Same rules as Homework 1.

Problem 91 Your signature below indicates you have:

(a.) I read Chapter 5 of Cook's lecture notes: ________________.

Problem 92 Remember, we have many properties to use in addition to the cofactor formulae,

(a.) Calculate $\det(A)$ where $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 5 & 3 & 1 \end{bmatrix}$

(b.) Calculate $\det(B)$ where $B = \begin{bmatrix} 2 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 & 3 \\ 7 & 7 & 2 & 7 & 7 \\ 5 & 3 & 0 & 0 & 0 \end{bmatrix}$

(c.) Let A, B be as given in the previous problems. If $M = \left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & 3B \end{array}\right]$ then calculate $\det(M)$ via application of properties of determinants given in the lecture notes and the results of the previous pair of problems.

Problem 93 For which values of x is the matrix $M = \begin{bmatrix} x & 2 & 2 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix}$ invertible?

Problem 94 Square root of a matrix?

(a.) Consider the problem $A^2 = X$ if there is a matrix A for which this matrix equation has a solution then it would be reasonable to think of $A = \sqrt{X}$. Consider, if $\det(X) = -1$ is it possible to find a real matrix A for which $A^2 = X$?

(b.) Consider $X = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$. Find a real matrix A for which $A^2 = X$.

Problem 95 Solve $Ax + 3y = 7$ and $5x - By = 6$ by Cramer's rule. Comment on needed conditions on A, B for the solution to exist.

Problem 96 Let A be a matrix which is similar to $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. In other words, suppose there exists an invertible matrix P for which $B = P^{-1}AP$. Calculate $\det(A)$ and $\mathrm{trace}(A)$.

Problem 97 Calculate $\det \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix}$ by row reduction paired with the identities derived in the notes for elementary row operations.

Problem 98 Determinant calculates volume

(a.) Find the volume of the paralell piped with edges $(1,2,3), (2,3,3), (-1,-2,0)$.

(b.) Let P be a paralell piped with edges $u, v, w \in \mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$. Show that $\text{Vol}(T(P)) = |\det([T])|\text{Vol}(P)$. Use the concatenation proposition! This problem need not be difficult.

Problem 99 The cross product: For all $a, b, c \in \mathbb{R}^3$ we define

$$T(a,b) = \sum_{j=1}^{3} \left(\det[a|b|e_j]\right) e_j.$$

Show $T(a,b) \bullet c = \det[a|b|c]$ and thus $T(a,b) = -T(b,a)$ and $a \bullet T(a,b) = 0$.

Problem 100 A natural candidate for the cross product in $\mathbb{R}^4$ is given by extending the formula in the previous problem: for all $a, b, c \in \mathbb{R}^4$ we define

$$T(a,b,c) = \sum_{j=1}^{4} \left(\det[a|b|c|e_j]\right) e_j$$

Show: Show $T(a,b,c) \bullet d = \det[a|b|c|d]$. Use this identity to help prove the **claim**:
Claim: if $\{a,b,c\}$ is LI then $\{a,b,c,T(a,b,c)\}$ is LI.

Note: I think Problem 100 is easily as hard as the rest of these problems combined.

P92

(a.) $\det \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 5 & 3 & 1 \end{bmatrix} = 2(2-6) - 2(0-10) = -8+20 = \boxed{12} = \underline{\det A}.$

(b.) $\det \left[\begin{array}{cc|c|cc} 2 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 & 3 \\ \hline 7 & 7 & 2 & 7 & 7 \\ \hline 5 & 3 & 0 & 0 & 0 \end{array}\right] = -2\det \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 2 & -1 & 0 \\ 2 & 2 & 0 & 3 \\ 5 & 3 & 0 & 0 \end{bmatrix}$

$= -2(2)\det \begin{bmatrix} 0 & 2 & 0 \\ 2 & 2 & 3 \\ 5 & 3 & 0 \end{bmatrix} - 2(1)\det \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 3 \\ 5 & 3 & 0 \end{bmatrix}$

$= -4(-2)[0-15] - 2[2(0-9) - 2(0-15) + 2(6-10)]$

$= -120 - 2[-18+30-8]$

$= -120 - 2(4)$

$= \boxed{-128 = \det B}$

(c.) $\det \left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & 3B \end{array}\right] = \det(2A)\det(3B)$

$= 2^3(\det A)\, 3^5 \det(B)$

$= 8(12)(243)(-128)$

$= \boxed{-2,985,984}$

weird.

$M = \begin{bmatrix} x & 2 & 2 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix}$ for which values of x does M^{-1} exist?

