Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapter 7 of the notes: ________________.

Problem 109 Let $W_1 = \text{span}\{(1,1,1,0),(0,0,0,1)\}$ and $W_2 = \text{span}\{(1,-1,0,0),(0,1,-1,0)\}$. Show $W_1 \oplus W_2 = \mathbb{R}^4$.

Problem 110 Suppose V is a finite dimensional vector space and $V = W_1 \oplus W_2$. Does it follow that $ann(W_1) \oplus ann(W_2) = V^*$? Prove or disprove.

Problem 111 Let $T(f(x)) = f(x) + xf'(x)$ for $f(x) \in P_3(\mathbb{R})$. Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ provide bases for $W_1 = \text{span}(\beta_1)$ and $W_2 = \text{span}(\beta_2)$.

(a.) show W_1 and W_2 are invariant subspaces of T,

(b.) Find $[T_{W_1}]_{\beta_1,\beta_1}$ and $[T_{W_2}]_{\beta_2,\beta_2}$

(c.) verify $[T]_{\beta,\beta} = [T_{W_1}]_{\beta_1,\beta_1} \oplus [T_{W_2}]_{\beta_2,\beta_2}$ where $\beta = \beta_1 \cup \beta_2$

Problem 112 Let $A \in \mathbb{F}^{n\times n}$ and define $T : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $T(A) = A + A^T$

(a.) show that $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$ is an invariant subspace of T.

(b.) show that $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ is contained in $\text{Ker}(T)$.

(c.) Let β_s and β_a be bases for the symmetric and antisymmetric $n \times n$ matrices over $\mathbb{F}$. Form basis $\beta = \beta_s \cup \beta_a$ and find the block-structure of the matrix $[T]_{\beta,\beta}$

Problem 113 Suppose V is a finite dimensional vector space over a field $\mathbb{F}$. If $W_1 + W_2 = V$ and $\dim(W_1) + \dim(W_2) = \dim(V)$ then prove $V = W_1 \oplus W_2$.

Problem 114 If $p(t) = a_o + a_1 t + \cdots + a_n t^n \in \mathbb{R}[t]$ then we define $p(A) = a_o I + a_1 A + \cdots + a_n A^n$. Prove that if v is eigenvector of A with eigenvalue λ then v is also an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.

Problem 115 A square matrix A is nilpotent of degree k if $A^{k-1} \neq 0$ yet $A^k = 0$. Prove $\lambda = 0$ is the only eigenvalue of A.

Problem 116 Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$. Find the real eigenvalues and eigenvectors of A. Also, calculate A^n.

Problem 117 Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Is A diagonalizable as a complex matrix ?

Problem 118 Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Find the Jordan form associated with A.

Problem 119 Let $A = \begin{bmatrix} -6 & -4 \\ 10 & 6 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Is A diagonalizable as a complex matrix? Find the real Jordan form associated with A.

Problem 120 Let $A = \begin{bmatrix} 2 & 4 & -4 \\ -1 & 2 & -1 \\ 1 & 4 & -3 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable as a real matrix? Is A diagonalizable as a complex matrix? Find the real Jordan form associated with A.

Problem 121 Let $A = \begin{bmatrix} -5 & 4 & 0 & -4 \\ -11 & 8 & -2 & -7 \\ 2 & -1 & 1 & 3 \\ -1 & 1 & -2 & 0 \end{bmatrix}$. You can check this matrix has eigenvalues of $\lambda = 1 \pm 2i$ repeated. In fact (and please, understand, I do **not** want you to actually find these vectors) there exist nonzero vectors $v_1 = a_1 + ib_1$ and $v_2 = a_2 + ib_2$ such that:

$$(A - (1+2i)I)v_1 = 0 \qquad \& \qquad (A - (1+2i)I)v_2 = v_1$$

If $L(x) = Ax$ then find $[L]_{\beta,\beta}$ and $[L]_{\gamma,\gamma}$ with respect to the bases $\beta = \{a_1, b_1, a_2, b_2\}$ and $\gamma = \{v_1, v_2, \overline{v_1}, \overline{v_2}\}$.

Problem 122 Suppose V is a vector space of dimension 4 over $\mathbb{R}$ and $T : V \to V$ is a linear transformation and there exist nonzero vectors v_1, v_2, v_3, v_4 such that:

$$T(v_1) = 7v_1 + v_2, \qquad T(v_2) = 7v_2, \qquad (T - 4Id_V)(v_3) = 0, \qquad T(v_4) = 4v_4$$

Add a needed condition (if any) and find a Jordan basis β for T and calculate $[T]_{\beta,\beta}$. Also, calculate $\det(T)$ and $\mathrm{trace}(T)$.

Problem 123 Suppose A is a 6×6 real matrix with characteristic polynomial $p(t) = (t-3)^3(t-2)^2(t-1)$. What are the possible Jordan forms associated to A. For each form determine the minimal polynomial for A.

Problem 124 Let $T : V \to V$ have basis $\beta = \{v_1, \dots, v_n\}$ for which the matrix of T is in Jordan form:

$$[T]_{\beta,\beta} = J_4(3) \oplus J_2(3) \oplus J_1(3) \oplus J_1(3) \oplus J_4(6)$$

Select vectors from β to construct the basis for each eigenspace and generalized eigenspace for T. That is, find $\beta_j \subset \beta$ for which $\mathcal{E}_{\lambda_j} = \mathrm{span}(\beta_j)$ and $\gamma_j \subset \beta$ for which $\mathrm{span}(\gamma_j) = K_\lambda$ for each eigenvalue of T.

Problem 125 Suppose $T : V \to V$ is a linear transformation such that $\lambda_1 \neq \lambda_2$ are eigenvalues of T. Let $\mathcal{E}_1 = \mathrm{Ker}(T - \lambda_1 Id_V)$ and $\mathcal{E}_2 = \mathrm{Ker}(T - \lambda_2 Id_V)$. Given $\mathcal{E}_1 + \mathcal{E}_2 = V$ show $V = \mathcal{E}_1 \oplus \mathcal{E}_2$ and show T is diagonalizable.

