Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 7 of the notes: ____________________.

Problem 91 Let $W_1 = \text{span}\{x + x^2, 1 + x^3\}$ and $W_2 = \text{span}\{1 + x, x^2 + x^3\}$. Find a basis for $W_1 \cap W_2$.

Problem 92 Find a basis for $W_1 + W_2$ where W_1, W_2 are the subspaces of $P_3(\mathbb{R})$ described in the previous problem. Do your calculations check against Theorem 6.7.8 ?

Problem 93 Example 7.6.3 shows a calculational technique to find bases β, γ for which $T : \mathbb{R}^n \to \mathbb{R}^m$ has a matrix $[T]_{\beta,\gamma} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $r = \text{rank}(T)$. Follow that example (use technology for the row reductions!) to find such β, γ for $T : \mathbb{R}^4 \to \mathbb{R}^3$ with

$$[T] = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 3 & -2 & 6 \\ 2 & 0 & -4 & 0 \end{bmatrix}$$

Problem 94 Let $v = (7, 9)$ and suppose $\beta = \{(2, 2), (-1, 1)\}$. Calculate $[v]_\beta$.

Problem 95 Consider bases $\beta = \{x^2, x, 1\}$ and $\bar{\beta} = \{1, x - 2, (x - 2)^2\}$. Find the coordinate change matrix $P_{\beta,\bar{\beta}}$ for which $[v]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[v]_\beta$ for each $v \in P_2(\mathbb{R})$

Problem 96 Consider $\mathbb{R}^{2\times 2}$. We have the usual basis

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and less usual basis

$$\bar{\beta} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

(a.) Find the coordinate change matrix $P_{\beta,\bar{\beta}}$ for which $[A]_{\bar{\beta}} = P_{\beta,\bar{\beta}}[A]_\beta$ for each $A \in \mathbb{R}^{2\times 2}$

(b.) Consider the mapping $L(A) = A^T$. Calculate $[L]_{\beta,\bar{\beta}}$.

Problem 97 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that:

$$T(v_1) = v_1, \qquad T(v_2) = 2v_1, \qquad T(v_3) = 3v_3$$

where $v_1 = (1, 1, 0)$ and $v_2 = (1, -1, 0)$ and $v_3 = (0, 0, 1)$. Find the standard matrix of T by an appropriate use of Proposition 7.5.7.

Problem 98 Suppose $T(f(x)) = f'(x) + f''(x)$ for $f(x) \in P_2(\mathbb{R})$.

(a.) Can you find a basis β for $P_2(\mathbb{R})$ such that $[T]_{\beta,\beta} = I_3$?

(b.) Find a subspace W with basis β_W and basis γ for $P_2(\mathbb{R})$ such that $T|_W : W \to P_2(\mathbb{R})$ has $[T|_W]_{\beta_W,\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Problem 99 Suppose T has matrix $[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ with respect to $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{E_{12} + E_{21}, I\} \subseteq \mathbb{R}^{2\times 2}$. Find the formula for $T : P_3(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ and find $[T]_{\bar{\beta},\bar{\gamma}}$ where $\bar{\beta} = \{x^3, x^2, x, 1\}$ and $\bar{\gamma} = \{2(E_{12} + E_{21}), 3I\}$.

Problem 100 Suppose $T : V \to W$ has $\text{Null}([T]_{\beta,\gamma}) = \text{span}\{(1,1,0),(0,1,2)\}$ and $\text{Col}([T]_{\beta,\gamma}) = \text{span}\{(1,0,1)\}$ where $\beta = \{1, x, x^2\}$ and $\gamma = \{e^t, \sin(t), \cos(t)\}$ are bases for V and W respective.

(a.) find $\text{Ker}(T)$ and $\text{Range}(T)$

(b.) find the formula for $T(a + bx + cx^2)$

Problem 101 Dual space has very nice applications to coordinate maps. In particular, given basis $\beta = \{v_1, \ldots, v_n\}$ for V we define dual basis $\beta^* = \{v^1, \ldots, v^n\} \subseteq V^*$ by the rule $v^i(v_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.

(a.) explain why $v^i(v_j) = \delta_{ij}$ for $1 \leq j \leq n$ suffices to define the linear map $v^i : V \to \mathbb{F}$,

(b.) prove $\Phi_\beta(x) = \sum_{i=1}^n v^i(x) v_i$ ← supposed to be e_i

(c.) explain why $[x]_\beta = (v^1(x), \ldots, v^n(x))$.

