Same rules as Homework 1. However, do keep in mind you are free to use technology to find e-vectors for 3×3 examples. Also, it would be wise to work out at least one of them by hand to increase your skill-level. Usually, students need some practice with factoring characteristic equations (actually, the most critical difficulty here is not finding $\text{char}(x) = \det(x - T)$, but, in the polynomial factoring which follows...)

Problem 101 Your signature below indicates you have:

(a.) I read Chapter 7 of Curtis: ____________________.

(b.) I read Chapter 7 supplemental by Cook: ____________________.

(c.) I watched the extra magic hour Lectures: ____________________.

Problem 102 Curtis §22 exercise #11 on page 193.

Problem 103 Curtis §23 exercise #3 on page 201.

Problem 104 Curtis §23 exercise #6 on page 201.

Problem 105 Curtis §23 exercise #7 on page 201.

Problem 106 Curtis §24 exercise #5 on page 215.

Problem 107 Curtis §24 exercise #8 on page 216.

Problem 108 Curtis §25 exercise #6 on page 226.

Problem 109 Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$. I'll be nice and tell you the eigenvalues are $\alpha = 1, -2$ and 3. Find an eigenbasis with respect to A. Diagonalize A and calculate A^n explicitly.

Problem 110 Continuing the previous problem, let $T = L_A$ and find the standard matrices of transformations E_1, E_2, E_3 for which $E_j\mathbb{R}^3 = \text{Ker}(T - \alpha_j)$ and $\mathbb{R}^3 = E_1\mathbb{R}^3 \oplus E_2\mathbb{R}^3 \oplus E_3\mathbb{R}^3$ where E_1, E_2, E_3 are idempotent and pairwise commuting with $E_1E_2 = E_1E_3 = E_2E_3 = 0$.

Problem 111 Consider $V = \text{span}_{\mathbb{R}}\{\cosh(x), \sinh(x), \cos(x), \sin(x)\}$ and let $T = D^2 + 1$ where $D = d/dx$. Find the eigenvalues of T and decompose V as the direct sum of eigenspaces.

Problem 112 Consider $V = \text{span}_{\mathbb{R}}\{\cosh(x), \sinh(x), \cos(x), \sin(x)\}$ and let $T = D$ where $D = d/dx$. Find the characteristic and minimal polynomials of T. Also, find the rational cannonical form of T.

Problem 113 Consider T of the previous problem once more. Complexify T and find a complex eigenbasis for the complexification of T. Also, determine the real Jordan form of T.

Problem 114 Find the companion matrix of $p(x) = x^5 + 3x^4 + 2x^2 - 3x - 9$ and $p(x)^2$ using the set-up as in Curtis' Chapter 7 (while prime polynomials are of primary interest to the classification results of Chapter 7, we can just as well calculate the companion matrix for any polynomial).

Problem 115 Suppose you are given a 5×5 matrix C with eigenvalues $\lambda = 2$ repeated five times. List the possible Jordan forms which are similar to the given C; that is $A = P^{-1}CP$ with $P = [\beta]$ where β is a Jordan basis for C. For each possibility, also find the minimal polynomial of C. To keep your list shorter, we'll consider two matrices with the same set of diagonal blocks the same Jordan form.

Problem 116 Suppose you are given a real 4×4 matrix C with complex eigenvalues $\lambda = 2 + 3i$ repeated twice. List the possible real Jordan forms which are similar to the given C. For each possibility, also find the minimal polynomial C. To keep your list shorter, we'll consider two matrices with the same set of diagonal blocks the same real Jordan form.

Problem 117 Suppose that If $A = \text{diag}\left(\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right)$ where this notation indicates that A is block-diagonal with the diagonal blocks as given. Find the eigenvalues of A and state the algebraic and geometric multiplicity of each eigenvalue. Recall we use the notation λ_j has algebraic multiplicity a_j and geometric multiplicity g_j.

Problem 118 Let V be a real 4 dimensional vector space. Suppose $T : V \to V$ is a linear transformation such that:

$$T(v_1) = v_1, \quad T(v_2) = 3v_2, \quad T(v_3) = 6v_3 - 7v_4, \quad T(v_4) = 6v_4 + 7v_3$$

Find the eigenvalues and complex eigenvalues of T. (technically, complex eigenvalues are the eigenvalues of the complexification of T) (assume $\{V_1, V_2, V_3, V_4\}$ LI)

Problem 119 Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ where A is as was given in Problem 102. Also, explicitly find the matrix exponential e^{tA}.

Problem 120 Suppose T is a real linear transformation on a vector space with basis $\vec{a}, \vec{b}$. Also, suppose the complexification of T has $T(\vec{u}) = (3 + 2i)\vec{u}$ where $\vec{u} = \vec{a} + i\vec{b}$ for real vectors $\vec{a}, \vec{b}$. Find the general real solution of $\frac{d\vec{v}}{dt} = T(\vec{v})$.

MISSION 7 SOLUTION

$\boxed{\text{P102}}$ Show $T \in L(V,V)$ has at most $n = \dim(V)$ distinct characteristic roots. (§22 #11 p. 193 CURTIS)

You could argue this several ways

1.) $P(\lambda) = 0 \Rightarrow (x-\lambda) \mid P(x)$

thus $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ <u>distinct</u> $\Rightarrow (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_m) \mid P(x)$.

Note, $P(x) = \det(T - x)$ is n^{th} order polynomial thus $m > n$ is absurd as it suggests an n^{th} order polynomial has a factor of larger degree.

2.) $\lambda_1, \dots, \lambda_m$ distinct and $m > n$ implies that the set of e-vectors $\{v_1, \dots, v_m\}$ belonging to these roots is LI. Since $\dim V = n$ this is impossible (can't have more than n-LI vect. in V).

Whichever argument you use we find the same result; there can be at most n distinct e-values.

