$\boxed{P100}$ Suppose $T: V \to W$ has $\mathrm{Null}([T]_{\beta,\gamma}) = \mathrm{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$ and $\mathrm{Col}([T]_{\beta,\gamma}) = \mathrm{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ where $\beta = \{1, x, x^2\}$ and $\gamma = \{e^t, \sin t, \cos t\}$ are bases for V, W over $\mathbb{R}$

(a.) $\mathrm{Ker}(T) = \Phi_\beta^{-1}(\mathrm{Null}([T]_{\beta,\gamma}))$

$$= \Phi_\beta^{-1}(\mathrm{span}\{(1,1,0),(0,1,2)\})$$

$$\overset{*}{=} \mathrm{span}\{\Phi_\beta^{-1}(1,1,0), \Phi_\beta^{-1}(0,1,2)\}$$

$$= \mathrm{span}\{1+x, x+2x^2\}$$

(Since $\Phi_\beta^{-1}(a,b,c) = a + bx + cx^2$)

Thus $\boxed{\mathrm{Ker}(T) = \mathrm{span}\{1+x, x+2x^2\}}$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi_\beta"'] & W \arrow[d, "\Phi_\gamma"] \\ \mathbb{R}^3 \arrow[r, "L_{[T]_{\beta,\gamma}}"] & \mathbb{R}^3 \end{tikzcd}$$

Ker T, Range(T), Null $[T]_{\beta,\gamma}$, Col $([T]_{\beta,\gamma})$

$\Phi_\beta(\mathrm{Ker}T) = \mathrm{Null}[T]_{\beta,\gamma}$

$\Phi_\gamma(\mathrm{Range}(T)) = \mathrm{Col}[T]_{\beta,\gamma}$

*: I proved isomorphisms behave as such in the notes. Also, the above picture was justified in Prop. 7.4.15

Likewise,

$$\mathrm{Range}(T) = \Phi_\gamma^{-1}(\mathrm{Col}[T]_{\beta,\gamma})$$

$$= \Phi_\gamma^{-1}(\mathrm{span}\{(1,0,1)\})$$

$$= \mathrm{span}\{\Phi_\gamma^{-1}(1,0,1)\}$$

$$= \mathrm{span}\{e^t + \cos t\}$$

Since $\Phi_\gamma^{-1}(a,b,c) = ae^t + b\sin t + c\cos t$ $\therefore$ $\boxed{\mathrm{Range}(T) = \mathrm{span}\{e^t + \cos t\}}$

(b.) Find formula for $T(a+bx+cx^2)$,

$$T(a+bx+cx^2) = \Phi_\gamma^{-1}([T]_{\beta,\gamma}[a+bx+cx^2]_\beta)$$

$$= \Phi_\gamma^{-1}\left([T]_{\beta,\gamma}\begin{bmatrix}a\\b\\c\end{bmatrix}\right)$$

Well, I'm not given enough to just calculate the above, I'll attack this from a different angle ⤵

P100

$$T(1+x) = 0 \quad \text{and} \quad T(x+2x^2) = 0$$

since $1+x,\ x+2x^2 \in \text{Ker}(T)$. Thus,

$$T(1) + T(x) = 0 \quad \text{and} \quad T(x) + 2T(x^2) = 0$$

Consequently, $T(x) = -T(1)$ and $T(x^2) = \frac{-1}{2}T(x) = \frac{1}{2}T(1)$

therefore, $T(a+bx+cx^2) = aT(1) + bT(x) + cT(x^2)$ yields

$$T(a+bx+cx^2) = (a - b + c/2)\,T(1)$$

But, Range (T) = span$\{e^t + \cos t\}$ thus

$$T(1) \in \text{Range}(T) \Rightarrow T(1) = \alpha(e^t + \cos t) \text{ for some } \alpha \in \mathbb{R}$$

Consequently,

$$\boxed{T(a+bx+cx^2) = (a-b+c/2)\,\alpha\,(e^t + \cos t)}$$

(the answer is not unique, any $\alpha \neq 0$ gives $T: V \to W$ which matches the given data for $\text{Null}[T]_{\beta,\gamma}$ & $\text{Col}[T]_{\beta,\gamma}$)

$$[T(a+bx+cx^2)]_\gamma = (a-b+c/2)\,\alpha\,[e^t + \cos t]_\gamma$$

$$= (a-b+c/2)\,\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a - \alpha b + c\alpha/2 \\ 0 \\ \alpha a - \alpha b + c\alpha/2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \alpha & -\alpha & \alpha/2 \\ 0 & 0 & 0 \\ \alpha & -\alpha & \alpha/2 \end{bmatrix}}_{\text{for any } \alpha \neq 0} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

this $[T]_{\beta,\gamma}$ gives the initial data of this problem.

