MATH 321: SPECTRAL THEOREM, #, ♭ MULTILINEAR MAPS	MISSION 6 [20PTS]

Reading §6.4 – 6.6 of Friedberg, Insel and Spence 5th edition and Chapter 10 of my 2016 Math 321 Lecture Notes would be wise. (see Canvas for the 2016 notes)

Problem 81 Text §6.3#22a, d page 365. (minimal solution)

Problem 82 Text §6.4#2 page 372. (is it normal, self-adjoint, neither ?...)

Problem 83 Text §6.4#10 page 373. (self-adjoint operator behavior)

Problem 84 Text §6.5#25 page 393. (on reflections and rotations in plane)

Problem 85 Note that that trace : $\mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear function hence trace $\in (\mathbb{R}^{n\times n})^*$. Recall $\langle A, B\rangle = \text{trace}(AB^T)$ defines an inner product on $\mathbb{R}^{n\times n}$. Find the Riesz vector for the trace functional.

Problem 86 Consider $R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the e-values and real e-vectors of R_z. Use your work on R_z to answer the following: if $R \in SO(3)$ with trace$(R) = 0$, then by what angle does R rotate?

Problem 87 Let $SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^TR = I\}$. Show that: If $R \in SO(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 88 Find eigenvalues and orthonormal eigenvectors for $Q(x, y) = x^2+4xy$. Change the formula for Q to eigencoordinates (I used $\bar{x}, \bar{y}$ for this concept in lecture). Geometrically, what is $x^2 + 4xy = 1$?

Problem 89 Suppose $Q(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$. Write $Q(v) = v^TAv$ for a symmetric matrix A. Find an orthonormal eigenbasis for A and find coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$.
Hint: for this question to make sense, it must be that the matrix of Q has e-values $1, 1, 10$.

Problem 90 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3\times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 uu^T, \qquad E_2 = \lambda_2 vv^T, \qquad E_3 = \lambda_3 ww^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ and the fact you are given $Au = \lambda_1 u$ etc.

Problem 91 Notice $u = \frac{1}{\sqrt{3}}(1,-1,1)$ and $v = \frac{1}{\sqrt{2}}(0,1,1)$ and $w = \frac{1}{\sqrt{6}}(2,1,-1)$ form an orthonormal basis for $\mathbb{R}^3$. Find a matrix A with eigenvalues $12, 2, 18$ by making use of the construction of the last problem.

Problem 92 [illegible] define $\Upsilon(A,B) = AB + BA$ for all $A, B \in \mathbb{R}^{n\times n}$ show Υ is a symmetric, bilinear form.

moved to Quiz 6

Problem 93 Suppose (V, g) is a real geometry. Show (V^*, g^*) is also a real geometry given we define $g^*(\alpha, \beta) = g(\sharp\alpha, \sharp\beta)$.

Problem 94 Let (V, g) be a real geometry. Prove $\sharp \circ \flat = Id_V$ and $\flat \circ \sharp = Id_{V^*}$. See my notes for the necessary definitions.

Problem 93: Let V be a real vector space and $x, y \in V$. Define $x \otimes y : V^* \times V^* \to \mathbb{R}$ according to the rule $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Show $x \otimes y$ is a bilinear mapping on $V^* \times V^*$.

Problem 94 Continuing the construction in the last problem, if V has basis $\beta = \{v_1, \dots, v_n\}$ show $\Upsilon = \{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ serves as a basis for $\mathcal{B}(V^*)$. That is, show Υ is LI and that any bilinear mapping $V^* \times V^* \to \mathbb{R}$ can be expressed as a linear combination of the Υ maps.