$\boxed{\text{P103}}$ Show $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is diagonalizable in $\mathbb{C}^{3\times3}$ but, not in $\mathbb{R}^{3\times3}$

(§23 #3 p. 201)

Remember, $m(x)$ is product of distinct linear factors iff A diagonable. Also $m(x) \mid P(x)$ where each factor in $P(x)$ must appear in $m(x)$ (well, irreducible factor to be precise.)

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 + 1 = (1-\lambda)\underbrace{(\lambda^2 + \lambda + 1)}_{\text{irreducible over } \mathbb{R}}$$

Thus, for $A \in \mathbb{C}^{3\times3}$, $m(x) = (\lambda - 1)(\lambda + \frac{1}{2} - \frac{i\sqrt{3}}{2})(\lambda + \frac{1}{2} + \frac{i\sqrt{3}}{2})$ $\therefore$ A diagonable.

but for $A \in \mathbb{R}^{3\times3}$, $m(x) = (\lambda - 1)(\lambda^2 + \lambda + 1)$ $\therefore$ A not diagonable.

P103 Continued

I've answered the question already. But, now I'll show the details of <u>how</u> the diagonalization works out.

$\underline{\lambda = 1}$

$$A - I = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{observe} \quad \vec{V}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \text{Null}(A - I).$$

Alternatively, if you can't see it, $A - I \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ hence V_1 follows.

Continuing, since $A \in \mathbb{C}^{3\times 3}$ has $A^* = A$ we know if we solve $A\vec{V} = \lambda \vec{V}$ then $A\vec{V}^* = \lambda^* \vec{V}^*$ so just one more calculation to go. Note $(-1+i\sqrt{3})(-1-i\sqrt{3}) = 4$ and

$$\frac{-2-2i\sqrt{3}}{2-2i\sqrt{3}} = \frac{(-2-2i\sqrt{3})(1+i\sqrt{3})}{2(1+3)} = \frac{1}{4}(-1-i\sqrt{3})(1+i\sqrt{3}) = \frac{2-2i\sqrt{3}}{4} = \frac{1-i\sqrt{3}}{2}.$$

$\boxed{\lambda = \frac{-1+i\sqrt{3}}{2}}$ $\quad -\lambda = \frac{1-i\sqrt{3}}{2}$

$$A - \lambda I \sim 2A - 2\lambda I = \begin{bmatrix} 1-i\sqrt{3} & 0 & 2 \\ 2 & 1-i\sqrt{3} & 0 \\ 0 & 2 & 1-i\sqrt{3} \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 2+2i\sqrt{3} \\ 4 & 2-2i\sqrt{3} & 0 \\ 0 & 2 & 1-i\sqrt{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 0 & 2+2i\sqrt{3} \\ 0 & 2-2i\sqrt{3} & -2-2i\sqrt{3} \\ 0 & 2-2i\sqrt{3} & (1-i\sqrt{3})^2 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 2+2i\sqrt{3} \\ 0 & 2-2i\sqrt{3} & -2-2i\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Note: $(1-i\sqrt{3})^2 = 1 - 2i\sqrt{3} - 3 = -2-2i\sqrt{3}$

Thus, $A - \left(\frac{1-i\sqrt{3}}{2}\right)I \sim \begin{bmatrix} 2 & 0 & 1+i\sqrt{3} \\ 0 & 2 & 1-i\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} \longrightarrow 2u = -(1+i\sqrt{3})w \\ \longrightarrow 2v = -(1-i\sqrt{3})w \end{matrix}$

$\vec{v} = (u, v, w) \in \text{Null}(A - \lambda I)$ has form $w\left(\frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, -1\right) = \vec{V}$

Normalizing to length one,

$$\vec{V} = \frac{1}{\sqrt{3}}\left\langle \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, -1 \right\rangle \text{ for } \lambda = \frac{-1+i\sqrt{3}}{2}$$

$$\therefore \quad \vec{V}^* = \frac{1}{\sqrt{3}}\left\langle \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{2}}{2}, -1 \right\rangle \text{ for } \lambda^* = \frac{-1-i\sqrt{3}}{2}$$

P103 continued

$$P = \left[\vec{v}_1 \middle| \vec{v} \middle| \vec{v}^* \right] = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 & 1+i\sqrt{3} & 1-i\sqrt{3} \\ 2 & 1-i\sqrt{3} & 1+i\sqrt{3} \\ 2 & -2 & -2 \end{bmatrix}$$

$$(P^T)^* P = \frac{1}{12} \begin{bmatrix} 2 & 2 & 2 \\ 1-i\sqrt{3} & 1+i\sqrt{3} & -2 \\ 1+i\sqrt{3} & 1-i\sqrt{3} & -2 \end{bmatrix} \begin{bmatrix} 2 & 1+i\sqrt{3} & 1-i\sqrt{3} \\ 2 & 1-i\sqrt{3} & 1+i\sqrt{3} \\ 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We find $P^\dagger = (P^T)^*$ gives the inverse; $P^{-1} = (P^T)^*$.

Calculate if you dare,

$$P^{-1} A P = \frac{1}{12} \begin{bmatrix} 2 & 2 & 2 \\ 1-i\sqrt{3} & 1+i\sqrt{3} & -2 \\ 1+i\sqrt{3} & 1-i\sqrt{3} & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1+i\sqrt{3} & 1-i\sqrt{3} \\ 2 & 1-i\sqrt{3} & 1+i\sqrt{3} \\ 2 & -2 & -2 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 2 & 2 & 2 \\ 1+i\sqrt{3} & -2 & 1-i\sqrt{3} \\ 1-i\sqrt{3} & -2 & 1+i\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1+i\sqrt{3} & 1-i\sqrt{3} \\ 2 & 1-i\sqrt{3} & 1-i\sqrt{3} \\ 2 & -2 & -2 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i\sqrt{3}) & 0 \\ 0 & 0 & \frac{1}{2}(-1-i\sqrt{3}) \end{bmatrix}}.$$