Remark: *I couldn't the calculation below early this semester since we first discussed determinants before we studied linear transformations. I think now is the appropriate time to share this with you. It is considerably easier to understand than the technical proof I gave in terms of elementary matrices and such in the determinants chapter.*

Problem 126 Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ then we define $\Lambda^k T : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ where $\Lambda^k\mathbb{R}^n$ is the vector space of k-vectors over $\mathbb{R}^n$. In particular,

$$\Lambda^k T(v_1 \wedge \cdots \wedge v_k) = T(v_1) \wedge \cdots \wedge T(v_k).$$

Notice $\Lambda^n T(e_1 \wedge \cdots \wedge e_n) = T(e_1) \wedge \cdots \wedge T(e_n) = \det(T) e_1 \wedge \cdots \wedge e_n$.

(a.) if $S, T : \mathbb{R}^n \to \mathbb{R}^n$ show $\Lambda^k(T \circ S) = \Lambda^k T \circ \Lambda^k S$,

(b.) derive $\det(AB) = \det(A)\det(B)$ by examining (a.) with $k = n$ and T, S with $[T] = A$ and $[S] = B$.

Remark: the problem below didn't turn out quite as I had hoped... I accidentally wrote the solution in the problem statement and at the moment I can't think of a good way to fix it. That said, I include it to illustrate how the wedge product can be used to prove things about determinants directly if you have the computational courage:

(I.) The identity $\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A)\det(B)$ is important to several key theorems this semester. Let me outline an argument based on the wedge product technique I introduced in some of your previous homework. If $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where $A = [A_1|\cdots|A_m] \in \mathbb{F}^{m\times m}$ and $B = [B_1|\cdots|B_n] \in \mathbb{F}^{n\times n}$ then the standard basis $e_1, e_2, \dots, e_{m+n} \in \mathbb{F}^{m+n}$ gives

$$Me_1 = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \sum_{j_1=1}^{m} A_{j_1 1} e_{j_1}, \ \dots \ , Me_m = \begin{bmatrix} A_m \\ 0 \end{bmatrix} = \sum_{j_m=1}^{m} A_{j_m m} e_{j_m}$$

and

$$Me_{m+1} = \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = \sum_{k_1=1}^{n} B_{k_1 1} e_{m+k_1}, \ \dots \ , Me_{m+n} = \begin{bmatrix} 0 \\ B_n \end{bmatrix} = \sum_{k_n=1}^{n} B_{k_n n} e_{m+k_n}.$$

Hence calculate, $Me_1 \wedge \cdots \wedge Me_m \wedge Me_{m+1} \wedge \cdots \wedge Me_{m+n} =$

$$= \left(\sum_{j_1=1}^{m} A_{j_1 1} e_{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_m=1}^{m} A_{j_1 m} e_{j_m} \right) \wedge \left(\sum_{k_1=1}^{n} B_{k_1 1} e_{m+k_1} \right) \wedge \cdots \wedge \left(\sum_{k_n=1}^{n} B_{k_n 1} e_{m+k_n} \right)$$

$$= \left(\sum_{j_1=1}^{m} \cdots \sum_{j_m=1}^{m} A_{j_1 1} \cdots A_{j_1 m} e_{j_1} \wedge \cdots \wedge e_{j_m} \right)$$

$$\wedge \left(\sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} B_{k_1 1} \cdots B_{k_n 1} e_{m+k_1} \wedge \cdots \wedge e_{m+k_n} \right)$$

$$= \left(\sum_{j_1,\dots,j_m=1}^{m} A_{j_1 1} \cdots A_{j_1 m} \epsilon_{j_1 \cdots j_m} e_1 \wedge \cdots \wedge e_m \right)$$

$$\wedge \left(\sum_{k_1,\dots,k_n=1}^{n} B_{k_1 1} \cdots B_{k_1 n} \epsilon_{k_1 \cdots k_n} e_{m+1} \wedge \cdots \wedge e_{m+n} \right)$$

$$= \det(A)\det(B) e_1 \wedge \cdots \wedge e_m \wedge e_{m+1} \wedge \cdots \wedge e_{m+n}$$

Thus $\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A)\det(B)$.

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Consider the subspace $W = \text{span}\{(1,1)\}$ of $\mathbb{R}^2$. Suppose $W_1 \oplus W = W_2 \oplus W$ does it follow $W_1 = W_2$? Prove or disprove.

(II.) Consider the linear transformation $L(A) = A^T$ for $A \in \mathbb{F}^{2\times 2}$ where $1 \neq -1$ in $\mathbb{F}$. Find an eigenbasis for L and find a direct sum decomposition of $\mathbb{F}^{2\times 2}$ via the eigenspaces of L. Find the characteristic polynomial $p(x)$ for L.

(III.) Consider $T(f(x)) = -f(x) - f'(x)$ definining a linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$. Find a Jordan basis for T and express the matrix of T as a direct sum of Jordan blocks.

(IV.) Consider $T(f(x)) = f''(x)$ definining a linear transformation $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$. Find a Jordan basis for T and express the matrix of T as a direct sum of Jordan blocks.

(V.) Suppose $T : V \to V$ has eigenvalue λ show for any $n \in \mathbb{N}$ that T^n has eigenvalue λ^n.

(VI.) Prove a linear transformation T is invertible if and only if $\lambda = 0$ is **not** an eigenvalue of T

(VII.) Suppose A is invertible. Prove if A has eigenvalue λ then A^{-1} has eigenvalue $1/\lambda$.

(VIII.) Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable? Find the Jordan form associated with A.