$\boxed{P101}$ $V = \text{span}\,\beta$, $\beta = \{v_1, v_2, \dots, v_n\}$ and $\beta^* = \{v^1, \dots, v^n\}$ defined by $v^i(v_j) = \delta_{ij}$ $\forall i,j = 1,2,\dots,n$ extended linearly.

(a.) Suppose $v^i : V \longrightarrow \mathbb{F}$ is linear and $v^i(v_j) = \delta_{ij}$ $\forall i,j$. Let $x = \sum_{j=1}^{n} c_j v_j$ for $c_1, c_2, \dots, c_n \in \mathbb{F}$. We calculate,

$$\begin{aligned} v^i(x) &= v^i\Big(\sum_{j=1}^{n} c_j v_j\Big) \\ &= \sum_{j=1}^{n} c_j \underbrace{v^i(v_j)}_{\delta_{ij}} \\ &= c_i \end{aligned}$$

thus v^i defined on β provides a clear path to calculating $v^i(x)$ for any $x \in V$. Furthermore, we find that

$$\underline{x = \sum_{i=1}^{n} v^i(x)} \; *$$

(b.) $$\begin{aligned} \Phi_\beta(x) &= \Phi_\beta\Big(\sum_{i=1}^{n} v^i(x)\, v_i\Big) \quad : (\text{by } *) \\ &= \sum_{i=1}^{n} v^i(x)\, \Phi_\beta(v_i) \quad : \text{by linearity of } \Phi_\beta \\ &= \sum_{i=1}^{n} v^i(x)\, e_i \end{aligned}$$

aha, $\Phi_\beta(x) = (v^1(x), v^2(x), \dots, v^n(x))$.

(c.) oops, I already did it $\nearrow$ it's really immediate from (b.)

P102 If $W \leq V$ then $\text{ann}(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for } w \in W\}$

(a.) Let $\alpha, \beta \in \text{ann}(W)$ and $c \in \mathbb{F}$. If $w \in W$ then

$$(c\alpha + \beta)(w) = c\,\alpha(w) + \beta(w) = c(0) + 0 = 0$$

using $\alpha, \beta \in \text{ann}(W)$ means $\alpha(w) = 0$ and $\beta(w) = 0$ for any $w \in W$. Thus $c\alpha, \alpha + \beta \in \text{ann}(W)$. Finally note the zero map $0(x) = 0 \ \forall x \in V$ has $0(w) = 0 \ \forall w \in W$ thus $0 \in \text{ann}(W)$ and we conclude by the subspace test that $\text{ann}(W) \leq V^*$.

(b.) if $W_1 \leq W_2 \leq V$ then suppose $\alpha \in \text{ann}(W_2)$ if $w \in W_1$ then $w \in W_2$ since $W_1 \leq W_2$ thus $\alpha(w) = 0$. But, $w \in W_1$ was arbitrary hence $\alpha(w) = 0$ $\forall w \in W_1$ which proves $\alpha \in \text{ann}(W_1)$ $\therefore$ $\underline{\text{ann}(W_2) \subseteq \text{ann}(W_1)}$.

P103 Find isomorphism from $W^\perp = \{x \in \mathbb{R}^n \mid x \cdot w = 0 \ \forall w \in W\}$ and $\text{ann}(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \text{ for each } w \in W\}$

(I should clarify, $W \leq \mathbb{R}^n$ for this problem!)