Problem 102 The annihilator of a subspace is naturally constructed in the dual space. In particular, if $W \leq V$ then define $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for each } w \in W\}$

(a.) show $ann(W) \leq V^*$

(b.) if $W_1 \leq W_2 \leq V$ then show $ann(W_2) \subseteq ann(W_1)$

Remark: part (b.) of the above problem has a natural analog with the construction of the perpendicular space for a given $S \subseteq \mathbb{R}^n$. For example, the x-axis (W_1) is perpendicular to the yz-plane ($W_1^\perp$). Whereas the xy-plane (W_2) is perpendicular to the z-axis ($W_2^\perp$). So, note $W_1 \leq W_2$ has $W_2^\perp \leq W_1^\perp$. In view of this, perhaps the following problem is not too surprising:

Problem 103 Find an isomorphism from $W^\perp = \{x \in \mathbb{R}^n \mid x \bullet w = 0, \text{ for all } w \in W\}$ and $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for each } w \in W\}$.

Problem 104 Consider $V = P_3(\mathbb{R}) \times \mathbb{C}^{2\times 2}$ as a real vector space. If $S_n(\mathbb{R})$ denotes the symmetric $n \times n$ matrices then for what n (if any) is $V \cong S_n \times S_n$?

Problem 105 Consider $V = \mathbb{R}^3$ and the subspace $W = \text{span}\{(1,1,1)\}$. Find a basis and coordinate chart for V/W. Describe the geometry of the cosets in V/W

Problem 106 Consider $V = P_2(\mathbb{R})$ and the linear transformation $T(f(x)) = f'(x)$ find $\text{Ker}(T)$ and find the inverse mapping $S : P_2(\mathbb{R})/\text{Ker}(T) \to T(P_2(\mathbb{R}))$ given by $S(f(x) + \text{Ker}(T)) = T(f(x))$. This is a special case of what common slogan from calculus I?

Problem 107 Suppose S is a subset of V. If we define $S + W = \{s + W \mid s \in S\}$ for a subspace W of V.

(a.) if S is LI then is $S + W$ a LI in V/W ? Discuss.

(b.) if S is linearly dependent in V then is $S + W$ linearly dependent in V/W ? Discuss.

Problem 108 Show $\mathbb{R}^{n\times n}/A_n \cong S_n$ where S_n denoted the set of symmetric matrices and A_n denotes the set of antisymmetric matrices in $\mathbb{R}^{n\times n}$. *Hint: use the first isomorphism theorem wisely.*

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Is the set of rational functions over $\mathbb{R}$ a subspace of the set of continuous functions on $\mathbb{R}$?

(II.) Show $W = \{(a+bx^2, (a+2b, a-b)) \mid a, b \in \mathbb{R}\}$ is a subspace of $P_2(\mathbb{R}) \times \mathbb{R}^2$.

(III.) Consider $S = \{1+t^2, 1-t, 1+t+t^3, 2+t^3\} \subseteq \mathbb{R}[t]$. Find a basis β for $\text{span}(S)$. Also, find the formula for $[a+bt+ct^2+dt^3]_\beta$.

(IV.) Let $\beta = \{1, (x-1), (x-1)^2\}$. Calculate $[ax^2+bx+c]_\beta$.
Hint: be smart, use Taylor's Theorem you learned in Calculus II.

(V.) Consider the set of quadratic forms in two variables x, y. Let $\gamma = \{x^2, y^2, xy\}$ and define the set of trivariate homogeneous polynomials of order two by

$$W = \text{span}\{x^2, y^2, xy\}.$$

Observe W can be viewed as a function space and as it is a span we find $W \leq \mathcal{F}(\mathbb{R}^2, \mathbb{R})$. If $v = 3x^2 + 2(x-y)y$ then calculate $[v]_\gamma$.

(VI.) Suppose $T : U \to V$ and $S : V \to W$ are linear transformations. Show that:

(a.) $\text{Range}(S \circ T) \subseteq \text{Range}(S)$

(b.) $\text{Ker}(T) \subseteq \text{Ker}(S \circ T)$

(VII.) Consider $\text{Aut}(V) = \{\Psi : V \to V \mid \Psi \text{ an isomorphism}\}$. Is $\text{Aut}(V) \leq \mathcal{L}(V)$? Here $\mathcal{L}(V)$ denotes the set of all linear mappings from V to V.

(VIII.) Investigate relation of $ann(W_1 + W_2)$ and $ann(W_1 \cap W_2)$.