On the other hand $\vec{v} = \vec{a} + i\vec{b}$ where $\vec{a} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\vec{b} = \frac{1}{2\sqrt{3}} \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \\ 0 \end{bmatrix}$

and we can use $Q = \left[\vec{v}_1 \middle| \vec{a} \middle| \vec{b} \right]$ to transform A to

$$Q^{-1} A Q = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{bmatrix}}_{}$$

Real Jordan Form
(Next best thing to diagonal over $\mathbb{R}$)

$\boxed{P104}$ Suppose $V \neq 0$ is vector space of finite dimension over $\mathbb{C}$ and $T : V \longrightarrow V$ linear with $T^2 = 1$.
Let $V_+ = \{ v \in V \mid T(v) = v \}$ and $V_- = \{ v \in V \mid T(v) = -v \}$
Show that $V = V_+ \oplus V_-$ (#6 of p. 201 CURTIS)

Let $v \in V_+ \cap V_-$ then $T(v) = v$ and $T(v) = -v$ hence $v = -v \Rightarrow v = 0 \Rightarrow V_+ \cap V_- = \{0\}$ as $T(0) = 0 = \pm 0$. Let $x \in V$ and note

$$x = \underbrace{\tfrac{1}{2}\left(x + T(x)\right)}_{x_+} + \underbrace{\tfrac{1}{2}\left(x - T(x)\right)}_{x_-}$$

Observe, $T(x_+) = T\left(\tfrac{1}{2}(x + T(x))\right) = \tfrac{1}{2}T(x) + \tfrac{1}{2}T^2(x) = \tfrac{1}{2}\left(x + T(x)\right)$
thus $T(x_+) = x_+ \therefore x_+ \in V_+$. Likewise,
$T(x_-) = T\left(\tfrac{1}{2}(x - T(x))\right) = \tfrac{1}{2}T(x) - \tfrac{1}{2}T^2(x) = \tfrac{-1}{2}\left(x - T(x)\right)$
thus $T(x_-) = -x_- \therefore x_- \in V_-$. We've shown $x = x_+ + x_-$ for $x_+ \in V_+$ and $x_- \in V_- \therefore V = V_+ + V_-$ and as $V_+ \cap V_- = \{0\}$ we conclude $V = V_+ \oplus V_-$.

$\boxed{P105}$ Let $V = V_1 \oplus \cdots \oplus V_s$ where $V_j \neq 0$ $\forall j = 1, 2, .., s$.
Show $\exists$ $E_1, .., E_s \in L(V, V)$ such that $E_1 + \cdots + E_s = Id_V$
and $E_i E_j = E_j E_i = 0$ for $i \neq j$ and $V_j = E_j(V)$ for $j = 1, .., s$.

If $x \in V$ then $\exists!$ $x_j \in V_j$ for $j = 1, 2, .., s$ such that
$x = x_1 + x_2 + \cdots + x_s$ since $V = \bigoplus_{j=1}^{s} V_j$. Define $E_j : V \to V$
by $\underline{E_j(x) = x_j}_{*}$. Observe $\underline{x_j \in V_j \text{ has } E_j(0 + \cdots + x_j + \cdots + 0) = x_j}$ $**$
thus $E_j(V) = V_j$ as it is clear $\underbrace{E_j(V) \subseteq V_j}_{\text{by } *}$ and $\underbrace{V_j \subseteq E_j(V)}_{\text{by } **}$
Consider, $x = x_1 + \cdots + x_s \in V$,

$$\begin{aligned}(E_1 + \cdots + E_s)(x) &= E_1(x) + \cdots + E_s(x) &&: \text{def}^{\text{t}} \text{ of } E_1 + \cdots + E_s \\ &= x_1 + \cdots + x_s &&: \text{def}^{\text{t}} \text{ of } E_j \text{ for } j = 1, .., s \\ &= x\end{aligned}$$

Thus $E_1 + \cdots + E_s = Id_V$. Suppose $i \neq j$ and $x \in V$,

$$(E_i E_j)(x) = E_i(E_j(x)) = E_i(x_j) = E_i(0 + \cdots + \underset{i^{th}}{0} + \cdots + x_j + \cdots + 0) = 0$$

Likewise $(E_j E_i)(x) = 0$ hence $E_i E_j = 0$ and $E_j E_i = 0$.

$\boxed{P106}$ Suppose V is finite-dim'l over $\mathbb{C}$ and $T \in L(V, V)$
has $T^2 = T$. Does $\exists$ basis for V of characteristic vectors of T

Recall, T is diagonable $\iff$ T has basis of e-vectors for T
and we also know T is diagonable $\iff$ $m(x) = (x - \lambda_1) \cdots (x - \lambda_s)$
for $\lambda_1, ..., \lambda_s$ distinct. Notice, $T^2 = T \Rightarrow T^2 - T = 0$
thus $m(x) = x^2 - x = x(x-1)$ is the minimal polynomial
for T (it has $m(T) = 0$ and adding factors spoils minimality)
Thus T is diagonable.

$\boxed{P107}$ Let V be finite dim'l over $\mathbb{C}$ and suppose $T \in L(V)$ has Jordan decomposition $T = D + N$ (D diagonalizable, N nilpotent). Let $X \in L(V)$. Show $TX = XT \iff DX = XD$ and $NX = XN$.

Let $T = D+N$ be the Jordan Decomp. of $T \in L(V)$ then we know $DN = ND$ and $\exists f(x), g(x) \in \mathbb{C}[x]$ for which $f(T) = D$ and $g(T) = N$. Let $X \in L(V)$.