P93

$\det(M) = \det \begin{bmatrix} x & 2 & 2 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix} = x(3-5) - 2(3-7) + 2(5-7)$

$\det M = -2x + 8 - 4 = -2x + 4 = 0 \Rightarrow \underline{x = 2}.$

Thus $\det M \neq 0$ provided $x \neq 2$. $\boxed{\text{Thus } M^{-1} \text{ exists for } x \neq 2}$

when $x = 2$ clearly $row_1(M) = 2\, row_2(M) \Rightarrow M^{-1}$ d.n.e.

P94

Square root of matrix

(a.) $A^2 = X$ if $\exists$ such a matrix A then it is reasonable to think of $A = \sqrt{X}$. Suppose $\det X = -1$ is it possible to solve $A^2 = X$?

$$\det(A^2) = \det(X) \Rightarrow \det AA = -1$$
$$\Rightarrow (\det A)(\det A) = -1$$
$$\Rightarrow (\det(A))^2 = -1 \quad \therefore \quad A \notin \mathbb{R}^{n\times n}$$
$$\Rightarrow \det(A) \notin \mathbb{R}$$

(b.) Consider $X = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$. Solve $A^2 = X$ for $A \in \mathbb{R}^{2\times 2}$

$(\det A)^2 = \det X = 7(22) - 10(15) = 154 - 150 = 4 \Rightarrow \underline{\det A = \pm 2}$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$

We face the equations:

$$a^2 + bc = 7$$
$$ac + dc = 15 \Rightarrow (a+d)c = 15$$
$$ab + bd = 10 \Rightarrow (a+d)b = 10$$
$$bc + d^2 = 22$$
$$ad - bc = \pm 2$$

$\frac{15}{c} = \frac{10}{b}$ if $a+d \neq 0$

$15b = 10c$

$\underline{3b = 2c}$.

Then, if $a+d \neq 0$ then $b = \frac{2}{3}c$ so,

$$a^2 + bc = a^2 + \tfrac{2}{3}c^2 = 7$$
$$bc + d^2 = \tfrac{2}{3}c^2 + d^2 = 22$$

$\underline{a^2 - d^2 = -15}$ (I)

$ad - \frac{2}{3}c^2 = \pm 2 \longrightarrow \underline{ad + d^2 = 22 \pm 2}$ (II)

$\underline{d(a+d) = 20 \text{ or } 24}$ (III)

Thus, $\frac{15}{c} = \frac{10}{b} = \frac{20}{d}$ or $\frac{24}{d} \longrightarrow 10d = 20b$ OR $10d = 24b$

$\underline{d = 2b}$ (IIIa) $\qquad \underline{5d = 12b}$ (III.b)

$d = 2\left(\frac{2}{3}\right)c$

Hence, $ad - bc = -2 \longrightarrow$

$$a\tfrac{4}{3}c - \tfrac{2}{3}c^2 = -2$$
$$4ac - 2c^2 = -6$$
$$4(15 - dc) - 2c^2 = 4\left(15 - \tfrac{4}{3}c^2\right) - 2c^2 = -6$$

$$60 - \left(\tfrac{16}{3} + 2\right)c^2 = -6$$
$$-\tfrac{22}{3}c^2 = \cancel{} -66$$

continued

p.94 continued

$$-\frac{22c^2}{3} = -66 \Rightarrow \frac{2c^2}{3} = 6 \Rightarrow c^2 = 9 \Rightarrow \underline{c = \pm 3}.$$

But, $b = \frac{2}{3}c = \frac{2}{3}(\pm 3) = \underline{\pm 2 = b}$.

and $d = \frac{4}{3}c = \frac{4}{3}(\pm 3) = \underline{\pm 4 = d}$.

Thus a should be chosen s.t. $\det\begin{bmatrix} a & \pm 2 \\ \pm 3 & \pm 4 \end{bmatrix} = -2$

that is, $\pm 4a - 6 = -2 \Leftrightarrow \pm 4a = 4 \Leftrightarrow \underline{a = \pm 1}$.

Consequently, $\boxed{A = \pm \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$ gives us sol$^{\underline{n}}$'s with $\det(A) = -2$

Notice, $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = X$ ✓

There are two more sol$^{\underline{n}}$'s which stem from (III.b).

$\underline{d = \frac{12}{5}b}$. or $\underline{b = \frac{5}{12}d}$.

$$b = \frac{2}{3}c \Rightarrow d = \frac{12}{5}\left(\frac{2}{3}c\right) = \underline{\frac{8}{5}c = d}.$$

$$ad - bc = 2 \Rightarrow a\left(\frac{8}{5}c\right) - \frac{2}{3}c^2 = 2$$

$$\frac{8}{5}ac - \frac{2}{3}c^2 = 2$$

$$\frac{8}{5}(15 - dc) - \frac{2}{3}c^2 = 2$$

$$24 - \frac{8}{5}c\left(\frac{8}{5}c\right) - \frac{2}{3}c^2 = 2$$

$$-\frac{64}{25}c^2 - \frac{2}{3}c^2 = 2 - 24 = -22$$

$$\left(\frac{-(64)(3) - 2(25)}{75}\right)c^2 = -22$$

$$\frac{-22(11)}{75}c^2 = -22 \therefore c^2 = \frac{75}{11}$$

Maybe an error in here ☹

to grader,
Sorry, I just found two answers, there may be two more to find
I'm moving on...

