Same rules as Homework 1. However, do keep in mind you are free to use technology to calculate row-reductions. There are many online resources to help you check your work. It would be wise to make use of them (Gram Schmidt has a lot of arithmetic, it's easy to make mistakes).

Problem 121 Your signature below indicates you have:

(a.) I read Sections 15, 30, 32 (just p.278 – 282) of Curtis: ____________________.

(b.) I read Chapter 8 of Cook's Lecture Notes: ____________________.

Problem 122 Consider $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Calculate $e^{J\theta}$ by explicit calculation from the definition of the matrix exponential. Simplify your answer using sine and cosine.

Problem 123 Let $S = \{(1,1,1),(0,2,2)\}$. Find an orthonormal basis β for $W = \text{span}(S)$. Also, extend β to an orthonormal basis for $\mathbb{R}^3$ and write $W^{\perp}$ as a span.

Problem 124 Let $S = \{(1,1,0,0),\ (1,2,2,2)\}$. Let $W = \text{span}(S)$.

(a) find an orthonormal basis β_1 for W

(b) find an orthonormal basis β_2 for $W^{\perp}$

Problem 125 Continuing the previous problem. Find the formula for $Proj_W : \mathbb{R}^4 \to W$. Also, find the point on W which is closest to (a,b,c,d). Likewise, find projection onto $W^{\perp}$ and the point which is closest to (a,b,c,d)

Problem 126 Define $\langle A, B\rangle = \text{trace}(AB^T)$ and $||A|| = \sqrt{\langle A, A\rangle}$. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Calculate $\langle A, B\rangle$, $||A||$ and $||B||$. Verify the inequality $|\langle A, B\rangle| \leq ||A||\,||B||$.

Problem 127 Let $S = \{x + x^2\} \subset P_2(\mathbb{R})$ where $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$. Find a basis for $S^{\perp}$.

Problem 128 Let $\mathcal{L}$ be the line which connects the points $(1,2,3,4)$ and $(5,5,5,5)$. Find the Euclidean distance from $(2,2,2,2)$ to $\mathcal{L}$.

Problem 129 Find the line closest to $(1,2),(2,0),(3,-2),(4,-7)$.

Problem 130 Curtis §15 exercise #10 on page 130.

Problem 131 If $L : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $L(x) \bullet L(y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$ then L is an **orthogonal transformation**. Show that if L is an orthogonal transformation on $\mathbb{R}^n$ then L is a linear transformation with $[L]^T[L] = I$.

Incidentally, $[L]^T[L] = I$ indicates $[L]$ is an **orthogonal matrix**; *the set of all orthogonal matrices is denoted $O(n) = \{R \mid R^T R = I\}$. The reason for this name is hopefully clear.*

Problem 132 Let $\mathrm{SO}(n) = \{R \in \mathbb{R}^{n \times n} \mid R^T R = I \ \& \ \det(R) = 1\}$ denote the set of **special orthogonal matrices.** Show that $\mathrm{SO}(n)$ forms a group under matrix multiplication. (see page 82 of Curtis for definition of group if you forgot)

Incidentally, the linear transformations with matrices in $SO(n)$ are **rotations**. *Consequently, we sometimes call $SO(n)$ the group of rotation matrices.*

Problem 133 If $(V, \langle\ ,\ \rangle)$ is a real inner product space and $L : V \to V$ is linear transformation then we say L is an **isometry** of V if $\langle L(x), L(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. Suppose $V = W_1 \oplus W_2$ where W_1, W_2 are orthogonal subspaces. Prove $L(W_1)$ and $L(W_2)$ are also orthogonal.

Problem 134 Suppose T is invertible and T has e-value λ. Show $\frac{1}{\lambda}$ is an e-value for T^{-1}.

Problem 135 Suppose $A \in \mathbb{C}^{n \times n}$. Show A^T and A have the same e-values. Do A and A^T share the same e-vectors for a given e-value? Prove or disprove.

Problem 136 Consider $R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the e-values and real e-vectors of R.

Problem 137 Let $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I\}$. Show that: If $R \in \mathrm{SO}(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 138 Let $R \in \mathrm{SO}(3)$ with $\mathrm{trace}(R) = 0$. By what angle does R rotate?