(IX.) Let V be a vector space and $M, N \leq V$ and $x, y \in V$. Prove:

$$x + M \subseteq y + N \quad \text{if and only if} \quad M \subseteq N \text{ and } x - y \in N.$$

SOLUTION TO MISSION 6 : COORDINATE CHANGE, & DUAL & QUOTIENT SPACE

P91 $W_1 = span\{x+x^2, 1+x^3\}$ and $W_2 = span\{1+x, x^2+x^3\}$
Find basis for $W_1 \cap W_2$

If $f(x) \in W_1 \cap W_2$ then $f(x) \in W_1$ and $f(x) \in W_2$ thus $\exists a,b,c,d \in \mathbb{R}$ s.t. $f(x) = a(x+x^2) + b(1+x^3) = c(1+x) + d(x^2+x^3)$
Thus, $b + ax + ax^2 + bx^3 = c + cx + dx^2 + dx^3$ from which we equate coeff. to find,

$$b = c, \quad a = c, \quad a = d, \quad b = d$$

hence $a = b = c = d$ so,

$$f(x) = a(x+x^2) + a(1+x^3) = a(1+x) + a(x^2+x^3)$$

any way, $f(x) = a(1+x+x^2+x^3)$ thus

$$W_1 \cap W_2 \subseteq span\{1+x+x^2+x^3\}$$

conversely, as $1+x+x^2+x^3 = (x+x^2) + (1+x^3) = (1+x) + (x^2+x^3)$ it is clear $1+x+x^2+x^3 \in W_1 \cap W_2 \Rightarrow span\{1+x+x^2+x^3\} \subseteq W_1 \cap W_2$.
In conclusion, $\underline{\beta = \{1+x+x^2+x^3\} \text{ is basis for } W_1 \cap W_2}$.

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Alternatively, $W_1 = \{a+bx+cx^2+dx^3 \mid b=d, a=c\}$
and $W_2 = \{a+bx+cx^2+dx^3 \mid a=b, c=d\}$ thus

$$\begin{aligned} W_1 \cap W_2 &= \{a+bx+cx^2+dx^3 \mid a=b, c=d, b=d, a=c\} \\ &= \{a+bx+cx^2+dx^3 \mid a=b=c=d\} \\ &= span\{1+x+x^2+x^3\} \end{aligned}$$

Hence $\beta = \{1+x+x^2+x^3\}$ is basis for $W_1 \cap W_2$.

$\boxed{P92}$ Find basis for $W_1 + W_2$ (W_1, W_2 from P91)

Let $f(x) \in W_1 + W_2$ then $\exists f_1(x) \in W_1$ and $f_2(x) \in W_2$ s.t. $f(x) = f_1(x) + f_2(x)$. But, by construction $W_1 = \text{span}\,\beta_1$ and $W_2 = \text{span}\,\beta_2$ where $\beta_1 = \{x+x^2, 1+x^3\}$ and $\beta_2 = \{1+x, x^2+x^3\}$ hence $\exists c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.

$$f(x) = c_1(1+x) + c_2(x^2+x^3) + c_3(x+x^2) + c_4(1+x^3)$$

So $\gamma = \beta_1 \cup \beta_2 = \{1+x, x^2+x^3, x+x^2, 1+x^3\}$ generates $W_1 + W_2$. To find basis we need to eliminate any linear dependencies from γ. I'll use coordinates w.r.t. $\beta = \{1, x, x^2, x^3\}$ to analyze γ,

$$M = \left[\, [1+x]_\beta \,\middle|\, [x^2+x^3]_\beta \,\middle|\, [x+x^2]_\beta \,\middle|\, [1+x^3]_\beta \,\right] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus $\{col_1(M), col_2(M), col_3(M)\}$ are LI and thus as the isomorphism Φ_β^{-1} transfers LI sets in $\mathbb{R}^4$ to LI sets in $P_3(\mathbb{R})$ we find

$$\boxed{\beta_3 = \{1+x, x^2+x^3, x+x^2\}} \hookrightarrow \underline{\dim(W_1+W_2) = 3}.$$

Serves as basis for $W_1 + W_2$. Finally,

$$\begin{aligned} \dim(W_1+W_2) &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \\ &= 2 + 2 - 1 \\ &\overset{\checkmark}{=} 3 \end{aligned}$$

P93

$$[T] = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 3 & -2 & 6 \\ 2 & 0 & -4 & 0 \end{bmatrix} \xrightarrow{r_2 - r_1} \xrightarrow{r_3 - 2r_1} \xrightarrow{r_2/3} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref[T]$$

Thus $x \in Null[T]$ has $x_1 = 2x_3$ and $x_2 = -2x_4$ which means $x = (2x_3, -2x_4, x_3, x_4) = x_3(2,0,1,0) + x_4(0,-2,0,1)$