(IX.) Show $A = \begin{bmatrix} -2 & 3 & -1 \\ -4 & 5 & -3 \\ -4 & 4 & -2 \end{bmatrix}$ has eigenvalues of $\lambda = 1$ and $\lambda = 2i$ by finding eigenvectors with the given eigenvalues for A.

(X.) Let $V = \text{span}_{\mathbb{R}}\{e^x, e^{2x}, \cos(x), \sin(x)\}$ and consider $T = D + 1$ where $D = d/dx : V \to V$. Show that $U = \text{span}_{\mathbb{R}}\{e^x, e^{2x}\}$ forms an invariant subspace of V with respect to T and find the matrix of $T|_U$ as well as $T_{V/U}$.

MISSION 7 SOLUTION

P109 $W_1 = \text{span}\{(1,1,1,0),(0,0,0,1)\} \leftarrow \beta_1$ so $W_1 = \text{span}\,\beta_1$

$W_2 = \text{span}\{(1,-1,0,0),(0,1,-1,0)\} \leftarrow \beta_2$ so $W_2 = \text{span}\,\beta_2$

Show $W_1 \oplus W_2 = \mathbb{R}^4$

Notice β_1 and β_2 are bases for W_1 & W_2 respective.

Consider $\beta = \beta_1 \cup \beta_2 \subseteq \mathbb{R}^4$ and notice

$$\det[\beta] = \det\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$: expanding on 4 row via Laplace's expansion by minors.

$$= \det\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= 1 \cdot \det\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} - 1 \cdot \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= 1(1-0) - 1[(-1)-1]$$

$$= 3 \neq 0$$ $\therefore$ $[\beta]^{-1}$ exists so $[\beta]$ has 4 LI columns and we find β is basis for $\mathbb{R}^4$ from which we observe $W_1 \oplus W_2 = \mathbb{R}^4$ as the concatenation of β_1 with β_2 gives basis $\beta = \beta_1 \cup \beta_2$ for all of $\mathbb{R}^4$!

$\boxed{P110}$ Suppose $\dim V < \infty$ and $W_1, W_2 \leq V$ where $V = W_1 \oplus W_2$. Let $\dim W_1 = N_1$ and $\dim W_2 = N_2$ hence $\exists\, \beta_1, \beta_2$ bases for W_1 & W_2 of form:

$$\beta_1 = \{V_{1,1}, \dots, V_{1,N_1}\} \quad \& \quad \beta_2 = \{V_{2,1}, \dots, V_{2,N_2}\}$$

with $\beta = \beta_1 \cup \beta_2$ a basis for V. Naturally β^* is basis for V^* and notice our notation implies $\beta^* = \beta_1^* \cup \beta_2^*$ where $\beta_1^* = \{V^{1,i}\}_{i=1}^{N_1}$ & $\beta_2^* = \{V^{2,j}\}_{j=1}^{N_2}$ moreover, for $1 \leq i, j \leq N_1$ and $1 \leq k, l \leq N_2$ we have:

$$① \; V^{1,i}(V_{1,j}) = \delta_{ij} \qquad ③ \; V^{2,k}(V_{1,i}) = 0$$

$$② \; V^{1,i}(V_{2,k}) = 0 \qquad ④ \; V^{2,k}(V_{2,l}) = \delta_{kl}$$

By our previous study of V^* we can decompose any $\alpha \in V^*$ via an expansion in β^* as follows:

$$\alpha = \sum_{i=1}^{N_1} \alpha(V_{1,i}) V^{1,i} + \sum_{j=1}^{N_2} \alpha(V_{2,j}) V^{2,j} \quad \star$$

Next, suppose $\alpha \in \operatorname{ann}(W_1)$ then $\alpha(V_{1,i}) = 0$ for $i = 1, 2, \dots, N_1$ hence $\star$ yields $\alpha = \sum_{j=1}^{N_2} \alpha(V_{2,j}) V^{2,j} \in \operatorname{span}(\beta_2^*)$.

Yet Eq: ③ also indicates β_2^* annihilates the basis for W_1 hence $V^{2,j} \in \operatorname{ann}(W_1)$ for each $j = 1, 2, \dots, N_2$ $\therefore$ $\operatorname{span}(\beta_2^*) \subseteq \operatorname{ann}(W_1)$ and we find both $\operatorname{ann}(W_1) \subseteq \operatorname{span}(\beta_2^*)$ and $\operatorname{ann}(W_1) \supseteq \operatorname{span}(\beta_2^*)$ thus $\operatorname{ann}(W_1) = \operatorname{span}(\beta_2^*)$. By the same argument interchanging $1 \longleftrightarrow 2$ we find $\operatorname{ann}(W_2) = \operatorname{span}(\beta_1^*)$.

However, $\#\beta_1^* = \#\beta_1$ and $\#\beta_2^* = \#\beta_2$ thus $\dim(\operatorname{ann}(W_1)) = \dim W_2$ and $\dim(\operatorname{ann} W_2) = \dim W_1$ thus $\dim V^* = \dim V = \dim W_1 + \dim W_2$ shows $\dim V^* = \dim(\operatorname{ann} W_2) + \dim(\operatorname{ann} W_1)$ $\therefore$ $\underline{\operatorname{ann} W_1 \oplus \operatorname{ann} W_2 = V^*}$ //

$\boxed{P111}$ $T(f(x)) = f(x) + x f'(x)$ for $f(x) \in P_3(\mathbb{R})$.
Let $\beta_1 = \{1, x^2\}$ and $\beta_2 = \{x, x^3\}$ be bases for W_1 & W_2 respective.

(a.) $T(1) = 1 \in W_1$

$T(x^2) = x^2 + x(2x) = 3x^2 \in W_1$

Since $T(\beta_1) \subseteq W_1 \Rightarrow T(\text{span}(\beta_1)) \subseteq W_1$ aka $T(W_1) \subseteq W_1$

hence W_1 is T-invariant.