Problem 139 Formally, we have the identity: for $-1 \leq x \leq 1$

$$\boxed{x^2 = \sum_{n=0}^{\infty} \left(\frac{\langle x^2, \cos(n\pi x)\rangle}{\langle \cos(n\pi x), \cos(n\pi x)\rangle} \right) \cos(n\pi x) + \sum_{n=1}^{\infty} \left(\frac{\langle x^2, \sin(n\pi x)\rangle}{\langle \sin(n\pi x), \sin(n\pi x)\rangle} \right) \sin(n\pi x)} \ (\star)$$

where $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\, dx$ in this context. I say *formally* since we've left the world of finite linear algebra here. If we truncated these sums at finite n then we'd have trigonmetric approximations of x^2 within a $2n+1$ dimensional subspace of function space. However, we consider the full infinite sum and technically we should justify that the trigonmetric series converges, and, that its limit function does reproduce x^2 on $[-1, 1]$. We leave the formalities to the analysis course. Show (formally) that $(\star)$ yields:

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$$

on $-1 \leq x \leq 1$. You can use a CAS to do the integrals which are needed. Use this result to show $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (the $p = 2$ series converges to this value).

Problem 140 (Hokage) Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ be the projection defined by $\pi_j(x) = x - (x \bullet e_j)e_j$ for each $x \in \mathbb{R}^n$ for $j = 1, \ldots, n$. Suppose $\mathcal{P}$ is an $(n-1)$-dimensional paralell-piped which is formed by the convex-hull of $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ suspended at base-point $p \in (0, \infty)^n$;

$$\mathcal{P} = \left\{ p + \sum_{j=1}^{n-1} \alpha_j v_j \;\middle|\; \alpha_j \in [0, 1] \ \& \ \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

Let $n \in \mathbb{R}^n$ be a unit-vector in $\{v_1, \cdots, v_{n-1}\}^\perp$. The $(n-1)$-**area** of $\mathcal{P}$ is given by $\textbf{area}(\mathcal{P}) = |\det[v_1|\cdots|v_{n-1}|n]|$. We can study the area of the **shadows** formed by $\mathcal{P}$ on the coordinate hyperplanes. Let $\mathcal{P}_j = \pi_j(\mathcal{P})$ define the shadow of $\mathcal{P}$ on the $x_j = 0$ coordinate plane. Notice,

$$\mathcal{P}_j = \left\{ \pi_j(p) + \sum_{i=1}^{n-1} \alpha_i \pi_j(v_i) \;\middle|\; \alpha_i \in [0,1] \;\&\; \sum_{i=1}^{n-1} \alpha_i \leq 1 \right\}$$

which shows $\mathcal{P}_j$ is formed by the convex-hull $\pi_j(v_1), \ldots, \pi_j(v_n)$ of attached at basepoint $\pi_j(p)$. It follows that the $(n-1)$-area of the $\mathcal{P}_j$ can be calculated as follows:

$$\textbf{area}(\mathcal{P}_j) = |\det[\pi_j(v_1)|\cdots|\pi_j(v_{n-1})|e_j]|.$$

since e_j is perpendicular to $\mathcal{P}_j$. In the case $n = 2$ the 1-dimensional paralell-piped is just a line-segment. For example, if $v_1 = (1,1)$ then $(1/\sqrt{2}, -1/\sqrt{2})$ is perpendicular to v_1 and

$$\det \begin{bmatrix} 1 & 1/\sqrt{2} \\ 1 & -1/\sqrt{2} \end{bmatrix} = -2/\sqrt{2} = -\sqrt{2} \quad \Rightarrow \quad \textbf{area}(\mathcal{P}) = \sqrt{2}.$$

Of course, this is actually the length of the line-segment. Also, notice

$$\textbf{area}(\mathcal{P}_1)^2 + \textbf{area}(\mathcal{P}_2)^2 = 1^2 + 1^2 = \sqrt{2}^2 = \textbf{area}(\mathcal{P})^2.$$

This is not suprising. However, perhaps the fact this generalizes to n-dimensions in the following sense is not already known to you:

$$\textbf{area}(\mathcal{P}_1)^2 + \textbf{area}(\mathcal{P}_2)^2 + \cdots \textbf{area}(\mathcal{P}_n)^2 = \textbf{area}(\mathcal{P})^2$$

Prove it. You might call this the generalized Pythagorean identity, I'm not sure its history or formal name. That said, the formula I give for generalized area could just as well be termed generalized volume. Also, you could **define**

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \left[\begin{array}{c|c|c|c|c} & & & & e_1 \\ & & & & e_2 \\ v_1 & v_2 & \cdots & v_{n-1} & \vdots \\ & & & & e_n \end{array} \right] \in \mathbb{R}^n$$

where we insist the determinant is calculated via the Laplace expansion by minors along the last column. You can show $v_1 \times v_2 \times \cdots \times v_{n-1} \in \{v_1, \ldots, v_{n-1}\}^\perp$. But, if n is a unit-vector which spans $\{v_1, \ldots, v_{n-1}\}^\perp$ then the $(n-1)$-ry cross-product must be a vector parallel to n and thus:

$$v_1 \times v_2 \times \cdots \times v_{n-1} = [(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n]\, n$$

