), \dots, \alpha(f_n)).$$

$**$: $[T(f_1)]_\gamma^T = [g^1(T(f_1)), g^2(T(f_1)), \dots, g^m(T(f_1))]$

Observe $[T(f_1)]_\gamma = (c_1, c_2, \dots, c_m)$ indicates that
$T(f_1) = c_1 g_1 + c_2 g_2 + \cdots + c_m g_m$. However, as
$g^j(g_i) = \delta_{ij}$ we deduce $g^j(T(f_1)) = c_j$
hence $[T(f_1)]_\gamma^T = [g^1(T(f_1)), \dots, g^m(T(f_1))]$.

Likewise, for any $w \in W$ we have

$$[w]_\gamma = (g^1(w), g^2(w), \dots, g^m(w))$$

where $\gamma^* = \{g^1, \dots, g^m\}$ is the dual-basis to $\gamma = \{g_1, \dots, g_m\}$.

__Remark__: you'll notice the notation for $[T]_{\beta,\gamma}$ is replaced with $[T]_\beta^\gamma$ in other texts such as Damiano & Little (see Prop. 2.2.15, page 78, $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ translate $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$)

$$\cancel{T(v) = \sum_{i=1}^{m} \sum_{j=1}^{n} v^j [T(f_j)]_{\gamma\, i}\, g_i}$$

Sorry too cluttered see

It is instructive to explore how the matrix of $T: V \to W$ depends explicitly on the basis β for V and γ^* for W^*.

Assume $\dim V = n$ and $\dim W = m$,

$$T(v) = \sum_{i=1}^{m} g^i[T(v)]\, g_i \quad (\text{by } **)$$

$$= \sum_{i=1}^{m} g^i\Big[T\Big(\sum_{j=1}^{n} f^j(v) f_j\Big)\Big] g_i \;(\text{by } ** \text{ for } V)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f^j(v)\, g^i[T(f_j)]\, g_i$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \Big(\underbrace{g^i[T(f_j)]}_{A^i_j}\, \underbrace{f^j(v)}_{v^j}\Big) g_i$$

$$\left([T(v)]_\gamma\right)^i = \sum_{i=1}^{m}\sum_{j=1}^{n} A^i_j\, v^j$$

Our notation has been $\left([T]_{\beta,\gamma}\right)_{ij} = g^i(T(f_j))$

But, you can see that $\left([T]_\beta^\gamma\right)^i_j = g^i(T(f_j))$ is more reflective of the role β and γ^* play in defining the matrix of T. In particular, to use other bases $[T]_{\bar\beta}^{\bar\gamma} = \bar g^i(T(\bar f_j))$

we see immediately the transformed matrix transforms the same way as $\bar\beta$ but, instead of γ, as $\bar\gamma^*$ relates to γ^*. It turns out $\bar\beta$ & $\bar\beta^*$ are inversely related to β and β^* respective. This must occur as ~~&~~ $f^i(f_j) = \delta_{ij}$ and $\bar f^i(\bar f_j) = \delta_{ij}$ as well...