Hence $c = \pm\sqrt{\frac{75}{11}}$, $b = \frac{\pm 5}{12}\sqrt{\frac{75}{11}}$, $d = \frac{\pm 8}{5}\sqrt{\frac{75}{11}}$ let $\alpha = \pm\sqrt{\frac{75}{11}}$

$$A = \begin{bmatrix} a & \frac{5}{12}\alpha \\ \alpha & \frac{8}{5}\alpha \end{bmatrix}$$

(I checked it and it failed...)

P45

Solve $Ax + 3y = 7$ and $5x - By = 6$ by Cramer's Rule. State any conditions needed on A, B

$$\left.\begin{aligned} Ax + 3y &= 7 \\ 5x - By &= 6 \end{aligned}\right\} \longrightarrow \begin{bmatrix} A & 3 \\ 5 & -B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$x = \frac{\det\begin{bmatrix} 7 & 3 \\ 6 & -B \end{bmatrix}}{\det\begin{bmatrix} A & 3 \\ 5 & -B \end{bmatrix}} = \frac{-7B - 18}{-AB - 15} = \boxed{\frac{7B + 18}{AB + 15} = x}$$

$$y = \frac{\det\begin{bmatrix} A & 7 \\ 5 & 6 \end{bmatrix}}{-AB - 15} = \frac{6A - 35}{-AB - 15} = \boxed{\frac{35 - 6A}{AB + 15} = y}$$

These sol^n's exist only if $AB + 15 \neq 0$.
This is simply the condition that $\det\begin{bmatrix} A & 3 \\ 5 & -B \end{bmatrix} \neq 0$.
Fwiw, when $\det\begin{bmatrix} A & 3 \\ 5 & -B \end{bmatrix} = 0$ we might have ∞ly many sol^n's, to see that multiply by 6 and 7 to obtain
$6Ax + 18y = 42$ and $35x - 7By = 42$. We'd need $6A = 35$ and $18 = -7B$ $\therefore$ $A = 35/6$ & $B = \frac{-18}{7}$
in which case $\left\{\left(x, \frac{42 - 6(\frac{35}{6})x}{18}\right) \,\middle|\, x \in \mathbb{R}\right\}$ is sol^n set.
But, I just asked for the sol^n from Cramer's Rule so this extra bit was not required.

P96 Consider $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ such that $\exists P$ invertible such that $B = P^{-1}AP$. Calculate $\det A$ and trace (A)

Observe $A = PBP^{-1}$ hence,

$$\det A = \det(PBP^{-1}) = \det P \det B \det P^{-1} = \det \cancelto{1}{PP^{-1}} \det B = \boxed{15}$$

$$tr(A) = tr(PBP^{-1}) = tr(P^{-1}PB) = tr(IB) = tr(B) = \boxed{9}$$

P97

Calculate $\det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix}$ via row-reduction technique.

Prop. 8.4.4 tells us
$$\det(E_{r_i \leftrightarrow r_j}) = -1$$
$$\det(E_{kr_i \to r_i}) = k$$
$$\det(E_{r_i + br_j}) = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix} \xrightarrow[\text{①②}]{\substack{r_2-2r_1 \\ r_3-3r_1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 5 \\ 0 & 4 & 7 \end{bmatrix} \xrightarrow[\text{③④}]{\substack{4r_2 \\ 3r_3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & 20 \\ 0 & 12 & 21 \end{bmatrix} \xrightarrow[\text{⑤}]{r_3-r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & 20 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{⑥}]{r_2-20r_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R}$$

$$R = E_6 E_5 E_4 E_3 E_2 E_1 A$$

$$\det R = 12 = \underbrace{\det E_1}_{1} \underbrace{\det E_2}_{1} \underbrace{\det E_3}_{3} \underbrace{\det E_4}_{4} \underbrace{\det E_5}_{1} \underbrace{\det E_6}_{1} \det A$$

$$12 = 12 \det A \quad \therefore \quad \boxed{\det A = 1}$$

Remark: when you get used to this technique you can do a lot less writing. I merely write much to explain the idea.