Note, $n \bullet n = 1$ as we assumed n is unit-vector and we can show

$$(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n = \det[v_1|v_2|\ldots|v_{n-1}|n]$$

Notice this generalized cross-product is just an extension of the heurstic determinant commonly used in multivariate calculus to define the standard cross-product. In particular, the following is equivalent to the column-based definition

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_n \\ & & v_1^T & \\ & & v_2^T & \\ & & \vdots & \\ & & v_{n-1}^T & \end{bmatrix}$$

where we insist the determinant is calculated via the Laplace expansion by minors along the first row. In any event, my point in this discussion is merely that we can calculate higher-dimensional volumes with determinants and these go hand-in-hand with generalized tertiary cross-products. In particular,

$$||v_1 \times v_2 \times \cdots \times v_{n-1}|| = \mathbf{vol}(\mathcal{P})$$

where $\mathcal{P}$ is formed by the convex hull of $v_1, \ldots, v_{n-1}$. When $n = 2$ this gives vector length, when $n = 3$ this is the familar result that the area of the parallelogram with sides $\vec{A}, \vec{B}$ is just $||\vec{A} \times \vec{B}||$.

Mission 8 Solution

P122 $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then $J^2 = -I$, $J^3 = -J$ etc...

$$e^{J\theta} = \sum_{n=0}^{\infty} \frac{(J\theta)^n}{n!} = \sum_{j=0}^{\infty} \frac{\theta^{2j} J^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{\theta^{2j+1} J^{2j+1}}{(2j+1)!} \qquad J^{2j} = (-1)^j I$$

$$= \left(\sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j}}{(2j)!} \right) I + \left(\sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \right) J$$

$$= \boxed{(\cos\theta) I + (\sin\theta) J}$$

P123 $S = \{(1,1,1), (0,2,2)\}$ let $W = \mathrm{span}(S)$ find orthonormal basis β for W. Then, extend to orth. basis for $\mathbb{R}^3$

$\underline{u_1 = \frac{1}{\sqrt{3}}(1,1,1)}$ is normalized $(1,1,1)$.

$$\tilde{u}_2 = (0,2,2) - \left[\frac{(0,2,2)\cdot(1,1,1)}{3} \right] (1,1,1) = (0,2,2) - (4/3, 4/3, 4/3)$$

$$\therefore \tilde{u}_2 = (-4/3, 2/3, 2/3) = \frac{2}{3}(-2,1,1) \longrightarrow \underline{u_2 = \frac{1}{\sqrt{6}}(-2,1,1)}$$

We find $\boxed{\beta = \left\{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(-2,1,1) \right\}}$ an orthonormal basis of W.

<u>Remark</u>: this is certainly not unique! If we set $v_1 = \frac{1}{\sqrt{2}}(0,1,1)$ then $(1,1,1) \xrightarrow[GSA]{} (1,0,0)$

To extend to basis of $\mathbb{R}^3$ there are many methods. I note $\mathbb{R}^3 = W \oplus W^\perp$ and seek $v \in W^\perp$

$$W^\perp = \left(\mathrm{Col} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \right)^\perp = \mathrm{Null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \mathrm{Null} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

we deduce $v = (v_1, v_2, v_3) \in W^\perp$ has $v_1 = 0$, $v_2 + v_3 = 0$ hence $v = v_3(0,-1,1)$ hence $u_3 = \frac{1}{\sqrt{2}}(0,-1,1)$ is the vector we seek to make $\boxed{\beta \cup \{ \frac{1}{\sqrt{2}}(0,-1,1) \} \text{ an orthonormal basis for } \mathbb{R}^3}$

P124 Let $S = \{(1,1,0,0), (1,2,2,2)\}$, $W = \text{span}(S)$

(a.) $\underline{u_1 = \frac{1}{\sqrt{2}}(1,1,0,0)}$

$$\tilde{u}_2 = (1,2,2,2) - \left[\frac{1}{2}(1,2,2,2)\cdot(1,1,0,0)\right](1,1,0,0)$$

$$\Rightarrow \tilde{u}_2 = (1,2,2,2) - (3/2, 3/2, 0, 0) = (-1/2, 1/2, 2, 2)$$

$$\tilde{u}_2 = \frac{1}{2}(-1,1,4,4) \quad \therefore \quad \underline{u_2 = \frac{1}{\sqrt{34}}(-1,1,4,4)}$$

$$\therefore \boxed{\beta_1 = \left\{ \frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{\sqrt{34}}(-1,1,4,4) \right\}}$$

(b.)
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$v = (v_1, v_2, v_3, v_4) \in W^{\perp}$ has $\begin{aligned} v_1 &= 2v_3 + 2v_4 \\ v_2 &= -2v_3 - 2v_4 \end{aligned}$

$$v = (2v_3 + 2v_4, -2v_3 - 2v_4, v_3, v_4)$$
$$= v_3 \underbrace{(2,-2,1,0)}_{\tilde{u}_3} + v_4 \underbrace{(2,-2,0,1)}_{\tilde{u}_4}$$

$\tilde{u}_3, \tilde{u}_4$ give basis for $W^{\perp}$, but $\tilde{u}_3 \cdot \tilde{u}_4 \neq 0$.
So, run GSA on them.

