Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 7 of the notes: ________________.

Problem 109 Let $W_1 = \text{span}\{(1,1,1,0),(0,0,0,1)\}$ and $W_2 = \text{span}\{(1,-1,0,0),(0,1,-1,0)\}$. Show $W_1 \oplus W_2 = \mathbb{R}^4$.

Problem 110 Suppose V is a finite dimensional vector space and $V = W_1 \oplus W_2$. Does it follow that $ann(W_1) \oplus ann(W_2) = V^*$? Prove or disprove.

Problem 111 Let $T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$. Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ provide bases for $W_1 = \text{span}(\beta_1)$ and $W_2 = \text{span}(\beta_2)$.

(a.) show W_1 and W_2 are invariant subspaces of T,

(b.) Find $[T_{W_1}]_{\beta_1,\beta_1}$ and $[T_{W_2}]_{\beta_2,\beta_2}$

(c.) verify $[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2}$ where $\beta = \beta_1 \cup \beta_2$

Problem 112 Let $A \in \mathbb{F}^{n\times n}$ and define $T : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $T(A) = A + A^T$

(a.) show that $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ is an invariant subspace of T.

(b.) show that $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ is contained in $\text{Ker}(T)$.

(c.) Let β_s and β_a be bases for the symmetric and antisymmetric $n \times n$ matrices over $\mathbb{F}$. Form basis $\beta = \beta_s \cup \beta_a$ and find the block-structure of the matrix $[T]_{\beta,\beta}$

Problem 113 Suppose V is a finite dimensional vector space over a field $\mathbb{F}$. If $W_1 + W_2 = V$ and $\dim(W_1) + \dim(W_2) = \dim(V)$ then prove $V = W_1 \oplus W_2$.

Problem 114 If $p(t) = a_o + a_1 t + \cdots + a_n t^n \in \mathbb{R}[t]$ then we define $p(A) = a_o I + a_1 A + \cdots + a_n A^n$. Prove that if v is eigenvector of A with eigenvalue λ then v is also an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.

Problem 115 A square matrix A is nilpotent of degree k if $A^{k-1} \neq 0$ yet $A^k = 0$. Prove $\lambda = 0$ is the only eigenvalue of A.

Problem 116 Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$. Find the real eigenvalues and eigenvectors of A. Also, calculate A^n.

Problem 117 Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Is A diagonalizable as a complex matrix ?

Problem 118 Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Find the Jordan form associated with A.

Problem 119 Let $A = \begin{bmatrix} -6 & -4 \\ 10 & 6 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Is A diagonalizable as a complex matrix? Find the real Jordan form associated with A.

Problem 120 Let $A = \begin{bmatrix} 2 & 4 & -4 \\ -1 & 2 & -1 \\ 1 & 4 & -3 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Is A diagonalizable as a complex matrix? Find the real Jordan form associated with A.

Problem 121 Let $A = \begin{bmatrix} -5 & 4 & 0 & -4 \\ -11 & 8 & -2 & -7 \\ 2 & -1 & 1 & 3 \\ -1 & 1 & -2 & 0 \end{bmatrix}$. You can check this matrix has eigenvalues of $\lambda = 1 \pm 2i$ repeated. In fact (and please, understand, I do **not** want you to actually find these vectors) there exist nonzero vectors $v_1 = a_1 + ib_1$ and $v_2 = a_2 + ib_2$ such that:

$$(A - (1+2i)I)v_1 = 0 \qquad \& \qquad (A - (1+2i)I)v_2 = v_1$$

If $L(x) = Ax$ then find $[L]_{\beta,\beta}$ and $[L]_{\gamma,\gamma}$ with respect to the bases $\beta = \{a_1, b_1, a_2, b_2\}$ and $\gamma = \{v_1, v_2, \overline{v_1}, \overline{v_2}\}$.