Let $x \in W^\perp$ and consider $\alpha_x : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\alpha_x(y) = x \cdot y \ \ \forall y \in \mathbb{R}^n$. Notice α_x is linear hence $\alpha_x \in (\mathbb{R}^n)^*$ and $\alpha_x(w) = x \cdot w = 0 \ \ \forall w \in W$ thus $\alpha_x \in \text{ann}(W)$. Thus define $\Psi : W^\perp \longrightarrow \text{ann}(W)$ by $\Psi(x) = \alpha_x$ and observe

$$\Psi(cx_1 + x_2)(y) = (cx_1 + x_2) \cdot y = c\, x_1 \cdot y + x_2 \cdot y = c\,\alpha_{x_1}(y) + \alpha_{x_2}(y)$$

for each $y \in \mathbb{R}^n$ thus $\Psi(cx_1 + x_2) = c\,\Psi(x_1) + \Psi(x_2)$.

P103 continued:

Let $\alpha \in \text{ann}(W)$ then construct the vector

$x = (\alpha(e_1), \alpha(e_2), \dots, \alpha(e_n))$ and notice for $w \in W$,

$$x \cdot w = (\alpha(e_1), \dots, \alpha(e_n)) \cdot (w_1, \dots, w_n)$$

$$= w_1 \alpha(e_1) + \dots + w_n \alpha(e_n)$$

$$= \alpha(w_1 e_1 + \dots + w_n e_n)$$

$$= \alpha(w) = 0 \quad (\text{remember, we assumed } \alpha \in \text{ann}(W))$$

Thus $\Psi : W^\perp \longrightarrow \text{ann}(W)$ is onto since $\Psi(x) = \alpha$ as discussed above. Furthermore,

For $x_1, x_2 \in W^\perp$,

$$\Psi(x_1) = \Psi(x_2) \Rightarrow \alpha_{x_1} = \alpha_{x_2}$$

$$\Rightarrow \alpha_{x_1}(y) = \alpha_{x_2}(y) \quad \forall y \in \mathbb{R}^n$$

$$\Rightarrow x_1 \cdot e_j = x_2 \cdot e_j \quad \forall j = 1, 2, \dots, n$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow \Psi \text{ is } 1\text{-}1.$$

Thus $\Psi : W^\perp \longrightarrow \text{ann}(W)$ is an <u>isomorphism</u> and this shows $W^\perp \cong \text{ann}(W)$. //

<u>Remark</u>: this sol^{n} is really just using the isomorphism we found of $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ given by $\alpha \longmapsto (\alpha(e_1), \dots, \alpha(e_n))$ restricted to $\text{ann}(W) \leq (\mathbb{R}^n)^*$. I think, $\Phi_{\beta^*}(\alpha) = [\alpha]_{\beta^*}$

$\Psi^{-1} = \Phi_{\beta^*}\big|_{\text{ann}(W)}$, oh, well also the codomain of Φ_{β^*} is reduced to $W^\perp$

P103 **Sol^n Two:** $\beta = \{e_1, e_2, \dots, e_n\}$ & $\beta^* = \{e^1, \dots, e^n\}$ as usual,

Consider $\varphi : \text{ann}(W) \longrightarrow \mathbb{R}^n$ given by $\Phi_{\beta^*}\big|_{\text{ann}(W)} = \varphi$

that is $\varphi(\alpha) = \Phi_{\beta^*}(\alpha) = (\alpha(e_1), \dots, \alpha(e_n))$ then

φ is restriction of linear map Φ_{β^*} and is thus linear.

Moreover, if $w \in W$ and we study $\varphi(\alpha) \cdot w$ for $\alpha \in \text{ann}(W)$,

$$\begin{aligned}
\varphi(\alpha) \cdot w &= (\alpha(e_1), \dots, \alpha(e_n)) \cdot w \\
&= w_1 \alpha(e_1) + \dots + w_n \alpha(e_n) \\
&= \alpha(w_1 e_1 + \dots + w_n e_n) \\
&= \alpha(w). \\
&= 0 \quad \Rightarrow \varphi(\alpha) \in W^{\perp} \therefore \underline{\text{Range}(\varphi) \subseteq W^{\perp}}.
\end{aligned}$$