$u_3 = \frac{1}{3}(2,-2,1,0)$ has $\|u_3\| = 1$.

$$\tilde{\tilde{u}}_4 = (2,-2,0,1) - \left[\frac{1}{9}(2,-2,0,1)\cdot(2,-2,1,0)\right](2,-2,1,0)$$

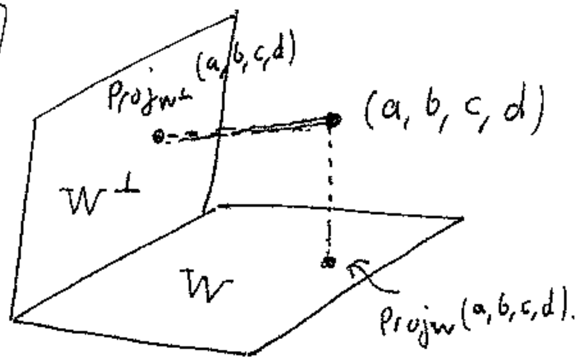
$$\tilde{\tilde{u}}_4 = (2,-2,0,1) - (16/9, -16/9, 8/9, 0)$$

$$\tilde{\tilde{u}}_4 = (2/9, -2/9, -8/9, 1) = \frac{1}{9}(2,-2,-8,9)$$

$$u_4 = \frac{1}{\sqrt{153}}(2,-2,-8,9)$$

$\therefore \boxed{\beta_2 = \left\{ \frac{1}{3}(2,-2,1,0), \frac{1}{\sqrt{153}}(2,-2,-8,9) \right\} \text{ is orthonormal basis for } W^{\perp}}$

$\boxed{P125}$



$$Proj_W(v) = (v\cdot u_1)u_1 + (v\cdot u_2)u_2$$

$$Orth_W(v) = Proj_{W^\perp}(v) = (v\cdot u_3)u_3 + (v\cdot u_4)u_4$$

$$Proj_W(a,b,c,d) = \left[\tfrac{1}{\sqrt{2}}(1,1,0,0)\cdot(a,b,c,d)\right]\tfrac{1}{\sqrt{2}}(1,1,0,0) + \left[\tfrac{1}{\sqrt{34}}(-1,1,4,4)\cdot(a,b,c,d)\right]\tfrac{1}{\sqrt{34}}(-1,1,4,4)$$

$$= \tfrac{1}{2}(a+b)(1,1,0,0) + \tfrac{1}{34}(-a+b+4c+4d)(-1,1,4,4)$$

$$= \left(\tfrac{9}{17}a + \tfrac{8}{17}b - \tfrac{2}{17}c - \tfrac{2}{17}d,\ \tfrac{8}{17}a + \tfrac{9}{17}b + \tfrac{2}{17}c + \tfrac{2}{17}d,\ \tfrac{2}{17}(-a+b+4c+4d),\ \tfrac{2}{17}(-a+b+4c+4d)\right)$$

$$= \boxed{\tfrac{1}{17}(9a+8b-2c-2d,\ 8a+9b+2c+2d,\ -2a+2b+8c+8d,\ -2a+2b+8c+8d)}$$

$$Proj_{W^\perp}(a,b,c,d) = \tfrac{1}{9}(2a-2b+c)(2,-2,1,0) + \tfrac{1}{153}(2a-2b-8c+9d)(2,-2,-8,9)$$

$$= \left(\tfrac{4}{9}a - \tfrac{4}{9}b + \tfrac{2}{9}c,\ \tfrac{-4}{9}a + \tfrac{4}{9}b - \tfrac{2}{9}c,\ \tfrac{2}{9}a - \tfrac{2}{9}b + \tfrac{1}{9}c,\ 0\right)$$

$$+ \left(\tfrac{4}{153}a - \tfrac{4}{153}b - \tfrac{16}{153}c + \tfrac{18}{153}d,\ \tfrac{-4a+4b+16c-18d}{153},\ \tfrac{-16a+16b+64c-72d}{153},\ \tfrac{18a-18b-72c+81d}{153}\right)$$

$$= \boxed{\tfrac{1}{17}(8a-8b+2c+2d,\ -8a+8b-2c-2d,\ 2a-2b+9c-8d,\ 2a-2b-8c+9d)}$$

Notice, $Proj_W(a,b,c,d) + Proj_{W^\perp}(a,b,c,d) = (a,b,c,d)$.