For W_2 I'll use more straight-forward argument, if $ax + bx^3 \in W_2 = \text{span}\,\beta_2$ then

$$T(ax+bx^3) = ax + bx^3 + x(a + 3bx^2)$$
$$= 2ax + 4bx^3 \in W_2 \quad \therefore \underline{T(W_2) \subseteq W_2}.$$

//

(b.) $\beta_1 = \{1, x^2\}$ and $T(a+bx^2) = a + 3bx^2 \Rightarrow [T_{W_1}]_{\beta_1\beta_1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\beta_2 = \{x, x^3\}$ and $T(ax+bx^3) = 2ax + 4bx^3 \Rightarrow [T_{W_2}]_{\beta_2,\beta_2} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(c.) If $\beta = \beta_1 \cup \beta_2 = \{1, x^2, x, x^3\}$

$$[T]_{\beta\beta} = \Big[[T(1)]_\beta \,\Big|\, [T(x^2)]_\beta \,\Big|\, [T(x)]_\beta \,\Big|\, [T(x^3)]_\beta \Big]$$

$$= \Big[[1]_\beta \,\Big|\, [3x^2]_\beta \,\Big|\, [2x]_\beta \,\Big|\, [4x^3]_\beta \Big]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \oplus \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = [T_{W_1}]_{\beta_1\beta_1} \oplus [T_{W_2}]_{\beta_2\beta_2}.$$

✓

$\boxed{P112}$ $T: \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}^{n\times n}$ define by $T(A) = A + A^T$

(a.) $S_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = A\}$. If $A \in S_n$ then

$T(A) = A + A^T = A + A = 2A \in S_n$ since

$(2A)^T = 2A^T = 2A$ thus $T(S_n) \subseteq S_n$

and $T_{S_n} = 2Id_{S_n}$. $*$

(b.) $A_n = \{A \in \mathbb{F}^{n\times n} \mid A^T = -A\}$ observe for $A \in A_n$ we find $T(A) = A + A^T = A - A = 0$ thus $A \in Ker(T) \Rightarrow$ $\underline{A_n \subseteq Ker(T)}$. Btw, as $0 \in A_n$ we also may note $T(A_n) \subseteq A_n \therefore A_n$ is T-invariant. and $T_{A_n} = 0$. $**$

(c.) If $S_n = span\, \beta_s$ and $A_n = span\, \beta_a$ recall $S_n \oplus A_n = \mathbb{F}^{n\times n}$ thus by T-invariance of S_n & A_n we calculate, for $\beta = \beta_s \cup \beta_a$

$$
\begin{aligned}
[T]_{\beta,\beta} &= [T_{S_n}]_{\beta_s,\beta_s} \oplus [T_{A_n}]_{\beta_a,\beta_a} \\
&= [2Id_{S_n}]_{\beta_s,\beta_s} \oplus [0]_{\beta_a,\beta_a} \\
&= \left[\begin{array}{c|c} 2I_k & 0 \\ \hline 0 & 0 \end{array}\right]
\end{aligned}
$$

Here we can calculate $k = \dim(S_n) = \frac{1}{2}(n^2 - n)$ since $\beta_s = \{E_{ij} + E_{ji} \mid 1 \leq i < j \leq n\} \cup \{E_{ii}\}_{i=1}^{n}$ serves as basis for S_n.

$\boxed{P112}$ I hope you see what follows was <u>not</u> needed given the argument I shared in cc.).

$$A \in S_n \implies A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \\ A_{1n} & \cdots & & A_{nn} \end{bmatrix}$$

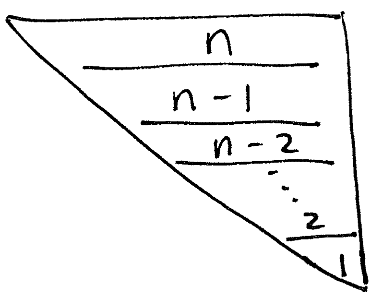
$$= \sum_{i<j} A_{ij}(E_{ij}+E_{ji}) + \sum_{i=1}^{n} A_{ii}E_{ii}$$

It follow $\beta_s = \{E_{12}+E_{21}, E_{13}+E_{31}, \dots, E_{n-1,n}+E_{n,n-1}\} \cup$ ∂ ∂ $\cup \{E_{11}, E_{22}, \dots, E_{nn}\}$ is LI spanning set for S_n

aka $\beta_s = \{E_{ij}+E_{ji} \mid 1 \le i < j \le n\} \cup \{E_{11}, \dots, E_{nn}\}$

Counting, $\#\beta_s = n + n-1 + \cdots + 2 + 1 \underset{\uparrow}{=} \frac{1}{2}n(n-1)$

Gauss' Kindergarten f-la



independent parameters in symmetric matrix this is where my "counting" comes from.

Likewise, $A \in A_n$

$$A = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1n} \\ -A_{12} & 0 & & \vdots \\ -A_{13} & & \ddots & \\ \vdots & & & A_{n-1,n} \\ -A_{1n} & & -A_{n-1,n} & 0 \end{pmatrix} = \sum_{i<j} A_{ij}(E_{ij}-E_{ji})$$

$$\Rightarrow \underline{\beta_a = \{E_{ij}-E_{ji} \mid 1 \le i < j \le n\}}.$$

$\boxed{P113}$ ① Suppose $\dim_{\mathbb{F}}(V) < \infty$ and $W_1 + W_2 = V$ with $\dim W_1 + \dim W_2 = \dim V$. Since we're given $W_1 + W_2 = V$ to show $V = W_1 \oplus W_2$ it remains to show $W_1 \cap W_2 = \{0\}$. (at least this will suffice... but, other argument exist!)