$\Leftarrow$ Suppose $DX = XD$ and $NX = XN$. Consider,

$$TX = (D+N)X = DX + NX = XD + XN = X(D+N) = XT.$$

$\Rightarrow$ Suppose $XT = TX$. Let $h(x) \in \mathbb{C}[x]$ and suppose $h(x) = \alpha_p x^p + \cdots + \alpha_1 x + \alpha_0$ then observe

$$\begin{aligned} h(T)X &= (\alpha_p T^p + \cdots + \alpha_1 T + \alpha_0)X \\ &= \alpha_p T^p X + \cdots + \alpha_2 T T X + \alpha_1 T X + \alpha_0 X \\ &= \alpha_p T^{p-1} X T + \cdots + \alpha_2 T X T + \alpha_1 X T + \alpha_0 X \\ &\;\vdots \\ &= X(\alpha_p T^p + \cdots + \alpha_1 T + \alpha_0) \\ &= X h(T). \end{aligned}$$

(btw, the proof of Lemma 22.2 is much the same as the argument here)

Thus $D = f(T)$ and $N = g(T)$ both commute with X as $TX = XT$. //

$\boxed{P108}$ §25 #6 on pg. 226

Find Jordan canonical forms over $\mathbb{C}$ for the matrices below. Also comment on similarity of given pairs

(a.) $\underbrace{A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}$ & $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftarrow$ <u>already</u> <u>in</u> <u>Jordan</u> <u>form</u>

$\lambda_1 = \lambda_2 = 1$ and $A - I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is e-vector

and hence A has one gen. e-vector $\Rightarrow A \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

It follows <u>A & B are similar</u>.

<u>Details</u>: $(A - I)\vec{v}_2 = \vec{v}_1 = e_1 \Rightarrow \left[\begin{array}{cc|c} 0 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$

$P = [\vec{v}_1 | \vec{v}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ as theory indicated

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b.) $\underbrace{A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{Not in Jordan form over } \mathbb{C}}$ and $\underbrace{B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{\text{diagonal, Jordan form}}$

$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \Rightarrow \underline{\lambda = \pm i}$.

Thus $A \sim \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and we find $A \sim B$.

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details (not needed if you use the theory). Consider $\lambda = i$

$A - iI = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

hence $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ & $P^{-1} = \frac{-1}{2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \underline{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}.$$

P108 continued

(c.) $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{\qquad\qquad}_{\text{Jordan Form}} \qquad\qquad \underbrace{\qquad\qquad}_{\text{Jordan form}}$$

Same e-values and blocks, just reorder to see $\underline{A \sim B}$.

P109 $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ has e-values $\alpha = 1, -2, 3$. Find an e-basis for A and calculate A^n explicitly

$\lambda = 1$ $\quad A - I = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ thus $\langle u, v, w\rangle \in \text{Null}(A - I)$ has $u + w = 0$ & $v - 4w = 0$ choose $w = 1$ to obtain $\underline{\vec{v}_1 = \langle -1, 4, 1\rangle}$.

$\lambda = -2$ $\quad A + 2I = \begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ if $\langle u, v, w\rangle \in \text{Null}(A + 2I)$ $u + w = 0$ and $v - w = 0$ So choose $w = 1$ and $\underline{\vec{v}_2 = \langle -1, 1, 1\rangle}$.

$\lambda = 3$ $\quad A - 3I = \begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \underline{\vec{v}_3 = \langle 1, 2, 1\rangle}$.

Hence $P = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3] = \begin{bmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ and $P^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 3 & 0 & 3 \end{bmatrix}$

thus, $P^{-1}AP = \begin{bmatrix} 1 & & \\ & -2 & \\ & & 3 \end{bmatrix} = D \Rightarrow A = PDP^{-1}$ and $A^2 = PDP^{-1}PDP^{-1}$

and $A^2 = PD^2P^{-1}$ continuing, $A^n = PD^nP^{-1}$ and clearly $D^n = \begin{bmatrix} 1 & & \\ & (-2)^n & \\ & & 3^n \end{bmatrix}$

$$\therefore \quad A^n = \begin{bmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}P^{-1} = \begin{bmatrix} -1 & -(-2)^n & 3^n \\ 4 & (-2)^n & 2(3)^n \\ 1 & (-2)^n & 3^n \end{bmatrix}\begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 3 & 0 & 3 \end{bmatrix}\frac{1}{6}$$

$$\therefore \quad \boxed{A^n = \frac{1}{6}\begin{bmatrix} 1 + (-1)^n 2^{n+1} + 3^{n+1} & -2 - (-2)^{n+1} & 3 - 6(-2)^n + 3^{n+1} \\ -4 + (-2)^{n+1} + 2(3)^{n+1} & 8 + (-2)^{n+1} & -12 + 6(-2)^n + 12(3)^n \\ -1 + (-2)^{n+1} + 3^{n+1} & 2 + (-2)^{n+1} & 3 + 3^{n+1} \end{bmatrix}}$$

$\boxed{P110}$ continuing $\underline{109}$ we study $T = L_A$. Observe,

$$m(x) = (x-1)(x+2)(x-3) \quad \text{thus}$$

$$q_1(x) = \frac{m(x)}{x-1} = (x+2)(x-3)$$

$$q_2(x) = \frac{m(x)}{x+2} = (x-1)(x-3)$$

$$q_3(x) = \frac{m(x)}{x-3} = (x-1)(x+2)$$

We seek $a_1(x), a_2(x), a_3(x)$ for which $a_1(x)q_1(x) + a_2(x)q_2(x) + a_3(x)q_3(x) = 1$
I hope A, B, C will suffice for $a_1(x), a_2(x), a_3(x)$ here,

$$\left.\begin{array}{l} A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2) = 1 \\[6pt] \underline{x=1|} \quad (3)(-2)A = 1 \quad \therefore \quad A = -1/6 \\ \underline{x=-2|} \quad (-3)(-5)B = 1 \quad \therefore \quad B = 1/15 \\ \underline{x=3|} \quad (2)(5)C = 1 \quad \therefore \quad C = 1/10 \end{array}\right\}$$

