Reading §6.4 – 6.6 of Friedberg, Insel and Spence 5th edition and Chapter 10 of my 2016 Math 321 Lecture Notes would be wise. (see Canvas for the 2016 notes)

Problem 81 Text §6.3#22a, d page 365. (minimal solution)

Problem 82 Text §6.4#2 page 372. (is it normal, self-adjoint, neither ?...)

Problem 83 Text §6.4#10 page 373. (self-adjoint operator behavior)

Problem 84 Text §6.5#25 page 393. (on reflections and rotations in plane)

Problem 85 Note that that trace : $\mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear function hence trace $\in (\mathbb{R}^{n\times n})^*$. Recall $\langle A, B\rangle = \text{trace}(AB^T)$ defines an inner product on $\mathbb{R}^{n\times n}$. Find the Riesz vector for the trace functional.

Problem 86 Consider $R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the e-values and real e-vectors of R_z. Use your work on R_z to answer the following: if $R \in \text{SO}(3)$ with $\text{trace}(R) = 0$, then by what angle does R rotate?

Problem 87 Let $\text{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^TR = I, det(R) = 1\}$. Show that: If $R \in \text{SO}(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 88 Find eigenvalues and orthonormal eigenvectors for $Q(x, y) = x^2 + 4xy + y^2$. Change the formula for Q to eigencoordinates (I used $\bar{x}, \bar{y}$ for this concept in lecture). Geometrically, what is $x^2 + 4xy + y^2 = 1$?

Problem 89 Suppose $Q(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$. Write $Q(v) = v^TAv$ for a symmetric matrix A. Find an orthonormal eigenbasis for A and find coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$.
Hint: for this question to make sense, it must be that the matrix of Q has e-values $1, 1, 10$.

Problem 90 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3\times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 uu^T, \qquad E_2 = \lambda_2 vv^T, \qquad E_3 = \lambda_3 ww^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ and the fact you are given $Au = \lambda_1 u$ etc.

Problem 91 Notice $u = \frac{1}{\sqrt{3}}(1,-1,1)$ and $v = \frac{1}{\sqrt{2}}(0,1,1)$ and $w = \frac{1}{\sqrt{6}}(2,1,-1)$ form an orthonormal basis for $\mathbb{R}^3$. Find a matrix A with eigenvalues $12, 2, 18$ by making use of the construction of the last problem.

Problem 92 Define $\Upsilon(A,B) = AB + BA$ for all $A, B \in \mathbb{R}^{n \times n}$ show Υ is a symmetric, bilinear form.

Problem 93 Let V be a real vector space and $x, y \in V$. Define $x \otimes y : V^* \times V^* \to \mathbb{R}$ according to the rule $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Show $x \otimes y$ is a bilinear mapping on $V^* \times V^*$.

Problem 94 Continuing the construction in the last problem, if V has basis $\beta = \{v_1, \dots, v_n\}$ show $\Upsilon = \{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ serves as a basis for $\mathcal{B}(V^*)$. That is, show Υ is LI and that any bilinear mapping $V^* \times V^* \to \mathbb{R}$ can be expressed as a linear combination of the Υ maps.