Please follow the format which was announced in Blackboard. Thanks!

Problem 127 Suppose $\beta = \{\frac{1}{\sqrt{12}}(1,1,3,1), \frac{1}{\sqrt{12}}(1,-3,1,-1), \frac{1}{\sqrt{6}}(-1,-1,0,2), v_4\}$. Find v_4 such that β forms an orthonormal basis for $\mathbb{R}^4$ then calculate $[(0,0,1,0)]_\beta$.

Problem 128 Let $W = \text{span}\{(2,2,1,0),(1,1,1,0)\}$.

(a.) Find an orthonormal basis β_W for W in $\mathbb{R}^4$ with the standard Euclidean geometry,

(b.) Calculate $\text{Proj}_W(a,b,c,d)$,

(c.) Find the point on W closest to $(3,3,2,4)$.

Problem 129 Let $W = \text{span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ for a subspace in $\mathbb{R}^{2\times 2}$ with the standard Frobenius inner product $\langle A,B\rangle = \text{trace}(AB^T)$.

(a.) Find an orthonormal basis for $W^\perp$

(b.) Calculate $Proj_{W^\perp}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(c.) Calculate $Proj_W\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(d.) Find the matrix in W which is closest to $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

Problem 130 Consider the plane $\mathcal{P}$ in $\mathbb{R}^4$ given by $w+x+y+z=0$ and $w-x-y-z=0$. Notice $(1,1,1,1)\notin\mathcal{P}$. Find the point in $\mathcal{P}$ which is closest to $(1,1,1,1)$.

Problem 131 Show that $\langle f(x),g(x)\rangle = \int_0^1 f(x)g(x)dx$ defines an inner product on $\mathbb{R}[x]$. Explain why the same formula fails to define an inner product on the space of continuous real-valued functions on $\mathbb{R}$.

Problem 132 Let $W = P_2(\mathbb{R})$ and use $\langle f(x),g(x)\rangle = \int_0^1 f(x)g(x)dx$ for the inner product on W. Run the GSA on $\{1,x,x^2\}$ with respect to the given inner product.

Problem 133 Consider $V = P_2(\mathbb{R})\cup\{e^x\}$ this is naturally a subspace of the continuous functions on $\mathbb{R}$ as $V = \text{span}\{1,x,x^2,e^x\}$. Furthermore, V with $\langle f(x),g(x)\rangle = \int_0^1 f(x)g(x)dx$ is an inner product space (the counter-example for two problems back requires continuous functions not found in V). Calculate the projection of e^x onto the subspace $P_2(\mathbb{R})$ of V.
please give an approximate answer two two decimal places, you may use technology to compute the relevant integrals

Problem 134 **Cauchy Schwarz Inequality** Let V be an inner product space over $\mathbb{F}$ (either $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$) with inner product $\langle,\rangle$ and induced norm $\|x\| = \sqrt{\langle x,x\rangle}$. Prove $|\langle x,y\rangle| \le \|x\|\,\|y\|$ for all $x,y\in V$. Let me give you a path:

(a.) show $0 \le \|x-cy\|^2 = \|x\|^2 - c\langle y,x\rangle - \bar{c}\langle x,y\rangle + |c|^2\|y\|^2$ for all $x,y\in V$ and $c\in\mathbb{F}$,

(b.) when $y \neq 0$ can set $c = \frac{\langle x,y\rangle}{\langle y,y\rangle}$ in the inequality in $(a.)$

(c.) rearrange the inequality to derive the Cauchy Schwarz inequality in the case $y \neq 0$.

This is just a sketch, you need to connect these thoughts with a proper proof narrative.

Problem 135 **Triangle Inequality:** Let $(V, \langle , \rangle)$ be an inner product space with the usual induced norm $\|x\| = \sqrt{\langle x, x\rangle}$. Prove $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Problem 136 Let V be a complex inner product space with inner product $\langle , \rangle$. If $x, y \in V$ then there are at least two competing ideas to describe the angle between these vectors:

(1.) we can calculate the so-called **complex angle** defined by $\cos\tilde{\theta} = \frac{|\langle x,y\rangle|}{\|x\|\|y\|}$ for x, y nonzero in V. Notice $0 \leq \tilde{\theta} \leq \pi/2$ since the cosine of the complex angle is by-construction non-negative.