<u>Remark:</u> these quantities are the closest points on W & $W^\perp$ to (a,b,c,d).

<u>Remark:</u> the construction of β_1 & β_2 in P124 is not at all unique. However, the formulas above are uniquely determined by W and $W^\perp$!

$\boxed{P126}$ Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$\langle A, B \rangle = \text{trace}(AB^T) = \text{trace}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \text{trace}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \boxed{0} = \langle A, B \rangle$$

$$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\text{trace}(AA^T)} = \sqrt{\text{trace}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}} = \sqrt{4} = \boxed{2 = \|A\|}$$

$$\|B\| = \sqrt{\text{trace}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}} = \sqrt{\text{trace}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} = \boxed{\sqrt{2} = \|B\|}$$

Indeed, $0 = |\langle A, B \rangle| \leq \|A\| \cdot \|B\| = 2\sqrt{2}$.

(sorry, I meant for $\langle A, B \rangle \neq 0$, this accident made my request rather silly.)

$\boxed{P127}$ Let $S = \{x + x^2\} \subset P_2(\mathbb{R})$ where $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$

If $ax^2 + bx + c \in S^\perp$ then $\langle ax^2 + bx + c, x + x^2 \rangle = 0$ thus,

$$0 = \int_0^1 (ax^2 + bx + c)(x + x^2)\,dx = \int_0^1 \left[a(x^4 + x^3) + b(x^3 + x^2) + c(x^2 + x)\right]dx$$

$$\Rightarrow \quad 0 = a\left(\tfrac{1}{5} + \tfrac{1}{4}\right) + b\left(\tfrac{1}{4} + \tfrac{1}{3}\right) + c\left(\tfrac{1}{3} + \tfrac{1}{2}\right)$$

$$\therefore \quad c = \frac{6}{5}\left(\frac{-9a}{20} - \frac{7b}{12}\right) = \frac{-27}{50}a - \frac{7}{10}b$$

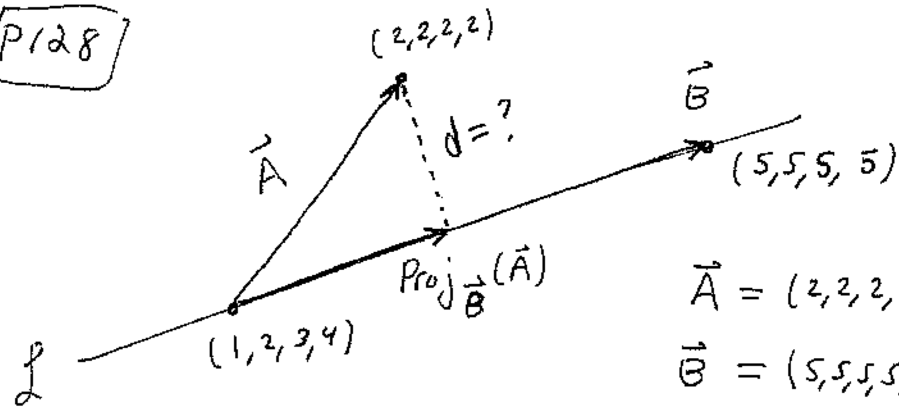
Thus
$$\begin{aligned} ax^2 + bx + c &= ax^2 + bx + \left(\tfrac{-27}{50}a - \tfrac{7}{10}b\right) \\ &= a\left(x^2 - \tfrac{27}{50}\right) + b\left(x - \tfrac{7}{10}\right) \end{aligned}$$

$$\Rightarrow \quad S^\perp = \text{span}\left\{x^2 - \tfrac{27}{50},\ x - \tfrac{7}{10}\right\}$$

Moreover, $\boxed{\left\{x^2 - \tfrac{27}{50},\ x - \tfrac{7}{10}\right\} \text{ is basis for } S^\perp}$

<u>Remark</u>: I'm glad I didn't ask for an orthonormal basis for $S^\perp$. That'd be a pain.

P128



$\vec{A} = (2,2,2,2) - (1,2,3,4) = (1, 0, -1, -2)$

$\vec{B} = (5,5,5,5) - (1,2,3,4) = (4, 3, 2, 1)$

$$\text{Proj}_{\vec{B}}(\vec{A}) = (\vec{A} \cdot \hat{B})\hat{B} = \left(\frac{\vec{A} \cdot \vec{B}}{\vec{B} \cdot \vec{B}}\right)\vec{B} = \frac{4-2-2}{16+9+4+1} = 0$$

Ha. In fact, $\vec{A} \perp \vec{B}$ so the point $\boxed{(1,2,3,4)}$ is the closest point.