Recall,

$$(*) \quad \underbrace{\dim(W_1 + W_2)} = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$$

$$\Rightarrow \quad \cancel{\dim(V)} = \cancel{\dim W_1 + \dim W_2} - \dim(W_1 \cap W_2)$$

$$\Rightarrow \quad 0 = \dim W_1 \cap W_2$$

Thus $W_1 \cap W_2 = \{0\}$ and so we conclude $W_1 \oplus W_2 = V$.

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② Alternatively, if β_1, β_2 are bases for W_1 and W_2 respective then $\beta = \beta_1 \cup \beta_2$ is generating set for $W_1 + W_2$ and $\# \beta = \# \beta_1 + \# \beta_2$ implies $\# \beta = \dim W_1 + \dim W_2 = \dim V$. So, β spans V and has $\dim(V)$ # of vectors which proves β is <u>basis</u> for V. Hence the concatenation of β_1 & β_2 to form β give basis $\therefore$ $V = W_1 \oplus W_2$.

<u>Remark</u>: argument ② is also nice here as it does not require the somewhat technical (*).

[P114] Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{R}[t]$
we define $p(A) = a_0 I + a_1 A + \cdots + a_n A^n$ for $A \in \mathbb{R}^{n\times n}$.
Suppose v is e-vector of A with e-value λ.
Then $Av = \lambda v \Rightarrow AAv = A\lambda v \Rightarrow \underline{A^2 v = \lambda^2 v}$.
Suppose inductively $A^k v = \lambda^k v$ and consider,

$$A^{k+1} v = A A^k v = A \lambda^k v = \lambda^k A v = \lambda^k \lambda v = \lambda^{k+1} v$$

Thus $\underline{A^k v = \lambda^k v \quad \forall k \in \mathbb{N}}$. $*$ With $*$ in mind we calculate,

$$\begin{aligned} p(A)v &= (a_0 I + a_1 A + \cdots + a_n A^n)v \\ &= a_0 Iv + a_1 Av + \cdots + a_n A^n v \\ &= a_0 v + a_1 \lambda v + \cdots + a_n \lambda^n v \\ &= (a_0 + a_1 \lambda + \cdots + a_n \lambda^n) v \\ &= p(\lambda) v. \end{aligned}$$

[P115] Suppose $A^{k-1} \neq 0$ yet $A^k = 0$ for some $n\times n$ matrix A.
Assume (towards a $\rightarrow\leftarrow$) that $\lambda \neq 0$ is an e-value of A and $v_1 \neq 0$ is an e-vector s.t. $Av_1 = \lambda v_1$. But, by $*$ from P114, we have $\underline{A^k v_1 = \lambda^k v_1}$ $**$ and $\lambda^k \neq 0$ and $v_1 \neq 0$ so $\lambda^k v_1 \neq 0$ yet $A^k v_1 = 0$ $\rightarrow\leftarrow$ $**$.
Consequently $\lambda = 0$ is only e-value possible for A.

P116 $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$

$$\det \begin{bmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix} = (\lambda-4)(\lambda-1)+2$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-3)(\lambda-2) = 0 \Rightarrow \underline{\lambda_1 = 3}, \ \underline{\lambda_2 = 2}.$$

$$(A-3I)V_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u+v=0 \quad \therefore \quad \underline{V_1 = \begin{bmatrix} u \\ -u \end{bmatrix}}$$

for $u \neq 0$

$$(A-2I)V_2 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2u+v=0 \quad \therefore \quad \underline{V_2 = \begin{bmatrix} u \\ -2u \end{bmatrix}}.$$

for $u \neq 0$

Thus $\mathcal{E}_{\lambda_1=3} = \text{span}\{(1,-1)\}$

and $\mathcal{E}_{\lambda_2=2} = \text{span}\{(1,-2)\}$. Let $\beta = \{(1,-1),(1,-2)\}$

$$A^n = [\beta]\, D^n\, [\beta]^{-1} \qquad \left(\text{since } D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = [\beta]^{-1} A [\beta]\right)$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3^n & 0 \\ 0 & 2^n \end{bmatrix} \frac{1}{-2+1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2\cdot 3^n & 3^n \\ -2^n & -2^n \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} 2\cdot 3^n - 2^n & 3^n - 2^n \\ -2\cdot 3^n + 2^{n+1} & -3^n + 2^{n+1} \end{bmatrix}}$$

P117 $\quad A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)\left[\lambda(\lambda-3)-4\right] - 2\left[2(3-\lambda)-8\right] + 4\left[4+4\lambda\right]$$

$$= (3-\lambda)\left[\lambda^2 - 3\lambda - 4\right] - 2\left[6 - 2\lambda - 8\right] + 16 + 16\lambda$$

$$= -\lambda^3 + 3\lambda^2 + \underline{4\lambda} + 3\lambda^2 - \underline{9\lambda} - 12 + 4 + \underline{4\lambda} + 16 + \underline{16\lambda}$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

$$= (\lambda - 8)(\lambda^2 + 2\lambda + 1)(-1)$$

$$= (8-\lambda)(\lambda+1)^2 \quad \therefore \quad \underline{\lambda_1 = 8}, \quad \underbrace{\lambda_2 = -1}_{\text{algebraic mult. 2.}}$$

$$A - 8I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{pmatrix} u = w \\ v = w/2 \\ \text{for } v_1 = (u,v,w) \end{pmatrix}} \Rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Leftarrow \text{basis vector for}$$

$$\therefore \boxed{\mathcal{E}'_{\lambda_1 = 8} = \text{span}\{\underbrace{(2,1,2)}_{v_1}\}}$$

$$A + I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

thus e-vector (u,v,w) with e-value $\lambda_2 = -1$ has $v = -2u - 2w$

Thus
$$(u,v,w) = (u, -2u - 2w, w)$$
$$= u(1,-2,0) + w(0,-2,1)$$

$$\therefore \boxed{\mathcal{E}'_{\lambda_2 = -1} = \text{span}\{\underbrace{(1,-2,0)}_{v_2}, \underbrace{(0,-2,1)}_{v_3}\}}$$