$Ker\, T = span\{\underbrace{(2,0,1,0), (0,-2,0,1)}_{\beta_0}\}$. I wish to extend β_0 with vectors in $\{e_1, e_2, e_3, e_4\}$.

$$rref\left[\begin{array}{cc|cccc} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right]$$

Apparently any two will do ($\nexists$ linear dep of β_0 and e_1, e_2, e_3, e_4)

So, I'll use $\boxed{\beta = \{e_1, e_2, (2,0,1,0), (0,-2,0,1)\}}$ — *

Let γ be basis for $\mathbb{R}^3$ then,

$$[T]_{\beta,\gamma} = \left[\, [T(e_1)]_\gamma \,\middle|\, [T(e_2)]_\gamma \,\middle|\, [T(2,0,1,0)]_\gamma \,\middle|\, [T(0,-2,0,1)]_\gamma \right]$$

0 0

we want $[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Simply use

$\gamma = \{\underbrace{(1,1,2)}_{T(e_1)}, \underbrace{(0,3,0)}_{T(e_2)}, (a,b,c)\}$ then $\underline{[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]}$.

we have many choices for (a,b,c), but $\det\begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 2 & 0 & c \end{bmatrix} \neq 0$ is needed. Set $a = b = 0$ then $c = 1$ does nicely.

Let's use $\boxed{\gamma = \{(1,1,2), (0,3,0), (0,0,1)\}}$ — **

Remark: choice of * and ** NOT UNIQUE!

P94 $\quad v = (7,9)$ and $\beta = \{(2,2),(-1,1)\}$

$$[v]_\beta = [\beta]^{-1} v = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$= \frac{1}{2+2} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 16 \\ 4 \end{bmatrix}$$

$$\therefore \quad \boxed{[v]_\beta = \begin{bmatrix} 4 \\ 1 \end{bmatrix}}$$

Let's check, $4(2,2) + 1(-1,1) = (8,8) + (-1,1) \overset{\checkmark}{=} (7,9)$.

P95 $\quad \beta = \{x^2, x, 1\}$ and $\bar{\beta} = \{1, x-2, (x-2)^2\}$

$$V = ax^2 + bx + c = f(x) = \underline{f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2} \; *$$

$$\begin{tikzcd}
& \boxed{\begin{array}{c} V = ax^2+bx+c \\ P_2(\mathbb{R}) \end{array}} \arrow[dl, "\Phi_\beta"'] \arrow[dr] & \\
\boxed{[v]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \arrow[rr, "P_{\beta,\bar{\beta}}", "\text{multiply}"'] & & \boxed{[v]_{\bar{\beta}} = \begin{bmatrix} 4a+2b+c \\ 4a+b \\ a \end{bmatrix}}
\end{tikzcd}$$

Hence $\boxed{P_{\beta,\bar{\beta}} = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$. $\qquad \left([v]_{\bar{\beta}} = P_{\beta,\bar{\beta}} [v]_\beta \right)$

Alternatively,

$$P_{\beta,\bar{\beta}} = \left[[x^2]_{\bar{\beta}} \,\middle|\, [x]_{\bar{\beta}} \,\middle|\, [1]_{\bar{\beta}} \right] = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Notice $ax^2 + bx + c \overset{*}{=} (4a+2b+c)1 + (4a+b)(x-2) + a(x-2)^2$

so $\Phi_{\bar{\beta}}(ax^2+bx+c) = (4a+2b+c, 4a+b, a)$

$*$: Taylor's Th$^{\underline{m}}$ at 2.

P96

$$\beta = \{ E_{11}, E_{12}, E_{21}, E_{22} \}$$

$$E_{12} = \frac{1}{2}\left(E_{12} - E_{21} + E_{12} + E_{21} \right)$$

$$\bar{\beta} = \{ E_{11}, E_{22}, \underbrace{E_{12} + E_{21}}_{f_3}, \underbrace{E_{12} - E_{21}}_{f_4} \}$$

$$E_{21} = \frac{f_3 - f_4}{2}$$

(a.) $P_{\beta,\bar{\beta}} = \left[[E_{11}]_{\bar{\beta}} \,\middle|\, [E_{12}]_{\bar{\beta}} \,\middle|\, [E_{21}]_{\bar{\beta}} \,\middle|\, [E_{22}]_{\bar{\beta}} \right]$

$$\boxed{P_{\beta,\bar{\beta}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix}}$$

since $E_{12} = \dfrac{f_3 + f_4}{2}$

$E_{21} = \dfrac{f_3 - f_4}{2}$

(b.) $L\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{A^T}$

$$[L(A)]_{\bar{\beta}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}_{\bar{\beta}} = \left[aE_{11} + dE_{22} + c\left(\frac{f_3 + f_4}{2}\right) + b\left(\frac{f_3 - f_4}{2}\right) \right]_{\bar{\beta}}$$

$$\Rightarrow \quad [L(A)]_{\bar{\beta}} = \left(a, d, \frac{b+c}{2}, \frac{c-b}{2}\right)$$