Thus, following proof of Th^{m} 24.9, $f_1(x) = \frac{-1}{6}(x+2)(x-3)$
and $f_2(x) = \frac{1}{15}(x-1)(x-3)$ and $f_3(x) = \frac{1}{10}(x-1)(x+2)$
define $E_j = f_j(T)$, for $j = 1, 2, 3$ thus,

$$[E_1] = \frac{-1}{6}(A+2I)(A-3I) = \frac{-1}{6}\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} = \frac{-1}{6}\begin{bmatrix} -1 & 2 & -3 \\ 4 & -8 & 12 \\ 1 & -2 & 3 \end{bmatrix}$$

$$[E_2] = \frac{1}{15}(A-I)(A-3I) = \frac{1}{15}\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 5 & 5 & -15 \\ -5 & -5 & 15 \\ -5 & -5 & 15 \end{bmatrix}$$

$$[E_3] = \frac{1}{10}(A-I)(A+2I) = \frac{1}{10}\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 5 & 0 & 5 \\ 10 & 0 & 10 \\ 5 & 0 & 5 \end{bmatrix}$$

following Th^{m} 24.9 (Primary Decomposition Th^{m}) we <u>know</u>, if
we calculated correctly, $E_1 + E_2 + E_3 = 1$ and $E_j^2 = E_j$ for $j = 1, 2, 3$
and $E_i E_j = 0$ for $i \neq j$. Moreover, $E_j(\mathbb{R}^3) = \text{Ker}(T - \alpha_j)$.

We can verify all these explicitly with understanding $[E_1^2] = [E_1]^2$
and $[1] = I$ etc... I'll say a bit more about structure ↲

P110 continued

We can understand the form of E_j in view of our expectation $E_j(\mathbb{R}^3) = \text{Col}\,[E_j] = \text{Ker}(A - \alpha_j I)$

$$[E_1] = \frac{-1}{6}\begin{bmatrix} -1 & 2 & -3 \\ 4 & -8 & 12 \\ 1 & -2 & 3 \end{bmatrix} = \frac{-1}{6}\underbrace{\Big[\vec{v}_1 \,\Big|\, -2\vec{v}_1 \,\Big|\, 3\vec{v}_1\Big]} = \frac{-1}{6}\vec{v}_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}^T$$

from P109 : $\vec{v}_1 = \langle -1, 4, 1 \rangle$

Hence $\text{Col}\,[E_1] = \text{span}\{\vec{v}_1\}$ also, I note $[E_1] = \vec{v}_1\,\text{row}_1(P^{-1})$

Next, as $\vec{v}_2 = \langle -1, 1, 1 \rangle$ from P109

$$[E_2] = \frac{1}{15}\begin{bmatrix} 5 & 5 & -15 \\ -5 & -5 & 15 \\ -5 & -5 & 15 \end{bmatrix} = \frac{1}{15}\Big[-5\vec{v}_2 \,\Big|\, -5\vec{v}_2 \,\Big|\, 15\vec{v}_2\Big] = \frac{-1}{3}\vec{v}_2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}^T$$

Again the pattern $[E_2] = \vec{v}_2\,\text{row}_2(P^{-1})$ is seen (see P109 for P^{-1})

And, $\text{Col}\,[E_2] = \text{Span}\{\vec{v}_2\} = \text{Null}(A + 2I)$.

Last, as $\vec{v}_3 = \langle 1, 2, 1 \rangle$

$$[E_3] = \frac{1}{10}\begin{bmatrix} 5 & 0 & 5 \\ 10 & 0 & 10 \\ 5 & 0 & 5 \end{bmatrix} = \frac{1}{2}\vec{v}_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T = \vec{v}_3 \underbrace{[\tfrac{1}{2}, 0, \tfrac{1}{2}]}_{\text{row}_3(P^{-1})}$$

and $\text{Col}\,[E]_3 = \text{Span}\{\vec{v}_3\} = \text{Null}(A - 3I)$.

Why these patterns ?

P110 continued

$$P^{-1}P = \left[row_i(P^{-1})\,\vec{V}_j \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}P = \vec{V}_1\, row_1(P^{-1}) + \vec{V}_2\, row_2(P^{-1}) + \vec{V}_3\, row_3(P^{-1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad **$$

I suppose, once we decide $Range(E_j) = Ker(T-\alpha_j)$ then as $Ker(T-\alpha_j) = span\{\vec{V}_j\}$ here $(T = L_A)$ the $(**)$ identity forces $row_j(P^{-1})$ to create $[E_j]$ as a column-row product. Any way given Th$^{\underline{m}}$ 24.9 and this discussion, if $A \in \mathbb{F}^{n\times n}$ is diagonalizable with <u>distinct</u> e-values then the construction of the idempotents $E_1, \dots, E_n$ can be achieved via outer-products using the e-basis & rows of the inverse of the e-basis matrix $P = [v_1|v_2|\cdots|v_n]$. In particular,

$$[E_j] = v_j\, row_j(P^{-1}) \quad \text{defines} \quad E_j : \mathbb{F}^n \longrightarrow \mathbb{F}^n$$

and we can verify $E_1 + \cdots + E_n = 1$ (follows from $**$ for n) and $E_j^2 = E_j$ & $E_iE_j = E_jE_i = 0$ & $Range(E_j) = span\{v_j\} = Ker(T-\alpha_j)$

For example,

$$[E_j^2] = [E_j]^2 = v_j \overbrace{row_j(P^{-1})\, v_j}^{1} row_j(P^{-1}) = v_j row_j(P^{-1}) = [E_j]$$

Likewise $[E_iE_j] = 0$ since $row_i(P^{-1})\, col_j(P) = \delta_{ij} = 0$ for $i \neq j$.

One of the things I take from this, is no accident. we can find <u>constants</u> to weight $f_j(x) = A_j\, q_j(x)$ for <u>distinct</u> diagonalizable case.

