Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 7 of the notes: ____________________.

Problem 91 Let $W_1 = \text{span}\{x + x^2, 1 + x^3\}$ and $W_2 = \text{span}\{1 + x, x^2 + x^3\}$. Find a basis for $W_1 \cap W_2$.

Problem 92 Find a basis for $W_1 + W_2$ where W_1, W_2 are the subspaces of $P_3(\mathbb{R})$ described in the previous problem. Do your calculations check against Theorem 6.7.8 ?

Problem 93 Example 7.6.3 shows a calculational technique to find bases β, γ for which $T : \mathbb{R}^n \to \mathbb{R}^m$ has a matrix $[T]_{\beta,\gamma} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $r = \text{rank}(T)$. Follow that example (use technology for the row reductions!) to find such β, γ for $T : \mathbb{R}^4 \to \mathbb{R}^3$ with

$$[T] = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 3 & -2 & 6 \\ 2 & 0 & -4 & 0 \end{bmatrix}$$

Problem 94 Let $v = (7, 9)$ and suppose $\beta = \{(2, 2), (-1, 1)\}$. Calculate $[v]_\beta$.

Problem 95 Consider bases $\beta = \{x^2, x, 1\}$ and $\bar{\beta} = \{1, x - 2, (x - 2)^2\}$. Find the coordinate change matrix $P_{\beta,\bar{\beta}}$ for which $[v]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[v]_\beta$ for each $v \in P_2(\mathbb{R})$

Problem 96 Consider $\mathbb{R}^{2\times 2}$. We have the usual basis

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and less usual basis

$$\bar{\beta} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

(a.) Find the coordinate change matrix $P_{\beta,\bar{\beta}}$ for which $[A]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[A]_\beta$ for each $A \in \mathbb{R}^{2\times 2}$

(b.) Consider the mapping $L(A) = A^T$. Calculate $[L]_{\beta,\bar{\beta}}$.

Problem 97 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that:

$$T(v_1) = v_1, \qquad T(v_2) = 2v_1, \qquad T(v_3) = 3v_3$$

where $v_1 = (1, 1, 0)$ and $v_2 = (1, -1, 0)$ and $v_3 = (0, 0, 1)$. Find the standard matrix of T by an appropriate use of Proposition 7.5.7.

Problem 98 Suppose $T(f(x)) = f'(x) + f''(x)$ for $f(x) \in P_2(\mathbb{R})$.

(a.) Can you find a basis β for $P_2(\mathbb{R})$ such that $[T]_{\beta,\beta} = I_3$?

(b.) Find a subspace W with basis β_W and basis γ for $P_2(\mathbb{R})$ such that $T|_W : W \to P_2(\mathbb{R})$ has $[T|_W]_{\beta_W,\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Problem 99 Suppose T has matrix $[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ with respect to $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{E_{12} + E_{21}, I\} \subseteq \mathbb{R}^{2\times 2}$. Find the formula for $T : P_3(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ and find $[T]_{\bar{\beta},\bar{\gamma}}$ where $\bar{\beta} = \{x^3, x^2, x, 1\}$ and $\bar{\gamma} = \{2(E_{12} + E_{21}), 3I\}$.

Problem 100 Suppose $T : V \to W$ has $\text{Null}([T]_{\beta,\gamma}) = \text{span}\{(1, 1, 0), (0, 1, 2)\}$ and $\text{Col}([T]_{\beta,\gamma}) = \text{span}\{(1, 0, 1)\}$ where $\beta = \{1, x, x^2\}$ and $\gamma = \{e^t, \sin(t), \cos(t)\}$ are bases for V and W respective.

(a.) find $\text{Ker}(T)$ and $\text{Range}(T)$

(b.) find the formula for $T(a + bx + cx^2)$

Problem 101 Dual space has very nice applications to coordinate maps. In particular, given basis $\beta = \{v_1, \dots, v_n\}$ for V we define dual basis $\beta^* = \{v^1, \dots, v^n\} \subseteq V^*$ by the rule $v^i(v_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.

(a.) explain why $v^i(v_j) = \delta_{ij}$ for $1 \leq j \leq n$ suffices to define the linear map $v^i : V \to \mathbb{F}$,

(b.) prove $\Phi_\beta(x) = \sum_{i=1}^n v^i(x)e_i$,

(c.) explain why $[x]_\beta = (v^1(x), \dots, v^n(x))$.