<u>Remark</u>: this was not intentional, but, it's hillarious.

P129 Find closest line to points $(1,2), (2,0), (3,-2), (4,-7)$

Our model is $y = mx + b$ we face

$$\begin{aligned} 2 &= m + b \\ 0 &= 2m + b \\ -2 &= 3m + b \\ -7 &= 4m + b \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -2 \\ -7 \end{bmatrix}}_{c}$$

$$Mu = c \Rightarrow M^T M u = M^T c \quad \text{(normal eq}^{\text{ns}}\text{)}$$

$$\Rightarrow \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} u = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ -7 \end{bmatrix} = \begin{bmatrix} -32 \\ -7 \end{bmatrix}$$

$$\therefore u = \frac{1}{120 - 100} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} -32 \\ -7 \end{bmatrix}$$

$$u = \frac{1}{20} \begin{bmatrix} -58 \\ 110 \end{bmatrix} = \begin{bmatrix} -2.9 \\ 5.5 \end{bmatrix}$$

$$\therefore \boxed{y = -2.9x + 5.5}$$

P130 §15 of CURTIS, #10 of p. 130

Let V be an inner product space and suppose W_1 & W_2 are subspaces of equal dimension. Show $\exists T$ orthogonal such that $T(W_1) = W_2$

Let $\beta_1 = \{v_1, .., v_k\}$ and $\beta_2 = \{w_1, ..., w_k\}$ be orthonormal bases for W_1 & W_2 respective. Next, extend β_1 to γ_1 an orthonormal basis for V and likewise β_2 to γ_2 an orthonormal basis for V.

Denote,

$$\gamma_1 = \{v_1, .., v_k, u_{k+1}, .., u_n\} = \{f_i\}_{i=1}^{n}$$

$$\gamma_2 = \{w_1, .., w_k, x_{k+1}, .., x_n\} = \{g_i\}_{i=1}^{n}$$

Define $T : V \longrightarrow V$ by linearly extending the following formulas,

$$\left.\begin{aligned} T(v_j) &= w_j \quad \text{for } j = 1, 2, .., k \\ T(u_l) &= x_l \quad \text{for } l = k+1, .., n. \end{aligned}\right\} \text{aka } T(f_j) = g_j \ \forall j = 1, 2, .., n$$

Consider,

$$\langle T(\alpha), T(\beta) \rangle = \langle T(\sum_i \alpha_i f_i), T(\sum_j \beta_j f_j) \rangle$$

$$= \sum_i \sum_j \alpha_i \beta_j \langle T(f_i), T(f_j) \rangle$$

$$= \sum_i \sum_j \alpha_i \beta_j \underbrace{\langle g_i, g_j \rangle}_{\delta_{ij}} = \underline{\sum_{i=1}^{n} \alpha_i \beta_i} \ *$$

$$\langle \alpha, \beta \rangle = \langle \sum_i \alpha_i f_i, \sum_j \beta_j f_j \rangle = \sum_i \sum_j \alpha_i \beta_j \overbrace{\langle f_i, f_j \rangle}^{\delta_{ij}}$$

$$= \underline{\sum_i \alpha_i \beta_i} \ **$$

We find $\langle T(\alpha), T(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in V$ and by construction $T(W_1) = W_2$.

$\boxed{P131}$ Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ with $L(x) \cdot L(y) = x \cdot y \quad \forall x, y \in \mathbb{R}^n$

Observe, $\beta = \{L(e_i)\}_{i=1}^n$ forms an orthonormal basis ☆ for $\mathbb{R}^n$ as $L(e_i) \cdot L(e_j) = e_i \cdot e_j = \delta_{ij}$.

Consider then, for $c \in \mathbb{R}$, $x, y, z \in \mathbb{R}^n$

$$L(cx+y) \cdot L(z) = (cx+y) \cdot z$$

$$\Rightarrow L(cx+y) \cdot L(z) = c(x \cdot z) + y \cdot z = cL(x) \cdot L(z) + L(y) \cdot L(z)$$

$$\Rightarrow \underline{[L(cx+y) - cL(x) - L(y)] \cdot L(z) = 0} \; *$$

Consider $z = e_j$ to observe $[L(cx+y) - cL(x) - L(y)]_\beta = (0, \dots, 0)$

Hence $L(cx+y) - cL(x) - L(y) = 0$

$$\therefore \; L(cx+y) = cL(x) + L(y) \quad \forall x, y \in \mathbb{R}^n$$

We've shown L is linear from $\mathbb{R}^n \to \mathbb{R}^n$.