Suppose $x \in W^{\perp}$ then define $\Psi(x) : \mathbb{R}^n \to \mathbb{R}$ by

$\Psi(x)(y) = x \cdot y \quad \forall y \in \mathbb{R}^n$. Observe $\Psi(x) \in V^*$

and $\Psi(x)(w) = x \cdot w = 0 \quad \forall w \in W$ as $x \in W^{\perp}$

thus $\Psi(x) \in \text{ann}(W)$ thus $\underline{\text{Range}(\Psi) \subseteq \text{ann}(W)}$. Let

$\tilde{\varphi} : \text{ann}(W) \longrightarrow W^{\perp}$ and $\tilde{\Psi} : W^{\perp} \longrightarrow \text{ann}(W)$ be defined

by $\tilde{\varphi}(\alpha) = \varphi(\alpha) \quad \forall \alpha \in \text{ann}(W)$ and $\tilde{\Psi}(x) = \Psi(x) \quad \forall x \in W^{\perp}$.

Calculate, for $x \in W^{\perp}$ or $\alpha \in \text{ann}(W)$,

$$\begin{aligned}
(\tilde{\varphi} \circ \tilde{\Psi})(x) &= \tilde{\varphi}(\tilde{\Psi}(x)) = \varphi(\Psi(x)) = (\Psi(x)(e_1), \dots, \Psi(x)(e_n)) \\
&= (x \cdot e_1, \dots, x \cdot e_n) \\
&= (x_1, \dots, x_n) \\
&= x
\end{aligned}$$

$$\begin{aligned}
(\tilde{\Psi} \circ \tilde{\varphi})(\alpha) &= \Psi(\varphi(\alpha)) \\
&= \Psi(\alpha(e_1), \dots, \alpha(e_n))
\end{aligned}$$

Then
$$\begin{aligned}
(\tilde{\Psi} \circ \tilde{\varphi})(\alpha)(z) &= \Psi(\alpha(e_1), \dots, \alpha(e_n))(z) \\
&= (\alpha(e_1), \dots, \alpha(e_n)) \cdot z \\
&= z_1 \alpha(e_1) + \dots + z_n \alpha(e_n) \\
&= \alpha(z)
\end{aligned}$$

thus $(\tilde{\Psi} \circ \tilde{\varphi})(\alpha) = \alpha$.
Thus $\tilde{\Psi}$ and $\tilde{\varphi}$ are isomorphisms of $W^{\perp}$, $\text{ann}(W)$.

$\boxed{P104}$ $V = P_3(\mathbb{R}) \times \mathbb{C}^{2\times 2}$ as real vector space. Is it possible to find an isomorphism to $S_n \times S_n$ for appropriate n

$(S_n = \{A \in \mathbb{R}^{n\times n} \,/\, A^T = A\})$

$$\dim(V) = \dim(P_3(\mathbb{R})) + \dim_{\mathbb{R}}(\mathbb{C}^{2\times 2}) = 4 + 4(2) = \underline{12}.$$

$$\dim(S_n \times S_n) = \dim S_n + \dim S_n = 2\dim(S_n)$$

Can we find S_n with $\dim(S_n) = 6$? YES!

$$S_3 = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$$

Thus choose $\boxed{n = 3}$

QUESTION: what is $\dim(S_n)$ for arbitrary n ? Can you calculate it ?

Btw,

$$\Psi\left(\left(a + bx + cx^2 + dx^3, \left[\begin{array}{c|c} e+hi & j+ik \\ \hline l+mi & p+iq \end{array}\right]\right)\right) =$$

$$= \left(\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & h \end{bmatrix}, \begin{bmatrix} j & k & l \\ k & m & p \\ l & p & q \end{bmatrix} \right)$$

is the explicit isomorphism (notice I didn't require you find this, dimension alone suffices to allow or disallow the existence of an isomorphism!)

$\boxed{P105}$ Consider $V = \mathbb{R}^3$ and $W = \text{span}\{(1,1,1)\}$.
Find basis and coordinate chart for V/W. Describe geometrically.

$$V/W = \{ (a,b,c) + W \mid (a,b,c) \in \mathbb{R}^3 \}$$

Here $(a,b,c) + W = (\bar{a}, \bar{b}, \bar{c}) + W \iff (a-\bar{a}, b-\bar{b}, c-\bar{c}) \in W$
that is, $\exists k \in \mathbb{R}$ s.t. $a - \bar{a} = b - \bar{b} = c - \bar{c} = k$ which
means $a = \bar{a} + k$, $b = \bar{b} + k$, $c = \bar{c} + k$ for some $k \in \mathbb{R}$.