MISSION 6 SOLUTION

P81 §6.3 # 22a, d p. 365 (minimal sol^{n})

(a.) $x + 2y - z = 12 \leftrightarrow \underbrace{[1, 2, -1]}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 12$

(Solve $(AA^{*})u = b$ then $s = A^{*}u$ is the minimal sol^{n} by Th^{m} 6.13)

$AA^{T}u = [1, 2, -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u = 6u = 12 \quad \therefore \quad \underline{u = 2}.$

Then, $s = A^{T}u = \boxed{(2, 4, -2)}$ (textbook sol^{n} incorrect)

(d.) $\begin{matrix} x + y + z - w = 1 \\ 2x - y + w = 1 \end{matrix} \leftrightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$AA^{T} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \therefore \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/6 \end{bmatrix}$

Then $s = A^{T}u = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/4 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1/4 + 2/6 \\ 1/4 - 1/6 \\ 1/4 \\ -1/4 + 1/6 \end{bmatrix} = \boxed{(7/12, 1/12, 1/4, -1/12)}$

(textbook agrees this time)

$\boxed{P82}$ §6.4 #2 p. 372

For each T, determine if T is normal, self-adjoint or neither. If possible find orthonormal eigenbasis for T

(a.) $T(a,b) = \underbrace{\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix}$ observe $A^T = A \Rightarrow \underbrace{T^* = T}_{T \text{ self-adjoint \& normal}}$ $(V = \mathbb{R}^2)$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix} = (\lambda-2)(\lambda-5) - 4$$
$$= \lambda^2 - 7\lambda + 6$$
$$= (\lambda-1)(\lambda-6) \quad \therefore \quad \underline{\lambda_1 = 1, \ \lambda_2 = 6}.$$

$A - I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \Rightarrow \underline{u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ has } \lambda_1 = 1}.$

$A - 6I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \Rightarrow \underline{u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ has } \lambda = 6}.$

$\{u_1, u_2\}$ is orthonormal e-basis for T.

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(b.) $V = \mathbb{R}^3$

$T(a,b,c) = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ clearly $A^T \neq A$ $\therefore$ <u>A not self-adjoint</u>

$AA^T = \begin{bmatrix} 2 & 5 & -6 \\ 5 & 25 & -10 \\ -6 & -10 & 45 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 17 & -9 & 20 \\ -9 & 30 & -10 \\ 20 & -10 & 25 \end{bmatrix}$ $\therefore$ <u>A is not normal</u>.

$\boxed{P82}$ continued

(c.) $V = \mathbb{C}$, $T(a,b) = (2a+ib,\ a+2b) = \underbrace{\begin{bmatrix} 2 & i \\ 1 & 2 \end{bmatrix}}_{A}\begin{bmatrix} a \\ b \end{bmatrix}$

$$A^* = \begin{bmatrix} 2 & -i \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ -i & 2 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 2 & 1 \\ -i & 2 \end{bmatrix}\begin{bmatrix} 2 & i \\ 1 & 2 \end{bmatrix} = \left[\begin{array}{c|c} 5 & 2i+2 \\ \hline -2i+2 & 5 \end{array}\right]$$

$$AA^* = \begin{bmatrix} 2 & i \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -i & 2 \end{bmatrix} = \left[\begin{array}{c|c} 5 & 2+2i \\ \hline 2-2i & 5 \end{array}\right] = A^*A \quad \therefore \underline{A \text{ is normal}}$$

$$\Rightarrow \boxed{T \text{ is normal}}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & i \\ 1 & 2-\lambda \end{bmatrix} = \underbrace{(\lambda-2)^2 - i}_{\lambda = 2 \pm \sqrt{i}} = \lambda^2 - 4\lambda + 4 - i$$

$$\sqrt{i} = \sqrt{e^{\pi i/2}} = e^{\pi i/4} = \frac{1+i}{\sqrt{2}} \qquad \lambda = 2 \pm \sqrt{i}$$

Then $\underline{\lambda_\pm = 2 \pm e^{\pi i/4}}$.

$$A - \lambda_+ I = \underbrace{\begin{bmatrix} -e^{\pi i/4} & i \\ 1 & -e^{\pi i/4} \end{bmatrix}}_{u - e^{\pi i/4} v = 0} \Rightarrow \vec{u}_+ = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ve^{\pi i/4} \\ v \end{bmatrix}$$

Next, the 2's cancel, $\qquad \therefore \underline{\vec{u}_+ = \begin{bmatrix} e^{\pi i/4} \\ 1 \end{bmatrix}}$.