Indeed A is diagonalizable both over $\mathbb{R}$ and $\mathbb{C}$. $\beta = \{v_1, v_2, v_3\}$ gives $[\beta]^{-1} A [\beta] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$\boxed{P118}$ Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 2 & 1-\lambda & 3 \\ 5 & 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)(1-\lambda)\right] \quad \therefore \quad \begin{array}{ll} \lambda_1 = 1 & \underline{\text{twice}}. \\ \lambda_2 = 2 & \underline{\text{once}}. \end{array}$$

$$A - I = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$v_1 \in Null(A - I)$ has $v_1 = (u, v, w)$ with has $u = 0$, $w = 0$ v free $\therefore v_1 = (0, v, 0)$

or $\boxed{\mathcal{E}'_{\lambda_1 = 1} = span\{(0,1,0)\}}$

Next,

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 3 \\ 5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -17/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$v_3 = (u, v, w) \in Null(A - 2I)$ has $u = w/5$, $v = 17w/5$

hence setting $w = 5$ gives nice e-vector $\underline{v_3 = (1, 17, 5)}$

$\boxed{\mathcal{E}_{\lambda_2 = 2} = span\{(1, 17, 5)\}}$

Notice $\dim(\mathcal{E}_{\lambda_1 = 1}) = 1 < 2 =$ algebraic mult. of $\lambda_1 = 1$

thus A is <u>not</u> diagonalizable over $\mathbb{R}$ (or $\mathbb{C}$ btw)

Calculate $(A - 2I)v_2 = v_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{l} u = 0 \\ 2u + 3w = 1 \quad \therefore w = 1/3 \\ v \text{ free} \end{array}$$

Hence $v_2 = (0, 0, 1/3)$ does nicely. $\beta = \{v_1, v_2, v_3\}$

yields $[\beta]^{-1} A [\beta] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right] = \underbrace{J_2(1) \oplus J_1(2)}.$

$\leftarrow$ Jordan Form assoc. to A.

P119 $\quad A = \begin{bmatrix} -6 & -4 \\ 10 & 6 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{bmatrix} -6-\lambda & -4 \\ 10 & 6-\lambda \end{bmatrix} = (\lambda - 6)(\lambda + 6) + 40$$
$$= \lambda^2 + 4 \Rightarrow \underline{\lambda = \pm 2i}.$$

Thus A is not diagonalizable as real matrix since it has an e-value which is not in $\mathbb{R}$. However, we can find the real Jordan form and diagonalize A over $\mathbb{C}$. Simply calculate e-vect. for $\lambda = 2i$

$$(A - 2iI)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -6-2i & -4 \\ 10 & 6-2i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus $10u + (6-2i)v = 0 \Rightarrow u = \left(\frac{6-2i}{-10}\right)v$

setting $v = -10$ yields $(u, v) = (6-2i, -10)$

Hence $\begin{bmatrix} 6-2i \\ -10 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix} + i\begin{bmatrix} -2 \\ 0 \end{bmatrix} = V_1$

we can check $AV_1 = \begin{bmatrix} -6 & -4 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} 6-2i \\ -10 \end{bmatrix} = \begin{bmatrix} -36+12i+40 \\ 60-20i-60 \end{bmatrix}$

Anyway, it follows $\qquad = \begin{bmatrix} 4+12i \\ -20i \end{bmatrix} \checkmark = 2i\begin{bmatrix} 6-2i \\ -10 \end{bmatrix}$

$$\boxed{\gamma = \{V_1, \bar{V}_1\} \text{ yields } [\gamma]^{-1}A[\gamma] = \left[\begin{array}{c|c} 2i & 0 \\ \hline 0 & -2i \end{array}\right]}$$ (complex diagonalized)

and $\beta = \{(6, -10), (-2, 0)\}$ yields

$$\boxed{[\beta]^{-1}A[\beta] = \begin{bmatrix} 0 & +2 \\ -2 & 0 \end{bmatrix} \leftarrow \text{REAL JORDAN FORM OF } A.}$$

$\boxed{P120}$

$$A = \begin{bmatrix} 2 & 4 & -4 \\ -1 & 2 & -1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = -(\lambda^3 - \lambda^2 + 4\lambda - 4) = (1-\lambda)(\lambda^2+4)$$

Thus e-values of $\lambda_1 = 1$, $\lambda_2 = 2i$, $\lambda_3 = -2i$

It follows A is NOT diagonalizable over $\mathbb{R}$

But, A is COMPLEX DIAGONALIZABLE since it has three distinct complex e-values $\Rightarrow \exists$ e-basis for A over $\mathbb{C}$.

The usual calculations reveal

$$\text{Null}(A - I) = \text{span}\{\overbrace{(0,1,1)}^{v_1}\} = \mathcal{E}'_{\lambda_1 = 1}$$

$$\text{Null}(A - 2iI) = \text{span}\{\underbrace{(2, 1+i, 2)}_{v_2}\} = \mathcal{E}'_{2i}$$

$$\text{Null}(A + 2iI) = \text{span}\{\underbrace{(2, 1-i, 2)}_{v_3}\} = \mathcal{E}'_{-2i}$$

That is $\beta = \{(0,1,1), (2, 1+i, 2), (2, 1-i, 2)\}$ is e-basis for $\mathbb{C}^3$ w.r.t. A and we can show

$$\boxed{[\beta]^{-1} A [\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{bmatrix}}$$

However, if we use $\gamma = \{(0,1,1), \underbrace{(2,1,2)}_{Re(V_2)}, \underbrace{(0,1,0)}_{Im(V_2)}\}$ then

$$\boxed{[\gamma]^{-1} A [\gamma] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \leftarrow \text{REAL JORDAN FORM FOR } A.}$$

$\boxed{P121}$ Suppose $v_1 = a_1 + ib_1$ and $v_2 = a_2 + ib_2$ such that

$$(A - (1+2i)I)v_1 = 0 \quad \& \quad (A-(1+2i)I)v_2 = v_1$$

then for $L = L_A$ where $L(x) = Ax \ \forall x \in \mathbb{C}^4$ we may calculate $[L]_{\beta\beta}$ and $[L]_{\gamma\gamma}$ w.r.t.

