Your printed name indicates you read Chapter 6 of the notes: ____________________.

Assume $\mathbb{F}$ is a field and $n \in \mathbb{N}$ and V is a vector space over $\mathbb{F}$ with dual space $V^* = \mathcal{L}(V, \mathbb{F})$.

Problem 51 If $x, y \in V$ and $\alpha, \beta \in V^*$ then $\alpha \otimes \beta : V \times V \to \mathbb{F}$ and $x \otimes y : V^* \times V^* \to \mathbb{F}$ are given by:

$$(\alpha \otimes \beta)(x, y) = \alpha(x)\beta(y) \qquad (x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y).$$

It is simple to prove $\alpha \otimes \beta : V \times V \to \mathbb{F}$ is bilinear on V and $x \otimes y : V^* \times V^* \to \mathbb{F}$ is bilinear on V^*. Furthermore, if $\alpha_1, \alpha_2, \beta \in V^*$ and $c \in \mathbb{F}$ then $(c\alpha_1 + \alpha_2) \otimes \beta = c(\alpha_1 \otimes \beta) + \alpha_2 \otimes \beta$ and $\beta \otimes (c\alpha_1 + \alpha_2) = c\beta \otimes \alpha_1 + \beta \otimes \alpha_2$. Likewise, if $x, y, z \in V$ and $c \in \mathbb{F}$ then $(cx + y) \otimes z = c(x \otimes z) + y \otimes z$ and $z \otimes (cx + y) = cz \otimes x + z \otimes y$.

Suppose V is a finite dimensional vector space with basis $\Upsilon = \{v_1, \dots, v_n\}$ and dual basis $\Upsilon^* = \{v^1, \dots, v^n\}$ defined by $v^i(v_j) = \delta_{ij}$ extended linearly. Prove the following:

(a.) If $B : V \times V \to \mathbb{F}$ is bilinear then $\exists B_{ij} \in \mathbb{F}$ for which $B = \displaystyle\sum_{i,j=1}^{n} B_{ij} v^i \otimes v^j$.

(b.) If $T : V^* \times V^* \to \mathbb{F}$ is bilinear then $\exists T_{ij} \in \mathbb{F}$ for which $T = \displaystyle\sum_{i,j=1}^{n} T^{ij} v_i \otimes v_j$.

Problem 52 Consider $\alpha \in V^*$ where $(V, \langle\ ,\ \rangle)$ is an inner product space. Then we define $\sharp\alpha$ to be the unique vector for which $\alpha(x) = \langle x, \sharp\alpha \rangle$. The vector $\sharp\alpha$ is sometimes called the **Riesz' vector** of α. Calculate $\sharp\alpha$ for the following:

(a.) $V = \mathbb{R}^3$, $\alpha(x) = x_1 + 2x_2 + 3x_3$

(b.) $V = \mathbb{C}^2$, $\alpha(z) = z_1 + iz_2$

(c.) $V = P_2(\mathbb{R})$ with $\langle f(x), h(x) \rangle = \int_0^1 f(t)h(t)\,dt$ given $\alpha(f) = f(0) + f'(1)$.

Problem 53 Suppose (V, g) is a real geometry meaning that V is a real vector space paired with a bilinear, symmetric, nondegenerate form g. Given basis $\Upsilon = \{v_1, \dots, v_n\}$ for V and $\alpha \in V^*$ we define $\sharp\alpha \in V$ by $\alpha(x) = g(x, \sharp\alpha)$ for each $x \in V$.

(a.) Prove $\sharp : V^* \to V$ defines an isomorphism.

(b.) If $\flat = \sharp^{-1}$ then find the explicit formula for $\flat(x)$ when $x = \sum_{i=1}^{n} x^i v_i$

(c.) Show (V^*, g^*) is also a real geometry given we define $g^*(\alpha, \beta) = g(\sharp\alpha, \sharp\beta)$.

Problem 54 Let S_n be the symmetric and A_n be the antisymmetric $n \times n$ matrices over $\mathbb{F}$. Prove that $\mathbb{R}^{n\times n}/S_n \approxeq A_n$ and $\mathbb{R}^{n\times n}/A_n \approxeq S_n$.

Problem 55 Consider $V = \mathbb{R}[x]$ and $W = x^4\mathbb{R}[x]$. Show V/W is a finite dimensional vector space. To begin, find a careful criteria for

$$f(x) + W = g(x) + W$$

then propose a basis and prove it is a linearly independent spanning set for V/W.

Problem 56 Suppose $W \leq V$ where V is a vector space over $\mathbb{F}$. Also, let $T : V \to V$ be a linear transformation. If we define $S(x+W) = T(x) + W$ for each $x + W \in V/W$ then does S define a linear transformation on V/W ? Discuss.

Problem 57 Let $(V, \langle \ , \ \rangle)$ be a finite dimensional real inner product space. Suppose $W \leq V$. Let $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for all } w \in W\}$. Construct an explicit isomorphism from $ann(W)$ to $W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in W\}$.

Problem 58 Let vector $v = \langle a, b, c\rangle$ and define

$$\omega_v = adx + bdy + cdz \qquad \& \qquad \Phi_v = ady \wedge dz + bdz \wedge dx + cdx \wedge dy.$$

Here we use the notation dx, dy, dz for the dual basis to the standard basis e_1, e_2, e_3 for $\mathbb{R}^3$. I usually call ω_v the **work form** and Φ_v the **flux form** corresponding to v. Show:

(a.) Show $\omega_v \wedge \omega_w = \Phi_{v \times w}$ where $v \times w$ denotes the usual cross-product of vectors in $\mathbb{R}^3$

(b.) Show $\omega_u \wedge \omega_v \wedge \omega_w = u \bullet (v \times w) dx \wedge dy \wedge dz$

Problem 59 Let $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where $A \in \mathbb{F}^{m \times m}$ and $B \in \mathbb{F}^{k \times k}$. Prove that

$$\det(A \oplus B) = \det(A)\det(B)$$

using the wedge product algebra definition of the determinant.

Problem 60 Suppose $S = \{(1, 0, 1, 0), (4, 3, 5, 2), (a, b, c, d)\}$. What condition(s) on (a, b, c, d) are needed for S to be linearly dependent ?

(a) find the condition(s) via the row-reduction technique,

(b) find the condition(s) from the fact that x, y, z linearly dependent iff $x \wedge y \wedge z = 0$.