Problem 122 Suppose V is a vector space of dimension 4 over $\mathbb{R}$ and $T : V \to V$ is a linear transformation and there exist nonzero vectors v_1, v_2, v_3, v_4 such that:

$$T(v_1) = 7v_1 + v_2, \qquad T(v_2) = 7v_2, \qquad (T - 4Id_V)(v_3) = 0, \qquad T(v_4) = 4v_4$$

Add a needed condition (if any) and find a Jordan basis β for T and calculate $[T]_{\beta,\beta}$. Also, calculate $\det(T)$ and $\text{trace}(T)$.

Problem 123 Suppose A is a 6×6 real matrix with characteristic polynomial $p(t) = (t-3)^3(t-2)^2(t-1)$. What are the possible Jordan forms associated to A. For each form determine the minimal polynomial for A.

Problem 124 Let $T : V \to V$ have basis $\beta = \{v_1, \dots, v_n\}$ for which the matrix of T is in Jordan form:

$$[T]_{\beta,\beta} = J_4(3) \oplus J_2(3) \oplus J_1(3) \oplus J_1(3) \oplus J_4(6)$$

Select vectors from β to construct the basis for each eigenspace and generalized eigenspace for T. That is, find $\beta_j \subset \beta$ for which $\mathcal{E}_{\lambda_j} = \text{span}(\beta_j)$ and $\gamma_j \subset \beta$ for which $\text{span}(\gamma_j) = K_\lambda$ for each eigenvalue of T.

Problem 125 Suppose $T : V \to V$ is a linear transformation such that $\lambda_1 \neq \lambda_2$ are eigenvalues of T. Let $\mathcal{E}_1 = \text{Ker}(T - \lambda_1 Id_V)$ and $\mathcal{E}_2 = \text{Ker}(T - \lambda_2 Id_V)$. Given $\mathcal{E}_1 + \mathcal{E}_2 = V$ show $V = \mathcal{E}_1 \oplus \mathcal{E}_2$ and show T is diagonalizable.

Remark: *I couldn't the calculation below early this semester since we first discussed determinants before we studied linear transformations. I think now is the appropriate time to share this with you. It is considerably easier to understand than the technical proof I gave in terms of elementary matrices and such in the determinants chapter.*

Problem 126 Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ then we define $\Lambda^k T : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ where $\Lambda^k\mathbb{R}^n$ is the vector space of k-vectors over $\mathbb{R}^n$. In particular,

$$\Lambda^k T(v_1 \wedge \cdots \wedge v_k) = T(v_1) \wedge \cdots \wedge T(v_k).$$

Notice $\Lambda^n T(e_1 \wedge \cdots \wedge e_n) = T(e_1) \wedge \cdots \wedge T(e_n) = \det(T) e_1 \wedge \cdots \wedge e_n$.

(a.) if $S, T : \mathbb{R}^n \to \mathbb{R}^n$ show $\Lambda^k(T \circ S) = \Lambda^k T \circ \Lambda^k S$,

(b.) derive $\det(AB) = \det(A)\det(B)$ by examining (a.) with $k = n$ and T, S with $[T] = A$ and $[S] = B$.

Remark: the problem below didn't turn out quite as I had hoped... I accidentally wrote the solution in the problem statement and at the moment I can't think of a good way to fix it. That said, I include it to illustrate how the wedge product can be used to prove things about determinants directly if you have the computational courage:

(I.) The identity $\det\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A)\det(B)$ is important to several key theorems this semester. Let me outline an argument based on the wedge product technique I introduced in some of your previous homework. If $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where $A = [A_1|\cdots|A_m] \in \mathbb{F}^{m\times m}$ and $B = [B_1|\cdots|B_n] \in \mathbb{F}^{n\times n}$ then the standard basis $e_1, e_2, \dots, e_{m+n} \in \mathbb{F}^{m+n}$ gives

$$Me_1 = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \sum_{j_1=1}^{m} A_{j_1 1} e_{j_1}, \ \dots, Me_m = \begin{bmatrix} A_m \\ 0 \end{bmatrix} = \sum_{j_m=1}^{m} A_{j_m m} e_{j_m}$$

and

$$Me_{m+1} = \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = \sum_{k_1=1}^{n} B_{k_1 1} e_{m+k_1}, \ \dots, Me_{m+n} = \begin{bmatrix} 0 \\ B_n \end{bmatrix} = \sum_{k_n=1}^{n} B_{k_n n} e_{m+k_n}.$$