Consider,

$$\begin{aligned} L(x) \cdot L(y) &= ([L]x) \cdot ([L]y) \\ &= ([L]x)^T [L] y \\ &= x^T [L]^T [L] y = x \cdot y = x^T I y \end{aligned}$$

Set $x = e_i$ and $y = e_j$ to obtain $e_i^T I e_j = \delta_{ij}$ and hence $([L]^T [L])_{ij} = \delta_{ij}$ or $\underline{[L]^T[L] = I}$.

$$\therefore \; \underline{[L] \in O(n)}.$$

☆: this observation is crucial to using dot-product of $L(z)$ in nontrivial fashion. $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C} \not\Rightarrow \vec{B} = \vec{C}$ generally. We need $\vec{A}_j \cdot \vec{B} = \vec{A}_j \cdot \vec{C}$ for $\vec{A}_1, \dots, \vec{A}_n$ basis.

P132 $SO(n) = \{ R \in \mathbb{R}^{n \times n} \mid R^T R = I,\ \det R = 1 \}$

$I^T I = I$ and $\det I = 1$ $\therefore$ $I \in SO(n) \neq \emptyset$.

If $A, B \in SO(n)$ then $A^T A = I$, $B^T B = I$ hence,

$$(AB)^T AB = B^T A^T A B = B^T I B = B^T B = I$$

and $\det(AB) = \det A \det B = (1)(1) = 1$ $\therefore$ $(AB) \in SO(n)$.

Furthermore, $A \in SO(n) \Rightarrow A^T A = I$

$$\Rightarrow A^{-1} = A^T$$

$$\Rightarrow (A^{-1})^T A^{-1} = (A^T)^T A^{-1} = A A^{-1} = I$$

And, $\det(A^{-1}) = \frac{1}{\det A} = 1$ $\therefore$ $A^{-1} \in SO(n)$. Since matrix multiplication is <u>associative</u> we're done, $SO(n)$ forms a <u>group</u>.

P133 Suppose $(V, \langle , \rangle)$ forms a real inner-product space and linear trans. $L : V \to V$ is an isometry; $\langle L(x), L(y) \rangle = \langle x, y \rangle\ \forall x, y \in V$. Also, suppose $V = W_1 \oplus W_2$ where $W_1 \perp W_2$.

Let $x_1 \in L(W_1)$ and $x_2 \in L(W_2)$ then $\exists\, w_1 \in W_1$ and $w_2 \in W_2$ for which $x_1 = L(w_1)$ and $x_2 = L(w_2)$. Consider,

$$\langle x_1, x_2 \rangle = \langle L(w_1), L(w_2) \rangle = \langle w_1, w_2 \rangle = 0$$

since $W_1 \perp W_2 \Rightarrow \langle w_1, w_2 \rangle = 0$. Thus $L(W_1) \perp L(W_2)$.

P134 Suppose $T : V \to V$ has e-value λ where T is invertible. Show $1/\lambda$ is e-value for T^{-1}

Observe $\lambda \neq 0$ since $T^{-1} \Rightarrow \ker(T) = \{0\} \Rightarrow \lambda = 0$ not e-value.

Note, $\exists\, x \neq 0$ such that $T(x) = \lambda x$ hence,

$$T^{-1}(T(x)) = T^{-1}(\lambda x) \Rightarrow x = \lambda T^{-1}(x) \Rightarrow T^{-1}(x) = \frac{1}{\lambda} x$$

thus x is e-vector with e-value $\frac{1}{\lambda}$ for T^{-1}. //

$\boxed{P135}$ Let $A \in \mathbb{C}^{n\times n}$. Show A and A^T have same e-values.

Note, $P(t) = \det(A - tI)$ is characteristic polynomial for A and $\det(A^T - tI) = \det((A-tI)^T) = \det(A - tI)$ hence $P(t)$ is <u>also</u> the char. poly. for A^T. Thus A and A^T share the same e-values. However, A & A^T do <u>not</u> generally share same e-vectors, for example $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1$, $\lambda_2 = 0$

with $\underbrace{V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ \& \ V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{e-vectors for } A}$

$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has $\lambda_1 = 1$, $\lambda_2 = 0$

with $\underbrace{W_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \& \ W_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{e-vectors for } A^T}$

You can see the e-vectors are not the same.

<u>Remark</u>: if $[A, A^T] = 0$ then it is possible. See Thm 23.13 on p. 200 of CURTIS.

$\boxed{P136}$ Consider $R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(R - \lambda I) = \det\begin{bmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = \left[(\lambda - \cos\theta)^2 + \sin^2\theta\right]\left[1 - \lambda\right]$$

thus R has e-values $\lambda = 1$ or $\lambda = \cos\theta \pm i\sin\theta$. If $\theta = n\pi$ for $n \in \mathbb{Z}$ then $\lambda = 1, 1, 1$ or $\lambda = 1, -1, -1$. In the case $\lambda = 1$ thrice we have $R = I$ and every nonzero vector is an e-vector. If $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$ then $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has $\lambda_1 = 1$ whereas $V_2 = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ for any $(a,b) \neq 0$ is e-vector of $\lambda_2 = -1$.