(2.) we could view V as a real vector space with inner product given by $\langle x, y\rangle_{\mathbb{R}} = \text{Re}\langle x, y\rangle$ then the **real angle** between x, y nonzero is given by $\cos\theta = \frac{\langle x,y\rangle_{\mathbb{R}}}{\|x\|_{\mathbb{R}}\|y\|_{\mathbb{R}}}$ where $\|x\|_{\mathbb{R}} = \sqrt{\langle x, x\rangle_{\mathbb{R}}}$. Notice, θ so-constructed ranges over $[-\pi/2, \pi/2]$.

For the standard complex vector spaces below and the given vectors calculate the complex and real angle between the vectors:

(a.) $x = \langle 1, 1+i\rangle$ and $y = \langle 1-i, 2\rangle$ in $\mathbb{C}^2$

(b.) $A = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} i & 4i \\ 4i & 4i \end{bmatrix}$ in $\mathbb{C}^{2\times 2}$

Problem 137 Consider the vectors studied in the above problem,

(a.) is $\{x, y\}$ a linearly independent set in $\mathbb{C}^2$ as a complex vector space ?

(b.) is $\{x, y\}$ a linearly independent set in $\mathbb{C}^2$ as a real vector space ?

(c.) is $\{A, B\}$ a linearly independent set in $\mathbb{C}^{2\times 2}$ as a complex vector space ?

(d.) is $\{A, B\}$ a linearly independent set in $\mathbb{C}^{2\times 2}$ as a real vector space ?

Problem 138 Suppose V is a finite dimensional inner product space over[1] $\mathbb{R}$. Let $g : V \times V \to \mathbb{R}$ denote the inner product for V. If $W \leq V$ is a nontrivial subspace then show $g|_W : W \times W \to \mathbb{R}$ defined by $g|_W(x, y) = g(x, y)$ for all $x, y \in W$ is an inner product.

Remark: *in words, the restriction of an inner product is once more an inner product. If we study natural generalizations of inner products then we find this restriction property fails. That is the point of the next problem.*

Problem 139 A **scalar product** or **metric** on $\mathbb{R}^4$ is a symmetric bilinear form which is **nondegenerate**. In particular, $g : V \times V \to \mathbb{R}$ is **nondegenerate** if $g(v, w) = 0$ for all $w \in V$ implies $v = 0$. Nondegeneracy is equivalent to the condition that the matrix of g has nonzero determinant. It can be shown that every inner product is a metric, however, the converse

[1]this problem likely makes sense over $\mathbb{C}$ as well, but, I limit our scope for your convenience.

fails. This problem intends to illustrate some of the differences. The metric given below is the so-called **Minkowski Metric** of time-space. Let $g : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be defined by:

$$g(v, w) = v^T \eta w \qquad \text{where} \qquad \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a.) show g is a symmetric bilinear form on $\mathbb{R}^4$

(b.) why is g is **not** an inner product on $\mathbb{R}^4$?

(c.) let $W = \text{span}\{e_2, e_3, e_4\}$ and show $g|_W$ is an inner product (and hence a metric).

(d.) let $C = \text{span}\{e_2 - e_1\}$ and show $g|_C$ is **not** an metric.

Problem 140 We define $\text{SO}(2, \mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} \mid A^T A = I, \ \ \det(A) = 1\}$.

(a.) show $A \in \text{SO}(2, \mathbb{R})$ has the form $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta \in \mathbb{R}$.

(b.) Given a linear isometry T of $\mathbb{R}^2$ has $\text{trace}(T) = \sqrt{2}$ and $\det(T) = 1$. By what angle does T rotate?

Problem 141 Suppose $\theta \neq n\pi$ for $n \in \mathbb{Z}$. Consider $R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the complex eigenvalues and real e-vectors of R.