$$A - \lambda_- I = \begin{bmatrix} e^{\pi i/4} & i \\ 1 & e^{\pi i/4} \end{bmatrix} \Rightarrow \underline{\vec{u}_- = \begin{bmatrix} -e^{\pi i/4} \\ 1 \end{bmatrix}}.$$

$$\langle \vec{u}_+, \vec{u}_- \rangle = [e^{\pi i/4}, 1]\begin{bmatrix} -e^{-\pi i/4} \\ 1 \end{bmatrix} = -(e^{\pi i/4})(e^{-\pi i/4}) + 1 = -1+1 = 0.$$

It remains to <u>normalize</u> $\vec{u}_\pm$,

$$\langle \vec{u}_+, \vec{u}_+ \rangle = |e^{\pi i/4}|^2 + |1|^2 = 2 = \langle \vec{u}_-, \vec{u}_- \rangle$$

$$\beta = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}}(1+i) \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}}(-1-i) \\ 1 \end{bmatrix} \right\}$$

$$\beta = \left\{ \left(\tfrac{1}{2}(1+i), 1/\sqrt{2}\right), \left(\tfrac{-1}{2}(1+i), 1/\sqrt{2}\right) \right\}$$

$$\lambda_+ = 2 + \frac{1+i}{\sqrt{2}} \qquad \lambda_- = 2 - \frac{1+i}{\sqrt{2}}$$

$\boxed{P82}$ continued

(d.) $V = P_2(\mathbb{R})$ define $T(f) = f'$ with $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$

Let $\beta = \{1, x, x^2\}$ and $A = [T]_{\beta,\beta}$

$$A = \left[\, [T(1)]_\beta \,\middle|\, [T(x)]_\beta \,\middle|\, [T(x^2)]_\beta \right] = \left[\, [0]_\beta \,\middle|\, [1]_\beta \,\middle|\, [2x]_\beta \right] = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{\substack{A \neq A^* \\ \text{not self-adjoint}}}$$

$$A^*A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$AA^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^*A \neq AA^*$

A not normal.

It follows T is likewise neither self-adjoint or normal.

(e.) $V = \mathbb{R}^{2\times 2}$ and $T(A) = A^T$

$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $[T]_{\beta\beta} = M$

$$M = \left[\, [E_{11}]_\beta \,\middle|\, [E_{21}]_\beta \,\middle|\, [E_{12}]_\beta \,\middle|\, [E_{22}]_\beta \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M^T \Rightarrow \boxed{T \text{ is self-adjoint}}$$

Notice $A^T = A \Rightarrow T(A) = A \therefore \lambda = 1$ (symmetric matrices)

Likewise $A^T = -A \Rightarrow T(A) = -A \therefore \lambda = -1$ (antisymmetric matrices)

$$\gamma = \Big\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}}_{\lambda = 1}, \underbrace{\begin{bmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}}_{\lambda = -1} \Big\}$$

$\boxed{P82}$ continued

(f.) $V = \mathbb{R}^{2\times 2}$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \qquad \text{Let } \beta = \{E_{11}, E_{12}, E_{21}, E_{22}\} \text{ and } [T]_{\beta\beta} = M$$

$$\left.\begin{aligned} T(E_{11}) &= E_{21} \\ T(E_{12}) &= E_{22} \\ T(E_{21}) &= E_{11} \\ T(E_{22}) &= E_{12} \end{aligned}\right\} \longrightarrow M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M^T \quad \therefore \ M \text{ is self-adjoint}$$

$\Rightarrow$ T is self-adjoint.

$$\det\begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix} = -\lambda \det\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} + \det\begin{bmatrix} 0 & -\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$$= -\lambda\left[-\lambda(\lambda^2) + 1(\lambda)\right] + \lambda(-\lambda) + 1(1)$$

$$= \lambda^4 - \lambda^2 - \lambda^2 + 1 = \lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = (\lambda+1)^2(\lambda-1)^2$$

$$u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$$

$$M - I = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \rightarrow u_1 = u_3 \\ \rightarrow u_2 = u_4 \end{matrix} \longrightarrow \begin{bmatrix} u_1 & u_2 \\ u_1 & u_2 \end{bmatrix}$$

(going back to V)

$$M + I = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \rightarrow u_1 = -u_3 \\ \rightarrow u_2 = -u_4 \end{matrix} \longrightarrow \begin{bmatrix} u_1 & u_2 \\ -u_1 & -u_2 \end{bmatrix}$$

Hence,

$$\gamma = \Bigg\{ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{\lambda = 1}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}}_{\lambda = -1} \Bigg\}$$

<u>Remark</u>: the texbook answers must be for different problems. sorry.