$\boxed{P111}$ Let $\beta = \{\cosh x, \sinh x, \cos x, \sin x\}$ and let $T = D^2+1$ where $D = d/dx$ and $T: V \to V$ where $V = \text{span}_{\mathbb{R}}(\beta)$. Find e-values of T and decompose $V = \underbrace{W_1 \oplus \cdots \oplus W_f}_{\text{eigenspaces}}$

(I assume T diagonable by saying this)

Observe, $T(\cosh x) = D^2[\cosh x] + \cosh x = \cosh x + \cosh x = 2\cosh x$

and $T(\sinh x) = D^2[\sinh x] + \sinh x = \sinh x + \sinh x = 2\sinh x$

whereas $T(\cos x) = D^2[\cos x] + \cos x = -\cos x + \cos x = 0$

and $T(\sin x) = D^2[\sin x] + \sin x = -\sin x + \sin x = 0$.

All together $[T]_{\beta\beta} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (not needed here, but if β wasn't nice this would give us <u>how</u> to find e-values etc... by examining char. eq$^{\underline{n}}$...)

$W_1 = \text{span}_{\mathbb{R}}\{\cosh x, \sinh x\} = \text{Null}(T-2)$

$W_2 = \text{span}_{\mathbb{R}}\{\cos x, \sin x\} = \text{Null}(T)$

We find $V = \text{Null}(T) \oplus \text{Null}(T-2)$

or, if you prefer, $\underline{V = \text{span}\{\cosh x, \sinh x\} \oplus \text{span}\{\cos x, \sin x\}}$.

$\boxed{P112}$ $V = \text{span}\,\beta$ as in $\boxed{P111}$, but let $T = D$ where $D = d/dx$ find char. and min polynomials for T

$$\left.\begin{aligned} T(\cosh x) &= \sinh x \\ T(\sinh x) &= \cosh x \\ T(\cos x) &= -\sin x \\ T(\sin x) &= \cos x \end{aligned}\right\} \longrightarrow [T]_{\beta\beta} = A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\det(xI - A) = \det\left[\begin{array}{cc|cc} x & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & x & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & x & -1 \\ & & 1 & x \end{array}\right] = \boxed{(x^2-1)(x^2+1) = \text{char}_A(x)}$$

(could have $\pm$ this and factoring is good,

over $\mathbb{R}$, $\text{char}_A(x) = (x-1)(x+1)(x^2+1)$

P112 continued

Since $char_A(x) = (x-1)(x+1)(x^2+1)$ we cannot remove any factors over $\mathbb{R}$ (or $\mathbb{C}$ for that matter) we find $\boxed{m(x) = (x-1)(x+1)(x^2+1)}$

{this follows from $Th^{\underline{m}}$ that $m(x)$ and $char_A(x)$ share same prime factors (but not necessarily same multiplicity)}

$Th^{\underline{m}}$ 24.9 tells us,

$$V = Ker(T-1) \oplus Ker(T+1) \oplus Ker(T^2+1)$$

We could find γ a basis for V for which

$$\boxed{[T]_{\gamma\gamma} = \left[\begin{array}{cc|c} 1 & 0 & \bigcirc \\ 0 & -1 & \\ \hline \bigcirc & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{array}\right]}$$

← rational canonical form of T

what is below line is not technically needed. The theory already tells us $\exists \gamma$ such that

① funny $\gamma_2 = \{T(\sin x), \sin(x)\} = \{\cos x, \sin x\}$ is the T-cyclic basis for $Ker(T^2+1)$ for which T restricted to $Ker(T^2+1)$ has companion matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which we had from the outset. Unfortunately we'll not learn much about how to construct a basis for which the matrix is a companion matrix... oh well.

② On the other hand,

$e^x = \cosh x + \sinh x$

$e^{-x} = \cosh x - \sinh x$

Hence,

$\gamma_1 = \{e^x, e^{-x}\}$

or,

$\gamma_1 = \{\cosh x + \sinh x, \cosh x - \sinh x\}$

gives T restricted to span γ_1 the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

In summary,

$\gamma = \gamma_1 \cup \gamma_2$ gives the claimed rational canonical form

$\boxed{P113}$ Consider $T: V \to V$ as before. Complexify T and find a complex e-basis for T.

We found for $\beta = \{\cosh x, \sinh x, \cos x, \sin x\}$ $T = d/dx$ has

$$[T]_{\beta\beta} = A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and $\mathrm{char}_A(x) = (x^2-1)(x^2+1) \Rightarrow$ e-values $1, -1, i, -i$

we already found $v_1 = e^x = \cosh x + \sinh x$ has $\mathrm{Ker}(T-1) = \mathrm{span}\{v_1\}$ & $v_2 = e^{-x} = \cosh x - \sinh x$ with $\mathrm{Ker}(T+1) = \mathrm{span}\{v_2\}$. Now, consider $\lambda = i$,

$$A - iI = \begin{bmatrix} -i & 1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix} \sim \left[\begin{array}{cc|cc} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ \hline 0 & 0 & 1 & i \\ 0 & 0 & -1 & -i \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & i & 0 & 0 \\ 0 & -2i & 0 & 0 \\ \hline 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{array}\right]$$

thus $A - iI \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathrm{Ker}(A - iI) = \mathrm{span}\{\langle 0, 0, -i, 1\rangle\}$

$u_3 + iu_4 = 0$ set $\vec{u}_4 = 1$

Hence, returning to V, $\Phi_\beta^{-1}\begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix} = -i\cos(x) + \sin(x)$ should serve as complex e-vector with e-value $\lambda = i$. Let's check,

$$\begin{aligned} T_{\mathbb{C}}(-i\cos x + \sin x) &= -i\,T(\cos x) + T(\sin x) \\ &= i\sin x + \cos x \\ &= i(-i\cos x + \sin x) \end{aligned}$$

This is fine, but perhaps you already knew $e^{i\theta} = \cos\theta + i\sin\theta$ hence $e^{i\theta}$ & $e^{-i\theta}$ give the complex e-vectors. That is $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ continued

$\boxed{P113}$ I set $u_4 = 1$ to find $-i\cos x + \sin x$, if we had set $u_4 = i$ then we could also see $\ker(A - iI) = \text{span}\{\langle 0, 0, 1, i\rangle\}$