Problem 142 Let $\text{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I\}$. Show that: If $R \in \text{SO}(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 143 Let $R \in \text{SO}(3)$ with $\text{trace}(R) = 0$. By what angle does R rotate?
Hint: consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $[T] = R$ and study the basis where the third vector is the e-vector of unit-length whose existence you proved in the last problem

Problem 144 An n-parallel piped $\mathcal{P}$ with edges $v_1, \dots, v_n$ is the **convex-hull** of $v_1, \dots, v_n$. That is:

$$\mathcal{P} = \{c_1 v_1 + \cdots + c_n v_n \mid 0 \leq c_1, \dots, c_n \leq 1 \ \ \& \ \ c_1 + \cdots + c_n = 1\}.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry of the Euclidean geometry of $\mathbb{R}^n$. Show $T(\mathcal{P})$ is an n-parallel piped with the same n-volume as $\mathcal{P}$.
Reminder: the n-volume of an n-parallel piped with edges $v_1, \dots, v_n$ is given by $|det[v_1| \cdots |v_n]|$.

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Let M be a symmetric matrix and define $\Upsilon(A,B) = AB + BA$ for all $A, B \in \mathbb{R}^{n\times n}$ show Υ is a symmetric, bilinear form.

(II.) Let V be a complex vector space with inner product $\langle , \rangle$. Show $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$ for all $x, y \in V$ and $c \in \mathbb{C}$.

(III.) Let V be a real vector space with inner product $\langle , \rangle$ and let r be a positive constant. Define $g : V \times V \to \mathbb{R}$ by $g(x,y) = r\langle x, y\rangle$ for all $x, y \in V$. Show g defines an inner product on V. Comment on the geometry given by g as it relates to the geometry given by $\langle , \rangle$. In particular, compare and constrast the angles between vectors and the length of vectors as measured by $\langle , \rangle$ vs. g

(IV.) Let $\beta = \{E_{ii} \mid 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} \mid 1 \leq i < j \leq n\}$ form a basis for the symmetric matrices $S_n \leq \mathbb{R}^{n\times n}$. Show that β is an orthogonal basis with respect to the Frobenius inner product.

(V.) Suppose (V, g) forms a geometry and β is a basis for V for which G is the matrix of g. Furthermore, suppose the linear mapping $L : V \to V$ is a g-orthogonal map such that A is its matrix; $[L(x)]_\beta = A[x]_\beta$ or simply $[L]_{\beta,\beta} = A$. Show $A^T G A = G$.

(VI.) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. Find the angle between A and B as measured by the inner product $\langle A, B\rangle = \text{trace}(AB^*)$ where $B^* = \bar{B}^T$.

(VII.) Let $S = \{(1,1,1,1), (0,2,1,0), (1,2,0,1)\}$. Find an orthonormal basis β for span(S). If $(a,b,c,d) \in \text{span}(S)$ then find the coordinates of (a,b,c,d) with respect to β.

(VIII.) Consider $S = \{x, e^x\}$. Find an orthonormal basis for $W = \text{span}(S)$ where the inner product is given by $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$

(IX.) Formally, we have the identity: for $-1 \leq x \leq 1$

$$\boxed{x^2 = \sum_{n=0}^{\infty}\left(\frac{\langle x^2, \cos(n\pi x)\rangle}{\langle \cos(n\pi x), \cos(n\pi x)\rangle}\right)\cos(n\pi x) + \sum_{n=1}^{\infty}\left(\frac{\langle x^2, \sin(n\pi x)\rangle}{\langle \sin(n\pi x), \sin(n\pi x)\rangle}\right)\sin(n\pi x)} \quad (\star)$$

where $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ in this context. I say *formally* since we've left the world of finite linear algebra here. If we truncated these sums at finite n then we'd have trigonmetric approximations of x^2 within a $2n+1$ dimensional subspace of function space. However, we consider the full infinite sum and technically we should justify that the trigonmetric series converges, and, that its limit function does reproduce x^2 on $[-1, 1]$. We leave the formalities to the analysis course. Show (formally) that $(\star)$ yields:

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2\pi^2}\cos(n\pi x)$$

on $-1 \leq x \leq 1$. You can use a CAS to do the integrals which are needed. Use this result to show $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (the $p = 2$ series converges to this value).