Hence calculate, $Me_1 \wedge \cdots \wedge Me_m \wedge Me_{m+1} \wedge \cdots \wedge Me_{m+n} =$

$$\begin{aligned}
&= \left(\sum_{j_1=1}^{m} A_{j_1 1} e_{j_1}\right) \wedge \cdots \wedge \left(\sum_{j_m=1}^{m} A_{j_1 m} e_{j_m}\right) \wedge \left(\sum_{k_1=1}^{n} B_{k_1 1} e_{m+k_1}\right) \wedge \cdots \wedge \left(\sum_{k_n=1}^{n} B_{k_n 1} e_{m+k_n}\right) \\
&= \left(\sum_{j_1=1}^{m} \cdots \sum_{j_m=1}^{m} A_{j_1 1} \cdots A_{j_1 m} e_{j_1} \wedge \cdots \wedge e_{j_m}\right) \\
&\quad \wedge \left(\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} B_{k_1 1} \cdots B_{k_n 1} e_{m+k_1} \wedge \cdots \wedge e_{m+k_n}\right) \\
&= \left(\sum_{j_1,\dots,j_m=1}^{m} A_{j_1 1} \cdots A_{j_1 m} \epsilon_{j_1 \cdots j_m} e_1 \wedge \cdots \wedge e_m\right) \\
&\quad \wedge \left(\sum_{k_1,\dots,k_n=1}^{n} B_{k_1 1} \cdots B_{k_1 n} \epsilon_{k_1 \cdots k_n} e_{m+1} \wedge \cdots \wedge e_{m+n}\right) \\
&= \det(A)\det(B) e_1 \wedge \cdots \wedge e_m \wedge e_{m+1} \wedge \cdots \wedge e_{m+n}
\end{aligned}$$

Thus $\det\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A)\det(B)$.

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Consider the subspace $W = \text{span}\{(1,1)\}$ of $\mathbb{R}^2$. Suppose $W_1 \oplus W = W_2 \oplus W$ does it follow $W_1 = W_2$? Prove or disprove.

(II.) Consider the linear transformation $L(A) = A^T$ for $A \in \mathbb{F}^{2\times 2}$ where $1 \neq -1$ in $\mathbb{F}$. Find an eigenbasis for L and find a direct sum decomposition of $\mathbb{F}^{2\times 2}$ via the eigenspaces of L. Find the characteristic polynomial $p(x)$ for L.

(III.) Consider $T(f(x)) = -f(x) - f'(x)$ definining a linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$. Find a Jordan basis for T and express the matrix of T as a direct sum of Jordan blocks.

(IV.) Consider $T(f(x)) = f''(x)$ definining a linear transformation $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$. Find a Jordan basis for T and express the matrix of T as a direct sum of Jordan blocks.

(V.) Suppose $T : V \to V$ has eigenvalue λ show for any $n \in \mathbb{N}$ that T^n has eigenvalue λ^n.

(VI.) Prove a linear transformation T is invertible if and only if $\lambda = 0$ is **not** an eigenvalue of T

(VII.) Suppose A is invertible. Prove if A has eigenvalue λ then A^{-1} has eigenvalue $1/\lambda$.

(VIII.) Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable? Find the Jordan form associated with A.

(IX.) Show $A = \begin{bmatrix} -2 & 3 & -1 \\ -4 & 5 & -3 \\ -4 & 4 & -2 \end{bmatrix}$ has eigenvalues of $\lambda = 1$ and $\lambda = 2i$ by finding eigenvectors with the given eigenvalues for A.

(X.) Let $V = \text{span}_{\mathbb{R}}\{e^x, e^{2x}, \cos(x), \sin(x)\}$ and consider $T = D + 1$ where $D = d/dx : V \to V$. Show that $U = \text{span}_{\mathbb{R}}\{e^x, e^{2x}\}$ forms an invariant subspace of V with respect to T and find the matrix of $T|_U$ as well as $T_{V/U}$.