$\beta = \{a_1, b_1, a_2, b_2\}$ vs. $\gamma = \{v_1, v_2, \bar{v}_1, \bar{v}_2\}$.

Following our derivations in class,

$$[L]_{\beta\beta} = \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}}_{\text{REAL JORDAN FORM OF } A.} \quad \text{and} \quad [L]_{\gamma\gamma} = \left[\begin{array}{cc|cc} 1+2i & 1 & 0 & 0 \\ 0 & 1+2i & 0 & 0 \\ \hline 0 & 0 & 1-2i & 1 \\ 0 & 0 & 0 & 1-2i \end{array}\right]$$

$$= J_2(1+2i) \oplus J_2(1-2i).$$

(these are not mystical, they follow from the easy calculations below:)

$$Av_1 = (1+2i)v_1 \qquad\qquad Av_2 = (1+2i)v_2 + v_1$$

$$A(a_1+ib_1) = (1+2i)(a_1+ib_1) \qquad A(a_2+ib_2) = (1+2i)(a_2+ib_2) + v_1$$

$$Aa_1 + iAb_1 = a_1 - 2b_1 + i(2a_1+b_1) \qquad Aa_2 + iAb_2 = a_2 - 2b_2 + a_1 + ib_1 + i(2a_2+b_2)$$

Hence,

$$\left.\begin{aligned} Aa_1 &= a_1 - 2b_1 &&\leftarrow \text{column } 1 \\ Ab_1 &= 2a_2 + b_1 &&\leftarrow \text{column } 2 \\ Aa_2 &= a_2 - 2b_2 + a_1 &&\leftarrow \text{column } 3 \\ Ab_2 &= b_1 + 2a_2 + b_2 &&\leftarrow \text{column } 4 \end{aligned}\right\} \text{ of } [L]_{\beta\beta} \text{ since } \beta = \{a_1, b_1, a_2, b_2\}.$$

$\boxed{P122}$ $T: V \to V$ is linear on $V = V(\mathbb{R})$ with $\dim(V) = 4$. Suppose $\exists\ v_1, v_2, v_3, v_4 \neq 0$ s.t.

$$T(v_1) = 7v_1 + v_2 \Rightarrow (T-7)(v_1) = v_2$$
$$T(v_2) = 7v_2 \Rightarrow (T-7)(v_2) = 0.$$
$$(T-4)(v_3) = 0 \Rightarrow (T-4)(v_3) = 0 \quad \text{(duh.)}$$
$$T(v_4) = 4v_4 \Rightarrow (T-4)(v_4) = 0.$$

We observe $\{v_1, v_2\}$ form 2-chain ($\lambda = 7$). But, $\{v_3, v_4\}$ both e-vectors with $\lambda = 4$. To be sure $\beta = \{v_2, v_1, v_3, v_4\}$ is basis need LI of $\{v_3, v_4\}$ given that LI of $\{v_3, v_4\}$ find

$$[T]_{\beta\beta} = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(β is Jordan Basis and of course $[T]_{\beta\beta} = J_2(7) \oplus J_1(4) \oplus J_1(4)$ is in Jordan Form)

We find,

$$\boxed{\det(T) = \det[T]_{\beta\beta} = (49)(16) = 784}$$

and

$$\boxed{\operatorname{tr}(T) = \operatorname{tr}([T]_{\beta\beta}) = 7+7+4+4 = 22}$$

$\boxed{P123}$ Suppose $A \in \mathbb{R}^{6\times 6}$ with

$$P_A(t) = (t-3)^3(t-2)^2(t-1)$$

What are possible Jordan forms associated to A?

To obtain $P_A(t) = \det(tI - A)$ as above,
Also, find $m_A(t)$ meaning smallest monic $m_A(t) \in \mathbb{R}[t]$ s.t. $m_A(A) = 0$.

$$J_1 = \begin{bmatrix} \boxed{3} & & & & & \\ & \boxed{3} & & & & \\ & & \boxed{3} & & & \\ & & & \boxed{2} & & \\ & & & & \boxed{2} & \\ & & & & & \boxed{1} \end{bmatrix} \Rightarrow \underbrace{\underline{m_A(t) = (t-3)(t-2)(t-1)}}_{\text{only this case has } A \text{ diagonalizable.}}.$$

$$J_2 = \begin{bmatrix} 3 & 1 & & & & \\ 0 & 3 & & & & \\ & & \boxed{3} & & & \\ & & & \boxed{2} & & \\ & & & & \boxed{2} & \\ & & & & & \boxed{1} \end{bmatrix} \Rightarrow \underline{m_A(t) = (t-3)^2(t-2)(t-1)}.$$

$$J_3 = \begin{bmatrix} 3 & 1 & 0 & & & \\ 0 & 3 & 1 & & & \\ 0 & 0 & 3 & & & \\ & & & \boxed{2} & & \\ & & & & \boxed{2} & \\ & & & & & \boxed{1} \end{bmatrix} \Rightarrow \underline{m_A(t) = (t-3)^3(t-2)(t-1)}.$$

$$J_4 = \begin{bmatrix} \boxed{3} & & & & & \\ & \boxed{3} & & & & \\ & & \boxed{3} & & & \\ & & & 2 & 1 & \\ & & & 0 & 2 & \\ & & & & & \boxed{1} \end{bmatrix} \Rightarrow \underline{m_A(t) = (t-3)(t-2)^2(t-1)}.$$