Need $[L(A)]_{\bar{\beta}} = [L]_{\beta,\bar{\beta}} [A]_\beta$ where $[A]_\beta = (a, b, c, d)$

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ d \\ \frac{1}{2}(c+b) \\ \frac{1}{2}(c-b) \end{bmatrix}$$

$$\therefore \quad \boxed{[L]_{\beta,\bar{\beta}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \end{bmatrix}}$$

P97 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear with

$$\underline{T(v_1) = v_1, \quad T(v_2) = 2v_1, \quad T(v_3) = 3v_3} \quad *$$

where $v_1 = (1,1,0)$, $v_2 = (1,-1,0)$ and $v_3 = (0,0,1)$.
Find $[T]$, use Prop. 7.5.7.

Let $\beta = \{v_1, v_2, v_3\}$, by Prop. 7.5.7, $(\gamma = \bar{\beta} = \beta)$

$$[T]_{\beta\beta} = [\beta]^{-1}[T][\beta]$$

So, $[T] = [\beta][T]_{\beta\beta}[\beta]^{-1}$. From $*$
we find $[T]_{\beta\beta} = \left[[T(v_1)]_\beta \mid [T(v_2)]_\beta \mid [T(v_3)]_\beta \right] =$

$$= \left[[v_1]_\beta \mid [2v_1]_\beta \mid [3v_3]_\beta \right]$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then,

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1^{-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 3/2 & -1/2 & 0 \\ 3/2 & -1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

$\boxed{P98}$ $T(f(x)) = f'(x) + f''(x)$ for $f(x) \in P_2(\mathbb{R})$

(a.) $T(ax^2+bx+c) = 2ax + b + 2a$

$Ker(T) = \{ax^2+bx+c \mid 2ax+b+2a = 0\} = span\{1\}$.

Thus $[T]_{\beta\beta} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is impossible. We

find $\nu(T) = dim(Ker(T)) = 1$ thus $rk(T) = 3-1 = 2$

and $dim(col[T]_{\beta\beta}) = dim(Range(T)) = 2$ so $[T]_{\beta\beta} = I_3$

is not possible.

(b.) Let $\beta_W = \{x^2, x\}$ then $T(x^2) = 2x+2 = W_1$

and $T(x) = 1 = W_2$. Letting $\gamma = \{W_1, W_2, x^2\}$

we have basis for $P_2(\mathbb{R})$ and

$$[T|_W]_{\beta_W,\gamma} = \Big[[T(x^2)]_\gamma \;\Big|\; [T(x)]_\gamma \Big]$$

$$= \Big[[2x+2]_\gamma \;\Big|\; [1]_\gamma \Big]$$

$$= \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}.$$

<u>Remark</u>: I chose to adjoin x^2 to $\{T(x), T(x^2)\} = \{1, 2x+2\}$ since clearly $x^2 \notin span\{1, 2x+2\} \Rightarrow \{1, 2x+2, x^2\}$ is LI hence serves as basis for 3-dim'l $P_2(\mathbb{R})$. My choices for constructing β_W and γ are certainly <u>not</u> unique.

$\boxed{P99}$ T has $[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ where $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ find formula for T and also calculate $[T]_{\bar{\beta},\bar{\gamma}}$ where $\bar{\beta} = \{x^3, x^2, x, 1\}$ and $\bar{\gamma} = \left\{ 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

From $[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \Big[[T(1)]_\gamma \,\Big|\, [T(x)]_\gamma \,\Big|\, [T(x^2)]_\gamma \,\Big|\, [T(x^3)]_\gamma \Big]$

we find $[T(1)]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow T(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and $[T(x)]_\gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow T(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus $\boxed{T(a + bx + cx^2 + dx^3) = \begin{bmatrix} b & a \\ a & b \end{bmatrix}}$

I'll calculate $[T]_{\bar{\beta},\bar{\gamma}}$ directly,

$$T(dx^3 + cx^2 + bx + a) = \begin{bmatrix} b & a \\ a & b \end{bmatrix} = \frac{a}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} + \frac{b}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

from which I read,

$$\boxed{[T]_{\bar{\beta},\bar{\gamma}} = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} a/2 \\ b/3 \end{bmatrix}$$