$\boxed{P83}$ § 6.4 # 10 p. 373

Let T be self-adjoint on finite dim'l inner-product space V. Prove $\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2$. Show $T-iI$ is invertible and $(T-iI)^{-1}$ has adjoint $(T+iI)^{-1}$

Suppose $T^* = T$ and let $x \in V$,

$$\underbrace{\langle T(x) \pm ix, T(x) \pm ix \rangle} = \langle T(x), T(x)\rangle + \langle T(x), \pm ix\rangle + \langle \pm ix, T(x)\rangle + \langle \pm ix, \pm ix\rangle$$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 \mp i\langle T(x), x\rangle \pm i\langle x, T(x)\rangle + \|\pm ix\|^2$$

$$= \|T(x)\|^2 \mp i\langle T(x), x\rangle \pm i\langle T^*(x), x\rangle + |\pm i|^2\|x\|^2$$

$$= \|T(x)\|^2 \mp i\langle T(x), x\rangle \pm i\langle T(x), x\rangle + \|x\|^2$$

$$= \underline{\|T(x)\|^2 + \|x\|^2}.$$

$T: V \to V$ is linear on finite dim'l V. Likewise, $T - iI : V \to V$ is linear on finite dim'l V. Let $x \in \mathrm{Ker}(T - iI)$ then $(T - iI)(x) = T(x) - ix = 0 \therefore T(x) = ix$.

But, $\|\underbrace{T(x) - ix}_{0}\|^2 = \|T(x)\|^2 + \|x\|^2 = \|ix\|^2 + \|x\|^2 = 2\|x\|^2 = 0$

hence $\|x\| = 0 \Rightarrow x = 0 \Rightarrow \mathrm{Ker}(T - iI) = 0 \therefore (T-iI)^{-1}$ exists since $\dim(V) = \dim(\mathrm{Ker}(T - iI)) + \dim(\mathrm{Range}(T - iI))$. implies $T - iI$ is onto (we can also use Th^m, $T: V \to V$ onto iff 1-1)

Consider,

$$(T - iI)^{-1}(T - iI) = I$$

$$\Rightarrow (T - iI)^*((T - iI)^{-1})^* = I^*$$

$$(T^* + iI^*)((T - iI)^{-1})^* = I$$

$$(T + iI)((T - iI)^{-1})^* = I \Rightarrow \underline{((T - iI)^{-1})^* = (T + iI)^{-1}}$$

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P84 §6.5 # 25, p. 393

Suppose T and U are reflections of $\mathbb{R}^2$ about the respective lines L and L' through origin with angles ϕ and ψ from the positive x-axis to L and L' respective. By exercise 24, UT is a rotation. Find the angle of UT's rotation.

From Example 5 we know the standard matrix of the reflection, hence

$$[T] = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \quad \& \quad [U] = \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{bmatrix}$$

Consider,

$$\begin{aligned}
[TU] &= [T][U] \\
&= \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{bmatrix} \\
&= \left[\begin{array}{c|c} \cos 2\phi \cos 2\psi + \sin 2\phi \sin 2\psi & \cos 2\phi \sin 2\psi - \sin 2\phi \cos 2\psi \\ \hline \sin 2\phi \cos 2\psi - \cos 2\phi \sin 2\psi & \sin 2\phi \sin 2\psi + \cos 2\phi \cos 2\psi \end{array}\right] \\
&= \left[\begin{array}{c|c} \cos(2\phi - 2\psi) & -\sin(2\phi - 2\psi) \\ \hline \sin(2\phi - 2\psi) & \cos(2\phi - 2\psi) \end{array}\right]
\end{aligned}$$

Thus $\boxed{2(\phi - \psi)}$ is the angle of the TU rotation.