Problem 102 The annihilator of a subspace is naturally constructed in the dual space. In particular, if $W \leq V$ then define $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for each } w \in W\}$

(a.) show $ann(W) \leq V^*$

(b.) if $W_1 \leq W_2 \leq V$ then show $ann(W_2) \subseteq ann(W_1)$

Remark: part (b.) of the above problem has a natural analog with the construction of the perpendicular space for a given $S \subseteq \mathbb{R}^n$. For example, the x-axis (W_1) is perpendicular to the yz-plane ($W_1^\perp$). Whereas the xy-plane (W_2) is perpendicular to the z-axis ($W_2^\perp$). So, note $W_1 \leq W_2$ has $W_2^\perp \leq W_1^\perp$. In view of this, perhaps the following problem is not too surprising:

Problem 103 Find an isomorphism from $W^\perp = \{x \in \mathbb{R}^n \mid x \bullet w = 0, \text{ for all } w \in W\}$ and $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for each } w \in W\}$.

Problem 104 Consider $V = P_3(\mathbb{R}) \times \mathbb{C}^{2\times 2}$ as a real vector space. If $S_n(\mathbb{R})$ denotes the symmetric $n \times n$ matrices then for what n (if any) is $V \cong S_n \times S_n$?

Problem 105 Consider $V = \mathbb{R}^3$ and the subspace $W = \text{span}\{(1, 1, 1)\}$. Find a basis and coordinate chart for V/W. Describe the geometry of the cosets in V/W

Problem 106 Consider $V = P_2(\mathbb{R})$ and the linear transformation $T(f(x)) = f'(x)$ find $\text{Ker}(T)$ and find the inverse mapping $S : P_2(\mathbb{R})/\text{Ker}(T) \to T(P_2(\mathbb{R}))$ given by $S(f(x)+\text{Ker}(T)) = T(f(x))$. This is a special case of what common slogan from calculus I?

Problem 107 Suppose S is a subset of V. If we define $S+W = \{s+W \mid s \in S\}$ for a subspace W of V.

(a.) if S is LI then is $S + W$ a LI in V/W ? Discuss.

(b.) if S is linearly dependent in V then is $S + W$ linearly dependent in V/W ? Discuss.

Problem 108 Show $\mathbb{R}^{n\times n}/A_n \cong S_n$ where S_n denoted the set of symmetric matrices and A_n denotes the set of antisymmetric matrices in $\mathbb{R}^{n\times n}$. *Hint: use the first isomorphism theorem wisely.*

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Is the set of rational functions over $\mathbb{R}$ a subspace of the set of continuous functions on $\mathbb{R}$?

(II.) Show $W = \{(a+bx^2, (a+2b, a-b)) \mid a, b \in \mathbb{R}\}$ is a subspace of $P_2(\mathbb{R}) \times \mathbb{R}^2$.

(III.) Consider $S = \{1+t^2, 1-t, 1+t+t^3, 2+t^3\} \subseteq \mathbb{R}[t]$. Find a basis β for span(S). Also, find the formula for $[a+bt+ct^2+dt^3]_\beta$.

(IV.) Let $\beta = \{1, (x-1), (x-1)^2\}$. Calculate $[ax^2+bx+c]_\beta$.
Hint: be smart, use Taylor's Theorem you learned in Calculus II.

(V.) Consider the set of quadratic forms in two variables x, y. Let $\gamma = \{x^2, y^2, xy\}$ and define the set of trivariate homogeneous polynomials of order two by

$$W = \text{span}\{x^2, y^2, xy\}.$$

Observe W can be viewed as a function space and as it is a span we find $W \leq \mathcal{F}(\mathbb{R}^2, \mathbb{R})$. If $v = 3x^2 + 2(x-y)y$ then calculate $[v]_\gamma$.

(VI.) Suppose $T : U \to V$ and $S : V \to W$ are linear transformations. Show that:

(a.) Range$(S \circ T) \subseteq$ Range(S)

(b.) Ker$(T) \subseteq$ Ker$(S \circ T)$

(VII.) Consider Aut$(V) = \{\Psi : V \to V \mid \Psi \text{ an isomorphism}\}$. Is Aut$(V) \leq \mathcal{L}(V)$? Here $\mathcal{L}(V)$ denotes the set of all linear mappings from V to V.

(VIII.) Investigate relation of $ann(W_1 + W_2)$ and $ann(W_1 \cap W_2)$.

(IX.) Let V be a vector space and $M, N \leq V$ and $x, y \in V$. Prove:

$$x + M \subseteq y + N \quad \text{if and only if} \quad M \subseteq N \text{ and } x - y \in N.$$