$$J_5 = \begin{bmatrix} 3 & 1 & & & & \\ 0 & 3 & & & & \\ & & \boxed{3} & & & \\ & & & 2 & 1 & \\ & & & 0 & 2 & \\ & & & & & \boxed{1} \end{bmatrix} \Rightarrow \underline{m_A(t) = (t-3)^2(t-2)^2(t-1)}.$$

$$J_6 = \begin{bmatrix} 3 & 1 & 0 & & & \\ 0 & 3 & 1 & & & \\ 0 & 0 & 3 & & & \\ & & & 2 & 1 & \\ & & & 0 & 2 & \\ & & & & & \boxed{1} \end{bmatrix} \Rightarrow \underline{m_A(t) = \underbrace{(t-3)^2(t-2)^2(t-1)}_{P_A(t)}}.$$

(they can match, but don't have to)

$\boxed{P124}$ $T : V \to V$ with basis $\beta = \{v_1, \dots, v_n\}$

has $[T]_{\beta,\beta} = J_4(3) \oplus J_2(3) \oplus J_1(3) \oplus J_1(3) \oplus J_4(6)$

$$e_1 \quad e_2 \; e_3 \; e_4 \qquad e_5 \; e_6 \qquad e_7 \qquad e_8 \qquad e_9 \; e_{10} \; e_{11} \; e_{12}$$

$\mathcal{E}'_3 = \text{span}(\beta_3)$ where $\beta_3 = \underbrace{\{e_1, e_5, e_7, e_8\}}_{\text{e-vectors with } \lambda = 3}$

$K_3 = \text{span}\{\gamma_3\}$ where $\gamma_3 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

$\mathcal{E}'_6 = \text{span}\{\beta_6\}$ where $\beta_6 = \{e_4\}$

$K_6 = \text{span}\, \gamma_6$ where $\gamma_6 = \{e_9, e_{10}, e_{11}, e_{12}\}$.

$\boxed{P125}$ Suppose $\lambda_1 \neq \lambda_2$ are e-values of $T : V \to V$. Let $\mathcal{E}'_1 = \text{Ker}(T - \lambda_1)$ and $\mathcal{E}'_2 = \text{Ker}(T - \lambda_2)$. Supposing $V = \mathcal{E}'_1 + \mathcal{E}'_2$ show $V = \mathcal{E}'_1 \oplus \mathcal{E}_2$. Let $\mathcal{E}'_j$ have basis β_j for $j = 1, 2$. Consider $\beta_1 = \{v_1, \dots, v_{n_1}\}$ and $\beta_2 = \{w_1, \dots, w_{n_2}\}$. Suppose $x \in \mathcal{E}_1 \cap \mathcal{E}_2$ then

$$T(x) = \lambda_1 x = \lambda_2 x \Rightarrow \underbrace{(\lambda_1 - \lambda_2)}^{(\lambda_1 \neq \lambda_2 \text{ so } \lambda_1 - \lambda_2 \neq 0)} x = 0 \Rightarrow x = 0$$

Thus $\mathcal{E}'_1 \cap \mathcal{E}'_2 = \{0\}$ and we find $\underline{V = \mathcal{E}'_1 \oplus \mathcal{E}'_2}$.

$\boxed{P126}$ $T: \mathbb{R}^n \to \mathbb{R}^n$, define $\Lambda^k T : \Lambda^k \mathbb{R}^n \to \Lambda^k \mathbb{R}^n$ via $\Lambda^k T(v_1 \wedge \cdots \wedge v_k) = T(v_1) \wedge \cdots \wedge T(v_k)$.

(a.) Suppose $S, T : \mathbb{R}^n \to \mathbb{R}^n$, calculate,

$$\left((\Lambda^k T) \circ (\Lambda^k S)\right)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) =$$

$$= (\Lambda^k T)(S(v_1) \wedge S(v_2) \wedge \cdots \wedge S(v_k))$$

$$= T(S(v_1)) \wedge T(S(v_2)) \wedge \cdots \wedge T(S(v_k))$$

$$= (T \circ S)(v_1) \wedge (T \circ S)(v_2) \wedge \cdots \wedge (T \circ S)(v_k)$$

$$= (\Lambda^k T \circ S)(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$$

Consequently, $\underline{\Lambda^k T \circ S = \Lambda^k T \circ \Lambda^k S}$.

(b.) Given $\Lambda^n T(e_1 \wedge \cdots \wedge e_n) = T(e_1) \wedge \cdots \wedge T(e_n) = \det(T) e_1 \wedge \cdots \wedge e_n$

Consider $T = L_A$ and $S = L_B$ so $[T] = A$, $[S] = B$,

Since $\Lambda^n(T \circ S) = \Lambda^n T \circ \Lambda^n S$ get

$$\Lambda^n(T \circ S)(e_1 \wedge \cdots \wedge e_n) = (\Lambda^n T)(\Lambda^n S(e_1 \wedge \cdots \wedge e_n))$$

$$(T \circ S)(e_1) \wedge \cdots \wedge (T \circ S)(e_n) = (\Lambda^n T)(\det(S)\, e_1 \wedge \cdots \wedge e_n)$$

$$\det(T \circ S)\, e_1 \wedge \cdots \wedge e_n = \det(S)\, \Lambda^n T(e_1 \wedge \cdots \wedge e_n)$$

$$\det(T \circ S)\, e_1 \wedge \cdots \wedge e_n = \det(S) \det(T)\, e_1 \wedge \cdots \wedge e_n$$

$$\Rightarrow \det(T \circ S) = \det S \det T = \det T \det S$$

$$\Rightarrow \underline{\det(AB) = \det(A)\det(B)}.$$

(Recall $[T \circ S] = [T][S] = AB$ & $[S] = B$, $[T] = A$.