Likewise, as I was supposed to calculate, the UT rotation has angle $2(\psi - \phi)$.

$\boxed{P85}$ Note that trace : $\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ defines a dual vector in $(\mathbb{R}^{n\times n})^*$. The Riesz' vector $\#$trace has $\underbrace{\langle x, \#\text{trace}\rangle}$ = trace(x)

$$\text{trace}\,(x\,(\#\text{trace})^T) = \text{trace}\,(x) \quad \forall x \in \mathbb{R}^{n\times n}$$

apparently $\boxed{\#\text{trace} = I}$ then

$$\langle x, I\rangle = \text{trace}\,(xI^T) = \text{trace}\,(x) \quad \forall x \in \mathbb{R}^{n\times n}.$$

$\boxed{P86}$ Consider $R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{R_2(\cos\theta + i\sin\theta) \oplus J_1(1)}^{\text{(using Real Jordan form discussion)}}$

$\lambda = \cos\theta \pm i\sin\theta$ and $\lambda = 1$

$$\det(R_z - \lambda I) = \det\begin{bmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = \left[(\lambda - \cos\theta)^2 + \sin^2\theta\right](1-\lambda) = 0$$

$$\Rightarrow \underline{\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta} \text{ or } \lambda = 1}.$$

Observe $R_z e_3 = e_3$ thus $\mathcal{E}_{\lambda=1} = \text{span}\{(0,0,1)\}$.

In $\boxed{P87}$ we show $R \in SO(3)$ s.t. $R \neq I$ has form R_z if the transformation has complex e-value. Since trace$(R) = 0$ is given it follows ~~one~~ two of the eigenvalues are complex hence $\exists\, \beta = \{v_1, v_2, v_3\}$ for which $L = L_R$ has

$$[L]_{\beta\beta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\beta]^{-1} R\,[\beta]$$

taking the trace of the equation above yields,

$$2\cos\theta + 1 = \text{trace}\,[\beta]^{-1}[\beta]R = \text{tr}\,(R) = 0$$

Hence $\cos\theta = -1/2$

$$\Rightarrow \boxed{\theta = \frac{2\pi}{3}}$$

$\boxed{P87}$ Let $SO(3) = \{ R \in \mathbb{R}^{3\times 3} \mid R^T R = I \}$

If $R \in SO(3)$ and $R \neq I$ then show R has only two e-vectors of unit-length with $\lambda = 1$

Let $\lambda \in \mathbb{R}$ be eigenvalue of R then as $R^T R = R R^T = I$ calculate for $Rx = \lambda x$, with $x \neq 0$

$$\langle x, x \rangle = \langle R^T R\, x, x \rangle = \langle Rx, Rx \rangle = \langle \lambda x, \lambda x \rangle = \lambda^2 \langle x, x \rangle$$

thus $\lambda^2 = 1$ and so $\underline{\lambda = \pm 1}$. If we regard $R \in SO(3)$ as the matrix for $L_R : \mathbb{R}^3 \to \mathbb{R}^3$ then we may complexify and then $R^T R = I = R R^T$ yields normality and hence R is complex diagonalizable. Anyway, we don't need all that. What we do need is to notice

$$\det(R) = \lambda_1 \lambda_2 \lambda_3$$

where either $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ or $\lambda_1 \in \mathbb{R}$ and $\overbrace{\bar{\lambda}_2 = \lambda_3 \in \mathbb{C}}^{\text{follows since } char_R(t) \text{ is real poly.}}$

① If $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then $[L_R]_{\beta,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [\beta]^{-1} R [\beta] = I$
hence $\lambda_1 = \lambda_2 = \lambda_3 = 1$ cannot happen as $R \neq I$. $\Rightarrow \underline{R = I}$.

② If $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$ then $\dim(\mathcal{E}_{\lambda=1}) = 1$ and so $\exists$ exactly two unit-vectors with $\lambda = 1$.