Moreover, $\tilde{\beta} = \{ e^x, e^{-x}, e^{ix}, e^{-ix} \}$ ← complex e-basis for T

$$\begin{array}{cccc} e^x & e^{-x} & e^{ix} & e^{-ix} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \lambda = 1 & \lambda = -1 & \lambda = i & \lambda = -i \end{array}$$

where

$e^x = \cosh x + \sinh x$

$e^{ix} = \cos x + i \sin x$

So to be explicit,

$$\boxed{\tilde{\beta} = \{ \cosh x + \sinh x,\ \cosh x - \sinh x,\ \cos x + i\sin x,\ \cos x - i \sin x \}}$$

We extract a <u>real</u> basis

$$\gamma = \{ \cosh x + \sinh x,\ \cosh x - \sinh x,\ \cos x,\ \sin x \}$$

for which

$$\underbrace{\boxed{[T]_{\gamma\gamma} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]}}_{\text{Real Jordan form of } T.}$$

<u>Remark</u>: the more you know about $e^{i\theta}$ & $e^x = \cosh x + \sinh x$ the less calculation is actually needed here. I could write <u>much</u> less.

$\boxed{P114}$ $P(x) = x^5 - (-3)x^4 - (0)x^3 - (-2)x^2 - 3x - 9$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 9 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} = A_{P(x)}$$

and, following Curtis' conventions, $A_{P(x)^2}$ is 10×10 with blocks,

$$A_{P(x)^2} = \begin{bmatrix} C & B \\ 0 & C \end{bmatrix} \quad \text{where} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\boxed{P115}$ C is 5×5 matrix and $\lambda = 2$ has algebraic multiplicity 5, this means $\text{Char}_A(x) = (x-2)^5$ or $(2-x)^5$ if you prefer.

$$m(x) = x - 2 \longrightarrow J_c = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{bmatrix}$$

$$m(x) = (x-2)^2 \longrightarrow J_c = \begin{bmatrix} 2 & 1 & & & \\ 0 & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{bmatrix}$$

$$J_c = \begin{bmatrix} 2 & 1 & & & \\ 0 & 2 & & & \\ & & 2 & 1 & \\ & & 0 & 2 & \\ & & & & 2 \end{bmatrix}$$

$$m(x) = (x-2)^3 \longrightarrow J_c = \begin{bmatrix} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ & & & 2 & \\ & & & & 2 \end{bmatrix}$$

$$J_c = \begin{bmatrix} 2 & 1 & 0 & & \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ & & & 2 & 1 \\ & & & 0 & 2 \end{bmatrix}$$

$$m(x) = (x-2)^4 \longrightarrow J_c = \begin{bmatrix} 2 & 1 & 0 & 0 & \\ 0 & 2 & 1 & 0 & \\ 0 & 0 & 2 & 1 & \\ 0 & 0 & 0 & 2 & \\ & & & & 2 \end{bmatrix} \quad \& \quad m(x) = (x-2)^5 \to J_c = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

there are 7 cases. Apparently, $\exists$ 7 classes of similar 5×5 matrices with e-value $\lambda = 2$ $\underbrace{\text{5-times repeated}}$ this language refers to algebraic multiplicity..

$\boxed{P116}$ $C \in \mathbb{R}^{4\times 4}$ with complex e-values $\lambda = 2+3i$ twice repeated
List possible real Jordan forms

$$J_c = \underbrace{\left[\begin{array}{cc|cc} 2 & 3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{array}\right]}_{} \quad \text{or} \quad J_c = \underbrace{\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{array}\right]}_{}$$

$$\underbrace{m(x) = (x-2)^2 + 9}_{} \qquad m(x) = \left[(x-2)^2+9\right]^2$$

$$\underbrace{m_{\mathbb{C}}(x) = (x-2+3i)(x-2-3i)}_{} \qquad \underbrace{m_{\mathbb{C}}(x) = (x-2+3i)^2(x-2-3i)^2}_{}$$

$T_{\mathbb{C}}$ diagonable $\qquad$ $T_{\mathbb{C}}$ not diagonable.

Here $m(x)$ is minimal polynomial for $C \in \mathbb{R}^{4\times 4}$
whereas $m_{\mathbb{C}}(x)$ is minimal poly. for $C \in \mathbb{C}^{4\times 4}$
where $T_{\mathbb{C}} : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$ is the complexification
of T and technically $T_{\mathbb{C}}$ has min. poly $m_{\mathbb{C}}(x)$.

$\boxed{P117}$ If $A = \text{diag}\left(\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right)$

$$\begin{array}{lll} \lambda_1 = 3, & \lambda_2 = 4, & \lambda_3 = 5 \\ a_1 = 2 & a_2 = 3 & a_3 = 2 \\ g_1 = 1 & g_2 = 1 & g_3 = 2 \end{array}$$

where a_j is algebraic multiplicity of λ_j
and $g_j = \dim(\text{Null}(A - \lambda_j I))$ = geometric multiplicity
For shame, I should have asked you to find

$$\text{char}_A(x) = (x-3)^2(x-4)^4(x-5)^2$$

$$m(x) = (x-3)^2(x-4)^4(x-5)$$

Oh well, maybe the final...