Consider,

$$(a,b,c) + W = (a-c, b-c, 0) + \underbrace{(c,c,c) + W}_{W}$$
$$= (a-c, b-c, 0) + W$$
$$= (a-c)\left[(1,0,0) + W\right] + (b-c)\left[(0,1,0) + W\right] \quad *$$

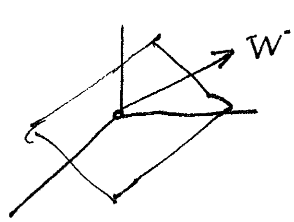
Thus, $V/W \subseteq \text{span}\{e_1 + W, e_2 + W\}$ and it follows
that $V/W = \text{span}\{e_1 + W, e_2 + W\}$. Moreover,

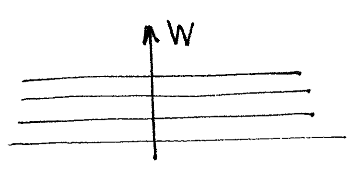
$$c_1(e_1 + W) + c_2(e_2 + W) = 0 \quad \Rightarrow \quad (c_1, c_2, 0) + W = 0$$

But, $0 = W$ in V/W thus $(c_1, c_2, 0) + W = W$ which
indicates $(c_1, c_2, 0) \in W \Rightarrow (c_1, c_2, 0) = k(1,1,1)$
hence $0 = k$ (3^{rd} component) and $c_1 = k$, $c_2 = k$ $\therefore c_1 = c_2 = 0$

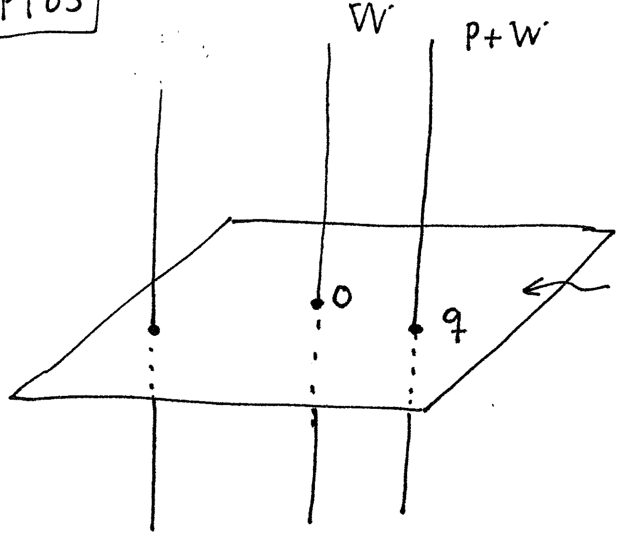
Thus $\boxed{\{e_1 + W, e_2 + W\} = \beta_{V/W} \text{ is basis for } V/W}$

$$\Phi_{\beta_{V/W}}((a,b,c) + W) = \Phi_{\beta_{V/W}}\Big(\underbrace{(a-c)(e_1 + W) + (b-c)(e_2 + W)}_{\text{using } * \text{ once more.}}\Big)$$
$$= (a-c, b-c).$$





P105



The coset space is a plane whose points correspond with ∥-translates of W in $\mathbb{R}^3$.

$x+y+z=0$ has W as its normal line.