③ If $\lambda_1 \in \mathbb{R}$ and $\lambda_2 = b + ic$ and $\lambda_3 = b - ic$. Note $\lambda_1 = \pm 1$ however, as $\det(R) = 1 = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 (b+ic)(b-ic) = \lambda_1 (b^2 + c^2)$ we find $\lambda_1 = \frac{1}{b^2 + c^2} > 0$ hence $\underline{\lambda_1 = 1}$.

Once more, $\dim(\mathcal{E}_{\lambda=1}) = 1$ and $\exists$ exactly two unit-vectors with $\lambda = 1$.

$\boxed{P88}$ $Q(x,y) = x^2 + 4xy + y^2$

$$Q(v) = [x,y]^T \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = v^T A v$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - 4 = (\lambda-3)(\lambda+1) = 0$$

$$A + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \& \quad A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\text{Null}(A+I) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad \text{Null}(A-3I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Let $\beta = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ then $\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = [\beta]^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$

or better yet $\begin{bmatrix} x \\ y \end{bmatrix} = [\beta] \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{x} + \bar{y} \\ \bar{y} - \bar{x} \end{bmatrix}$

We find $x = \frac{1}{\sqrt{2}}(\bar{x} + \bar{y})$ whereas $y = \frac{1}{\sqrt{2}}(\bar{y} - \bar{x})$

hence,

$$Q(x,y) = \left(\frac{1}{\sqrt{2}}(\bar{x}+\bar{y}) \right)^2 + 4\left(\frac{\bar{x}+\bar{y}}{\sqrt{2}} \right)\left(\frac{\bar{y}-\bar{x}}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}}(\bar{y}-\bar{x}) \right)^2$$

$$= \frac{1}{2}\left(\bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2 \right) + 2\left(\bar{y}^2 - \bar{x}^2 \right) + \frac{1}{2}\left(\bar{y}^2 - 2\bar{x}\bar{y} + \bar{x}^2 \right)$$

$$= \left(\frac{1}{2} - 2 + \frac{1}{2} \right)\bar{x}^2 + \left(\frac{1}{2} + 2 + \frac{1}{2} \right)\bar{y}^2$$

$$= \boxed{-\bar{x}^2 + 3\bar{y}^2} = 1$$

geometrically this is a hyperbola

No surprise here!

$$Q(v) = v^T A v = v^T [\beta] \underbrace{[\beta]^T A [\beta]} [\beta]^T v$$

$$= ([\beta]^T v)^T \text{diag}(\lambda_1, \lambda_2) [\beta]^T v$$

$$= [\bar{x}, \bar{y}] \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}.$$

$\boxed{P89}$

$$Q(x,y,z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$$

$$= [x,y,z] \underbrace{\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q(v)$$

Find eigen coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$

$$A - I = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow u = -v - \frac{w}{2} \Rightarrow \begin{bmatrix} -v - w/2 \\ v \\ w \end{bmatrix}$$

$$A - 10I = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} u = 2w \\ v = 2w \end{matrix} \Rightarrow \begin{bmatrix} 2w \\ 2w \\ w \end{bmatrix}$$

$$\text{Null}(A - 10I) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\text{Null}(A - I) = \text{span}\Big\{ \underbrace{\overset{v_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}}_{\text{not orthonormal, need to run G.S.A.}} \Big\}$$

$$v_1' = v_1$$

$$v_2' = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

Let $\beta = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$

then $\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = [\beta]^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ define eigen coordinates

for which $Q(v) = [x,y,z] A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\bar{x}, \bar{y}, \bar{z}] [\beta]^T A [\beta] \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$

$$= [\bar{x}, \bar{y}, \bar{z}] \begin{bmatrix} 1 & & \\ & 1 & \\ & & 10 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$$

$$= \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2.$$

P.90 Suppose $\{u, v, w\}$ forms an orthonormal e-basis for $A \in \mathbb{R}^{3\times 3}$ with e-values $\lambda_1, \lambda_2, \lambda_3$ respective ($Au = \lambda_1 u$, $Av = \lambda_2 v$, $Aw = \lambda_3 w$)