If $\theta \neq n\pi$ for some $n \in \mathbb{Z}$ then $\lambda = 1$ is the only <u>real</u> e-value and $Re_3 = e_3$ thus $V = k(0,0,1)$ for $k \neq 0$ are all the real e-vectors.

<u>Remark</u>: to grader, if students assumed $\theta \notin \pi\mathbb{Z}$ then give them full credit. That was my intention here.

$\boxed{P137}$ Suppose $R \in SO(3)$ and $R \neq I$ then R has only two e-vectors of unit-length for which $\lambda = 1$

Since $R \in SO(3)$ we have $R^T R = I$ and $\det(R) = 1$ thus $R^{-1} = R^T \Rightarrow R^{-1}$ and R have same e-values (by P135)

Also, if R has complex e-values $\{\lambda_1, \lambda_2, \lambda_3\}$ then R^{-1} has complex e-values $\{1/\lambda_1, 1/\lambda_2, 1/\lambda_3\}$ by (P134)

Thus, $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\}$

without loss of generality we may suppose $\lambda_1 = 1$ hence $\{\lambda_2, \lambda_3\} = \{\frac{1}{\lambda_2}, \frac{1}{\lambda_3}\}$. If $\lambda_2 = \frac{1}{\lambda_2}$ then $\lambda_3 = \frac{1}{\lambda_3}$ and $\lambda_2^2 = 1$ and $\lambda_3^2 = 1 \Rightarrow \lambda_2 = \pm 1$ and $\lambda_3 = \pm 1$ for $\lambda_2, \lambda_3 \in \mathbb{R}$

But, $\lambda_2 = \lambda_3 = 1 \Longrightarrow R = I$. On the other hand, if $\lambda_2 = 1$ and $\lambda_3 = -1$ then $\lambda_1 \lambda_2 \lambda_3 = (1)(1)(-1) = -1 \neq \det(R) \therefore$ not possible.

The other possibility is $\lambda_2 = \frac{1}{\lambda_3}$ and $\lambda_3 = \frac{1}{\lambda_2}$ and we find $\lambda_2 = 1$ or $\lambda_3 = 1$ forces $\lambda_2 = \lambda_3 = 1$ (and hence is not possible for $R \neq I$). In summary, $\lambda = 1$ has algebraic multiplicity 1 for $R \neq I \therefore$
dim (R $\dim(\mathrm{Null}(R - I)) = 1$

$$\Rightarrow \mathrm{Null}(R - I) = \mathrm{span}\{V\} \text{ for } RV = V,\ V \neq 0$$

Let $u = \frac{1}{\|V\|} V$ then $-u, u$ are both unit-vectors of e-value $\lambda = 1$.

(this is what we found in P136 for $\theta \neq 2\pi n$)
$Re_3 = e_3$ and $R(-e_3) = -e_3$ for R of last problem $\blacksquare$

$\boxed{P138}$ If $R \in SO(3)$ and $\text{trace}(R) = 0$ then we can determine the angle by which R rotates as follows, note $R \neq I$ as $\text{trace}(I) = 3$

$$P^{-1} R P = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

last problem we found
$\lambda = \cos\theta \pm i\sin\theta$, $\lambda = 1$
are e-values. Notice
$R \neq I$ and $\{\lambda_1, \lambda_2, \lambda_3\} = \{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\}$
with $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(R) \in \mathbb{R}$
forces λ_2, λ_3 to be conjugate pair
setting $\lambda_1 = 1$. But $\lambda_1\lambda_2\lambda_3 = 1$
$\Rightarrow \lambda_2 = e^{i\theta}$ and $\lambda_3 = \frac{1}{e^{i\theta}} = e^{-i\theta}$
thus there exists similarity transformation to P136 - type matrix

Finally, getting to the ~~exa~~good part,

$$\text{trace}(P^{-1}RP) = \text{trace}(R) = 2\cos\theta + 1$$

$$\Rightarrow 0 = 2\cos\theta + 1$$

$$\Rightarrow \cos\theta = \frac{-1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{2} + \frac{\pi}{6} = \boxed{\frac{2\pi}{3}}$$

$\boxed{P139}$ I leave the integration to the reader ☺
Set $x = 1$,

$$1 = \frac{1}{3} + \sum_{n=1}^{\infty} \cancel{(-1)^n} \frac{4}{n^2\pi^2} \cancel{\cos(n\pi)} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4}\left(1 - \frac{1}{3}\right) = \boxed{\frac{\pi^2}{6}}.$$