P98

(a.) $\text{Vol} = \left| \det\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 3 & 3 & 0 \end{bmatrix} \right| = |1(0+6) - 2(0+6) - 1(6-9)| = |-6+3| = \boxed{3}$

(b.) $\text{Vol}(P) = |\det[u|v|w]|$ where P has edges u, v, w.

$T(P)$ has edges $T(u), T(v), T(w)$. Consider then,

$$\begin{aligned} \text{Vol}(T(P)) &= |\det[T(u)|T(v)|T(w)]| \\ &= |\det[[T]u|[T]v|[T]w]| \\ &= |\det([T][u|v|w])| \\ &= |\det[T] \det[u|v|w]| \\ &= |\det[T]| \, |\det[u|v|w]| \\ &= \underline{|\det[T]| \, \text{Vol}(P)} \quad // \end{aligned}$$

Concatenation prop.

(notice, to preserve volume, need $\det[T] = \pm 1$.)

P99 For all $a, b, c \in \mathbb{R}^3$ we define

$$T(a,b) = \sum_{j=1}^{3} (\det[a|b|e_j])\, e_j$$

$$T(a,b) = \sum_{j=1}^{3} (\det[a|b|e_j])\, e_j$$

$$= -\sum_{j=1}^{3} (\det[b|a|e_j])\, e_j$$

$$= -T(b,a).//$$

$$a \cdot T(a,b) = \sum_{j=1}^{3} \det[a|b|e_j]\, a \cdot e_j$$

multilinearity gives linearity in 3rd slot.

$$= \det\left[a \,\middle|\, b \,\middle|\, \sum_{j=1}^{3} (a \cdot e_j)\, e_j\right]$$

$$= \det[a|b|a]$$

$$= \underline{0}.// \quad \text{as column is repeated}$$

$$T(a,b) \cdot c = \sum_{j=1}^{3} \det[a|b|e_j]\, e_j \cdot c$$

linearity of 3rd column in det.

$$= \det\left[a \,\middle|\, b \,\middle|\, \sum_{j=1}^{3} (e_j \cdot c)\, e_j\right]$$

$$= \underline{\det[a|b|c]}.//$$

<u>Remark</u>: better way

① prove $T(a,b) \cdot c = \det[a|b|c]$

② apply ① to case $c = a$ to get $T(a,b) \cdot a = \det(a|b|a) = 0$.

P100 For $a, b, c \in \mathbb{R}^4$ we define

$$T(a,b,c) = \sum_{j=1}^{4} \left(\det[a|b|c|e_j]\right) e_j$$

Show $\{a,b,c\}$ LI $\Rightarrow \{a,b,c, T(a,b,c)\}$ is LI

Observe, $T(a,b,c) \cdot w = \sum_{j=1}^{4} (\det(a|b|c|e_j) e_j \cdot w$

$$= \det\left(a|b|c|\sum_{j=1}^{4} (e_j \cdot w) e_j\right)$$

$$= \det[a|b|c|w].$$

Observe

$$T(a,b,c) \cdot a = \det[a|b|c|a] = 0 : \text{I}$$
$$T(a,b,c) \cdot b = \det[a|b|c|b] = 0 : \text{II}$$
$$T(a,b,c) \cdot c = \det[a|b|c|c] = 0 : \text{III}$$

Consider,

$$c_1 a + c_2 b + c_3 c + c_4 \, T(a,b,c) = 0 \quad *$$

Take dot-product of $*$ with a, b, c respective and use I II III to reveal

$$c_1(a \cdot a) = 0, \quad c_2(b \cdot b) = 0, \quad c_3(c \cdot c) = 0$$

But $\{a,b,c\}$ is LI hence $a, b, c \neq 0 \Rightarrow a \cdot a, b \cdot b, c \cdot c \neq 0$

thus $\underline{c_1 = 0}$, $\underline{c_2 = 0}$ and $\underline{c_3 = 0}$. Hence,

$$c_4 \, T(a,b,c) = 0$$

and as we may argue $T(a,b,c) \neq 0$ we find $\underline{c_4 = 0}$ hence $\{a,b,c, T(a,b,c)\}$ is LI.

It remains to argue $T(a,b,c) \neq 0$. Notice at <u>least</u> one of $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{bmatrix}$, $\det\begin{bmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$

and $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$ must be nonzero as if they are all zero and we are to expand $\det(a|b|c|w]$ for $w \notin \text{span}\{a,b,c\}$ then we'd have $\det[a|b|c|w] = 0$ despite LI of $\{a,b,c,w\}$ which is impossible thus at least one of M_1, M_2, M_3, M_4 is nonzero hence $T(a,b,c) = (M_1, -M_2, M_3, -M_4) \neq (0,0,0,0)$.

call these $M_1, \ldots, M_4$

where for example $M_1 = \det\begin{bmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$ etc...