-(what follows is not needed, I'm just playing a bit...)-

$q \in p+W \Rightarrow q = (p_1, p_2, p_3) + (k, k, k)$

then the intersection of $p+W$ with $x+y+z=0$ is given by q with k chosen to satisfy

$$(p_1+k)+(p_2+k)+(p_3+k) = 0$$

Thus $k = -\frac{1}{3}(p_1+p_2+p_3)$

$$q = (p_1, p_2, p_3) - \frac{1}{3}(p_1+p_2+p_3)(1,1,1)$$

$$q = \frac{1}{3}(2p_1-p_2-p_3,\ 2p_2-p_1-p_3,\ 2p_3-p_1-p_2)$$

We could use

$$\psi(p+W) = \left(\frac{1}{3}(2p_1-p_2-p_3), \frac{1}{3}(2p_2-p_1-p_3), \frac{1}{3}(2p_3-p_1-p_2)\right)$$

to create coordinate map

$$\psi_{12}(p+W) = \left(\frac{1}{3}(2p_1-p_2-p_3), \frac{1}{3}(2p_2-p_1-p_3)\right)$$

(or pick any two other components of ψ; ψ_{13}, ψ_{23}.

$\psi_{12}^{-1}(1,0) = p+W$ s.t. $2p_2 = p_1+p_3$ and $2p_1 = 3+p_2+p_3$

$p_3 = 2p_2 - p_1 = 2p_1 - 3 - p_2$

$\Rightarrow 3p_2 = 3p_1 - 3$, set $p_1 = 0$, $p_2 = -1$ and $p_3 = -2$

$= (0,-1,-2)+W$

$= (3,2,1)+W.$

P105 continued (again can skip this if not interested ☺)

I calculated $\psi_{12}^{-1}(1,0) = (3,2,1) + W$

then $\psi_{12}^{-1}(0,1) =$

$$\tfrac{1}{3}(2P_1 - P_2 - P_3) = 0, \quad \tfrac{1}{3}(2P_2 - P_1 - P_3) = 1$$

$$2P_1 = P_2 + P_3 \qquad 2P_2 = 3 + P_1 + P_3$$

$$P_3 = 2P_1 - P_2 = 2P_2 - P_1 - 3$$

$$3P_1 = 3P_2 - 3$$

Setting $P_1 = 1$ get $P_2 = 2$ and so $P_3 = 2(1) - 2 = 0$

$\therefore \quad \psi_{12}^{-1}(0,1) = (1,2,0) + W = (2,3,1) + W$

Thus $\beta_{12} = \{(3,2,1) + W, (2,3,1) + W\}$ basis has ψ_{12} as its coordinate map.

Remark: there are <u>many</u> choices for a basis on V/W.

Simple Algorithm to find basis for V/W:

1.) find basis for W say β_W and find β_{V-W} s.t. $\beta_W \cup \beta_{V-W}$ serves as basis for V

2.) $\beta_{V-W} + W = \{x + W \mid x \in \beta_{V-W}\}$ is basis for V/W.

$\boxed{P106}$ Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined $T(f(x)) = f'(x)$ for each $f(x) \in P_2(\mathbb{R})$. Consider, $f(x) \in Ker(T)$ provides

$f'(x) = 0 \Rightarrow f(x) = c \therefore Ker(T) = span\{1\}$. Let

$S : P_2(\mathbb{R})/Ker(T) \longrightarrow T(P_2(\mathbb{R}))$ be defined by

$$S(f(x) + Ker\,T) = T(f(x)) = f'(x)$$

Let $[f(x)] = f(x) + Ker(T)$ then

$$S([f(x)]) = f'(x)$$

In particular setting $f(x) = a + bx + cx^2$

$$S([a + bx + cx^2]) = \frac{d}{dx}\left[a + bx + cx^2\right] = b + 2cx$$

Define $G(a + bx) = [ax + \frac{1}{2}bx^2]$ ~~and~~ for each

$a + bx \in T(P_2(\mathbb{R}))$. Consider,

$$S(G(a+bx)) = S([ax + \tfrac{1}{2}bx^2]) = \frac{d}{dx}(ax + \tfrac{1}{2}bx^2) = a + bx$$

and,
$$G(S([a+bx+cx^2])) = G(b + 2cx) = [bx + cx^2]$$

but, $[a + bx + cx^2] = [bx + cx^2]$ since $(bx + cx^2) - (a+bx+cx^2) = -a$

is in $Ker(T) = span\{1\}$. Thus

$$G(S[a+bx+cx^2]) = [a+bx+cx^2]$$

we find $G = S^{-1}$. In summary

$$\boxed{S^{-1}(a+bx) = [ax + \tfrac{1}{2}bx^2]}$$

So the <u>inverse</u> of differentiation is integration, this being a literal inverse requires we work with cosets of $\frac{d}{dx}$'s kernel as we've done here.