Define: $E_1 = \lambda_1 uu^T$, $E_2 = \lambda_2 vv^T$, $E_3 = \lambda_3 ww^T$

Show: $E_1 + E_2 + E_3 = A$ and $Col(E_j) = Null(A - \lambda_j I)$ for $j = 1, 2, 3$

Consider $L(x) = (E_1 + E_2 + E_3)x$ and $L_A(x) = Ax$.
If L and L_A agree on a basis for $\mathbb{R}^3$ then $L = L_A$ and thus $E_1 + E_2 + E_3 = A$. Consider,

① $L_A(u) = Au = \lambda_1 u$

$$L(u) = (E_1 + E_2 + E_3)u = (\lambda_1 uu^T + \lambda_2 vv^T + \lambda_3 ww^T)u$$
$$= \lambda_1 u \underbrace{u^T u}_{0} + \lambda_2 v \underbrace{v^T u}_{0} + \lambda_3 w \underbrace{w^T u}_{0}$$

thus $L(u) = \lambda_1 u$ — Recall $x \cdot y = x^T y$.

and we find $L_A(u) = L(u) = \lambda_1 u$.

② $L_A(v) = Av = \lambda_2 v$

$$L(v) = \lambda_1 u \underbrace{u^T v}_{0} + \lambda_2 v \underbrace{v^T v}_{1} + \lambda_3 w \underbrace{w^T v}_{0} = \lambda_2 v = L_A(v).$$

③ $L_A(w) = Aw = \lambda_3 w$

$$L(w) = \lambda_1 u \underbrace{u^T w}_{0} + \lambda_2 v \underbrace{v^T w}_{0} + \lambda_3 w \underbrace{w^T w}_{1} = \lambda_3 w = L_A(w)$$

Therefore, $\boxed{A = E_1 + E_2 + E_3}$

scalars based on λ_1 & components of u

Observe $E_1 = \lambda_1 uu^T = [\,* u \,|\, * u \,|\, @ u\,]$ $\therefore$ $Col(E_1) = span\{u\}$

But, $Null(A - \lambda_1 I) = span\{u\}$ since λ_1 has geometric multiplicity 1. (oops! You need $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$)

Thus $Col(E_1) = Null(A - \lambda_1 I)$. Same argument works for 2 & 3.

P 91 $u = \frac{1}{\sqrt{3}}(1,-1,1)$

$v = \frac{1}{\sqrt{2}}(0,1,1)$

$w = \frac{1}{\sqrt{6}}(2,1,-1)$

construct A for which u, v, w are e-vectors with e-values $12, 2, 18$

$$A = 12uu^T + 2vv^T + 18ww^T$$

$$= \frac{12}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{2}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \frac{18}{6}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 12 & 6 & -6 \\ 6 & 3 & -3 \\ -6 & -3 & 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 16 & 2 & -2 \\ 2 & 8 & -6 \\ -2 & -6 & 8 \end{bmatrix}}$$

as a check on my calculation, trace $(A) = 16 + 8 + 8 = 32$ and $\lambda_1 + \lambda_2 + \lambda_3 = 12 + 2 + 18 = 32$ ✓

P92 Define $\Upsilon : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ by $\Upsilon(A,B) = AB + BA$

observe $\Upsilon(A,B) = AB + BA = BA + AB = \Upsilon(B,A)$ $\therefore$ Υ is symmetric.

Note, $\Upsilon(cA_1 + A_2, B) = (cA_1 + A_2)B + B(cA_1 + A_2)$

$= c(A_1B + BA_1) + A_2B + BA_2$

$= c\,\Upsilon(A_1, B) + \Upsilon(A_2, B)$ $(*)$

for all $c \in \mathbb{R}$ and $A_1, A_2, B \in \mathbb{R}^{n\times n}$ thus Υ is linear in its 1^{st} slot.