$\boxed{P118}$ $V(\mathbb{R})$ is 4-dim'l and $T: V \to V$ linear trans. with

$$\left.\begin{aligned} T(V_1) &= V_1 \\ T(V_2) &= 3V_2 \\ T(V_3) &= 6V_3 - 7V_4 \\ T(V_4) &= 6V_4 + 7V_3 \end{aligned}\right\} \qquad [T]_{\beta\beta} = A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & -7 & 6 \end{bmatrix}}$$

I meant to tell you, $\beta = \{V_1, V_2, V_3, V_4\}$ is LI (hence β a basis for V as $\dim V = 4$.

real Jordan form! I can read from this $\lambda_1 = 1$, $\lambda_2 = 3$ and there is a complex e-value $\lambda_3 = 6 \pm 7i$

$\boxed{P119}$ Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ for $\boxed{P109}$'s A where we found:

$$\boxed{A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \text{ where } \lambda_1 = 1,\ \vec{V}_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix},\ \lambda_2 = -2,\ \vec{V}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ \lambda_3 = 3,\ \vec{V}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

As $\frac{d}{dt}(e^{\lambda t}\vec{u}) = \lambda e^{\lambda t}\vec{u}$ and $A\vec{u} = \lambda\vec{u} \Rightarrow \frac{d}{dt}(e^{\lambda t}\vec{u}) = A(e^{\lambda t}\vec{u})$ we can find 3-LI sol$^{\text{n}}$'s to form general sol$^{\text{n}}$ just using e-vectors,

$$\boxed{\vec{x}(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

To be less elegant, use scalar notation (But DON'T)

$$\begin{aligned} x' &= x - y + 4z \qquad & x &= -c_1 e^{t} - c_2 e^{-2t} + c_3 e^{3t} \\ y' &= 3x + 2y - z \qquad & y &= 4c_1 e^{t} + c_2 e^{-2t} + 2c_3 e^{3t} \\ z' &= 2x + y - z \qquad & z &= c_1 e^{t} + c_2 e^{-2t} + c_3 e^{3t} \end{aligned}\Bigg\} \text{ coupled!}$$

$\boxed{P119}$ to calculate e^{tA} there are several methods. One sol^{n}, use e-vectors,

$$e^{tA}\vec{v} = \left(I + tA + \tfrac{1}{2}t^2A^2 + \cdots\right)\vec{v}$$
$$= \vec{v} + t\lambda\vec{v} + \tfrac{1}{2}t^2\lambda^2\vec{v} + \cdots$$
$$= \left(e^{t\lambda}\right)\vec{v}$$

Hence, for $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for $\lambda = 1, -2, 3$ as in $\boxed{P109}$

$$e^{tA}\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right] = \left[e^{tA}\vec{v}_1 \middle| e^{tA}\vec{v}_2 \middle| e^{tA}\vec{v}_3\right]$$

$$\Rightarrow \quad e^{tA}P = \left[e^{t}\vec{v}_1 \middle| e^{-2t}\vec{v}_2 \middle| e^{3t}\vec{v}_3\right]$$

$$e^{tA} = \left[e^{t}\vec{v}_1 \middle| e^{-2t}\vec{v}_2 \middle| e^{3t}\vec{v}_3\right]P^{-1}$$

$$= \begin{bmatrix} -e^{t} & -e^{-2t} & e^{3t} \\ 4e^{t} & e^{-2t} & 2e^{3t} \\ e^{t} & e^{-2t} & e^{3t} \end{bmatrix}\begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 3 & 0 & 3 \end{bmatrix}\left(\frac{1}{6}\right)$$

$$= \boxed{\begin{bmatrix} e^{t} - 2e^{-2t} - 3e^{3t} & -2e^{t} + 2e^{-2t} & 3e^{t} - 6e^{-2t} + 3e^{3t} \\ -4e^{t} - 2e^{-2t} + 6e^{3t} & 8e^{t} - 2e^{-2t} & -12e^{t} + 6e^{-2t} + 6e^{3t} \\ -e^{t} - 2e^{-2t} + 3e^{3t} & 2e^{t} - 2e^{-2t} & -3e^{t} + 3e^{3t} \end{bmatrix}}$$

$\boxed{P120}$ Suppose $T: V \to V$ where $V = \text{span}_{\mathbb{R}}\{\vec{a}, \vec{b}\}$ and suppose $T_{\mathbb{C}}(\vec{u}) = (3+2i)\vec{u}$ where $\vec{u} = \vec{a} + i\vec{b}$ find general <u>real</u> sol^{n} of $\frac{d\vec{v}}{dt} = T(\vec{v})$

Observe $\vec{z} = e^{(3+2i)t}\vec{u}$ has

$$\frac{d\vec{z}}{dt} = (3+2i)e^{(3+2i)t}\vec{u}$$

whereas

$$T_{\mathbb{C}}(\vec{z}) = e^{(3+2i)t}\,T_{\mathbb{C}}(\vec{u}) = (3+2i)e^{(3+2i)t}\vec{u}$$

thus $\vec{z} = e^{(3+2i)t}\vec{u}$ is complex-sol$^{\underline{n}}$ to

$\frac{d\vec{z}}{dt} = T_{\mathbb{C}}(\vec{z})$. Consider,

$$\begin{aligned}\vec{z} &= e^{3t}(\cos(2t) + i\sin(2t))(\vec{a} + i\vec{b}) \\ &= \underbrace{e^{3t}\cos(2t)\,\vec{a} - e^{3t}\sin(2t)\,\vec{b}}_{Re(\vec{z})} + i\underbrace{\left(e^{3t}\cos(2t)\vec{b} + e^{3t}\sin(2t)\vec{a}\right)}_{Im(\vec{z})}\end{aligned}$$

But, $\frac{d\vec{z}}{dt} = \frac{d\,Re(\vec{z})}{dt} + i\left(\frac{d\,Im(\vec{z})}{dt}\right)$ and

$T_{\mathbb{C}}(\vec{z}) = T(Re(\vec{z})) + i\,T(Im(\vec{z}))$ hence

$$\frac{d\vec{z}}{dt} = T_{\mathbb{C}}(\vec{z}) \Rightarrow \frac{d(Re\,\vec{z})}{dt} + i\frac{d\,Im(\vec{z})}{dt} = T(Re(\vec{z})) + i\,T(Im(\vec{z}))$$

Hence $Re(\vec{z})$ and $Im(\vec{z})$ are <u>real</u> sol^{n}'s to $\frac{d\vec{v}}{dt} = T(\vec{v})$

we conclude,

$$\boxed{\vec{v}(t) = c_1 e^{3t}(\cos(2t)\vec{a} - \sin(2t)\vec{b}) + c_2 e^{3t}(\sin(2t)\vec{b} + \cos(2t)\vec{a})}$$