$\boxed{P107}$ Suppose $S' \subseteq V$. Define $S'+W = \{s+W \mid s \in S'\}$

(a.) Suppose S' is LI. Is $S'+W$ LI in V/W? Well, Linear dependence in V/W for $S'+W$ requires we find a nontrivial sol^{n} to

$$c_1(s_1+W) + c_2(s_2+W) + \cdots + c_k(s_k+W) = 0$$

$*$ this requires $\exists\, c_1, c_2, .., c_n \in \mathbb{F}$ not all zero such that $c_1s_1 + c_2s_2 + \cdots + c_ns_n \in W$. Certainly this is possible unless $W \cap \text{span}(S') = \{0\}$.

For example, $\{e_1\} = S'$ is LI in $\mathbb{R}^2$. Let $W = \text{span}\{e_1\}$

$$e_1 + W = e_1 + \text{span}\{e_1\} = \text{span}\{e_1\} = W$$

thus $e_1 + W = 0$ in $\mathbb{R}^2/W$ hence $S'+W$ is <u>not</u> LI.

Again, $W \cap \text{span}(S') = \{0\}$ would imply $S'+W$ is LI, this is essentially clear from $*$.

(b.) Suppose $S' \subseteq V$ is linearly dependent. Then $\exists\, c_1, .., c_n \in \mathbb{F}$ not all zero s.t. $\sum_{i=1}^{n} c_i s_i = 0$. Since $0 \in W$ we also have

$$\sum_{i=1}^{n} c_i(s_i+W) = \left(\sum_{i=1}^{n} c_i s_i\right) + W = W$$

consequently $S'+W$ is likewise linearly dependent.

<u>Remark:</u> for (b.) the linear dep. in V/W could be rather silly for $S'+W$. We could have $S' \subseteq W$ hence $S'+W = \{W\}$ so $S+W$ is the set containing zero in V/W.

P108 Let $S_n = \{ A \in \mathbb{R}^{n\times n} \mid A^T = A \}$ and
$A_n = \{ A \in \mathbb{R}^{n \times n} \mid A^T = -A \}$

Consider $T : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ defined by

$$T(A) = \tfrac{1}{2}(A + A^T)$$

then $\ker(T) = \{ A \in \mathbb{R}^{n\times n} \mid \tfrac{1}{2}(A+A^T) = 0 \} = A_n$

and $T(\mathbb{R}^{n\times n}) = \{ \tfrac{1}{2}(A+A^T) \mid A \in \mathbb{R}^{n\times n} \} = S_n$

since $A \in S_n \Rightarrow A = A^T$ and thus $A = \tfrac{1}{2}(A+A^T)$

so, $T(\mathbb{R}^{n\times n}) = \{ A \mid A \in S_n \}$. Don't believe it yet?

fine, consider $B \in T(\mathbb{R}^{n\times n}) \Rightarrow B = \tfrac{1}{2}(A+A^T)$ for some $A \in \mathbb{R}^{n\times n}$ thus $B^T = \left(\tfrac{1}{2}(A+A^T)\right)^T = \tfrac{1}{2}(A^T + A) = B \therefore B \in S_n$ so $T(\mathbb{R}^{n\times n}) \subseteq S_n$. Conversely, $A \in S_n$ provides $A = \tfrac{1}{2}(A + A^T) \in T(\mathbb{R}^{n\times n})$ thus $S_n \subseteq T(\mathbb{R}^{n\times n})$

Therefore, $T(\mathbb{R}^{n\times n}) = S_n$.

Continuing, by 1st isomorphism Thm

$$\mathbb{R}^{n\times n} / \ker T \cong T(\mathbb{R}^{n\times n})$$

$$\therefore \boxed{\mathbb{R}^{n\times n} / A_n \cong S_n}$$

Notice, knowing $\dim(A_n)$ or $\dim(S_n)$ reveals the other since $\underbrace{\dim(\mathbb{R}^{n\times n})}_{n^2} - \dim(A_n) = \dim(S_n)$.