Likewise, for $c \in \mathbb{R}$, $A_1, A_2, B \in \mathbb{R}^{n\times n}$,

$\Upsilon(B, cA_1 + A_2) = \Upsilon(cA_1 + A_2, B)$: by symmetry

$= c\,\Upsilon(A_1, B) + \Upsilon(A_2, B)$: using $(*)$

$= c\,\Upsilon(B, A_1) + \Upsilon(B, A_2)$ $\therefore$ $\underline{\Upsilon \text{ is bilinear}}$.

$\boxed{P:93}$ $x \otimes y : V^* \times V^* \longrightarrow \mathbb{R}$ is defined by

$(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Consider,

$$(x \otimes y)(c\alpha_1 + \alpha_2, \beta) = (c\alpha_1 + \alpha_2)(x)\beta(y) \quad : \text{ def}^{=}$$

$$= (c\alpha_1(x) + \alpha_2(x))\beta(y) \quad : \text{def}^{n} \text{ of adding functions}$$

$$= c\alpha_1(x)\beta(y) + \alpha_2(x)\beta(y)$$

$$= c(x \otimes y)(\alpha_1, \beta) + (x \otimes y)(\alpha_2, \beta)$$

Likewise,

$$(x \otimes y)(\alpha, c\beta_1 + \beta_2) = \alpha(x)(c_1\beta_1 + \beta_2)(y)$$

$$= \alpha(x)(c\beta_1(y) + \beta_2(y))$$

$$= c\,\alpha(x)\beta_1(y) + \alpha(x)\beta_2(y)$$

$$= c(x \otimes y)(\alpha, \beta_1) + (x \otimes y)(\alpha, \beta_2)$$

Thus $\underbrace{x \otimes y}_{\text{bilinear map on } V^* \times V^*} \in \mathcal{B}(V^*) = \mathcal{B}(V^* \times V^*, \mathbb{R})$.

$\boxed{P94}$ V has basis $\beta = \{v_1, \dots, v_n\}$ then prove
$\Upsilon = \{ v_i \otimes v_j \mid 1 \le i, j \le n\}$ serves as basis for $\mathcal{B}(V^*)$

Suppose $\displaystyle\sum_{i,j=1}^{n} c_{ij}\, v_i \otimes v_j = 0$ then evaluate on (v^k, v^l)
where $v^k, v^l \in \beta^* \subset V^*$ defined as usual $v^k(v_i) = \delta_{ki}$
and $v^l(v_j) = \delta_{lj}$ etc. Thus,

$$0 = \left(\sum_{i,j} c_{ij}\, v_i \otimes v_j \right)(v^k, v^l) = \sum_{i,j} c_{ij} (v_i \otimes v_j)(v^k, v^l)$$

$$= \sum_{i,j} c_{ij}\, v^k(v_i)\, v^l(v_j)$$

$$= \sum_{i,j} c_{ij}\, \delta_{ki}\, \delta_{lj}$$

$$= c_{kl} \Rightarrow c_{ij} = 0 \;\; \forall i,j$$

$$\Rightarrow \underline{\Upsilon \text{ is LI}}.$$

If $b : V^* \times V^* \longrightarrow \mathbb{R}$ is bilinear then

$$b(\alpha, \beta) = b\left(\sum_i \alpha_i v^i, \sum_j \beta_j v^j \right)$$

$$= \sum_{i,j} \alpha_i \beta_j\, b(v^i, v^j)$$

$$= \sum_{i,j} b(v^i, v^j)\, \alpha(v_i)\, \beta(v_j)$$

$$= \left(\sum_{i,j} b(v^i, v^j)\, v_i \otimes v_j \right)(\alpha, \beta) \quad \forall \alpha, \beta \in V^*$$

$$\therefore\; b = \sum_{i,j} b(v^i, v^j)\, v_i \otimes v_j \quad \therefore\; \underline{\operatorname{span}(\Upsilon) = \mathcal{B}(V^*)}.$$

(we already showed $\Upsilon \subseteq \mathcal{B}(V^*)$ in previous problem hence $\operatorname{span} \Upsilon \subseteq \mathcal{B}(V^*)$.) (the calculation here shows $\mathcal{B}(V^*) \subseteq \operatorname{span} \Upsilon$)