Please follow the format which was announced in Blackboard. Thanks!

Problem 127 Suppose $\beta = \{\frac{1}{\sqrt{12}}(1,1,3,1), \frac{1}{\sqrt{12}}(1,-3,1,-1), \frac{1}{\sqrt{6}}(-1,-1,0,2), v_4\}$. Find v_4 such that β forms an orthonormal basis for $\mathbb{R}^4$ then calculate $[(0,0,1,0)]_\beta$.

Problem 128 Let $W = \text{span}\{(2,2,1,0),(1,1,1,0)\}$.

(a.) Find an orthonormal basis β_W for W in $\mathbb{R}^4$ with the standard Euclidean geometry,

(b.) Calculate $\text{Proj}_W(a,b,c,d)$,

(c.) Find the point on W closest to $(3,3,2,4)$.

Problem 129 Let $W = \text{span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ for a subspace in $\mathbb{R}^{2\times 2}$ with the standard Frobenius inner product $\langle A, B\rangle = \text{trace}(AB^T)$.

(a.) Find an orthonormal basis for $W^\perp$

(b.) Calculate $Proj_{W^\perp}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(c.) Calculate $Proj_W\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(d.) Find the matrix in W which is closest to $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

Problem 130 Consider the plane $\mathcal{P}$ in $\mathbb{R}^4$ given by $w+x+y+z=0$ and $w-x-y-z=0$. Notice $(1,1,1,1) \notin \mathcal{P}$. Find the point in $\mathcal{P}$ which is closest to $(1,1,1,1)$.

Problem 131 Show that $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$ defines an inner product on $\mathbb{R}[x]$. Explain why the same formula fails to define an inner product on the space of continuous real-valued functions on $\mathbb{R}$.

Problem 132 Let $W = P_2(\mathbb{R})$ and use $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$ for the inner product on W. Run the GSA on $\{1, x, x^2\}$ with respect to the given inner product.

Problem 133 Consider $V = P_2(\mathbb{R}) \cup \langle e^x \rangle$ this is naturally a subspace of the continuous functions on $\mathbb{R}$ as $V = \text{span}\{1, x, x^2, e^x\}$. Furthermore, V with $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$ is an inner product space (the counter-example for two problems back requires continuous functions not found in V). Calculate the projection of e^x onto the subspace $P_2(\mathbb{R})$ of V.
please give an approximate answer two two decimal places, you may use technology to compute the relevant integrals

Problem 134 **Cauchy Schwarz Inequality** Let V be an inner product space over $\mathbb{F}$ (either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) with inner product $\langle , \rangle$ and induced norm $||x|| = \sqrt{\langle x, x\rangle}$. Prove $|\langle x, y\rangle| \leq ||x||\,||y||$ for all $x, y \in V$. Let me give you a path:

(a.) show $0 \leq ||x - cy||^2 = ||x||^2 - c\langle y, x\rangle - \bar{c}\langle x, y\rangle + |c|^2||y||^2$ for all $x, y \in V$ and $c \in \mathbb{F}$, (b.) when $y \neq 0$ can set $c = \frac{\langle x,y\rangle}{\langle y,y\rangle}$ in the inequality in $(a.)$

(c.) rearrange the inequality to derive the Cauchy Schwarz inequality in the case $y \neq 0$.

This is just a sketch, you need to connect these thoughts with a proper proof narrative.

Problem 135 **Triangle Inequality:** Let $(V, \langle , \rangle)$ be an inner product space with the usual induced norm $||x|| = \sqrt{\langle x, x\rangle}$. Prove $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in V$.

Problem 136 Let V be a complex inner product space with inner product $\langle , \rangle$. If $x, y \in V$ then there are at least two competing ideas to describe the angle between these vectors:

(1.) we can calculate the so-called **complex angle** defined by $\cos\tilde{\theta} = \frac{|\langle x,y\rangle|}{||x||\,||y||}$ for x, y nonzero in V. Notice $0 \leq \tilde{\theta} \leq \pi/2$ since the cosine of the complex angle is by-construction non-negative.

(2.) we could view V as a real vector space with inner product given by $\langle x, y\rangle_{\mathbb{R}} = \mathrm{Re}\langle x, y\rangle$ then the **real angle** between x, y nonzero is given by $\cos\theta = \frac{\langle x,y\rangle_{\mathbb{R}}}{||x||_{\mathbb{R}}||y||_{\mathbb{R}}}$ where $||x||_{\mathbb{R}} = \sqrt{\langle x, x\rangle_{\mathbb{R}}}$. Notice, θ so-constructed ranges over $[-\pi/2, \pi/2]$.

For the standard complex vector spaces below and the given vectors calculate the complex and real angle between the vectors:

(a.) $x = \langle 1, 1+i\rangle$ and $y = \langle 1-i, 2\rangle$ in $\mathbb{C}^2$

(b.) $A = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} i & 4i \\ 4i & 4i \end{bmatrix}$ in $\mathbb{C}^{2\times 2}$

Problem 137 Consider the vectors studied in the above problem,

(a.) is $\{x, y\}$ a linearly independent set in $\mathbb{C}^2$ as a complex vector space ?

(b.) is $\{x, y\}$ a linearly independent set in $\mathbb{C}^2$ as a real vector space ?

(c.) is $\{A, B\}$ a linearly independent set in $\mathbb{C}^{2\times 2}$ as a complex vector space ?

(d.) is $\{A, B\}$ a linearly independent set in $\mathbb{C}^{2\times 2}$ as a real vector space ?

Problem 138 Suppose V is a finite dimensional inner product space over[1] $\mathbb{R}$. Let $g : V \times V \to \mathbb{R}$ denote the inner product for V. If $W \leq V$ is a nontrivial subspace then show $g|_W : W \times W \to \mathbb{R}$ defined by $g|_W(x, y) = g(x, y)$ for all $x, y \in W$ is an inner product.

Remark: *in words, the restriction of an inner product is once more an inner product. If we study natural generalizations of inner products then we find this restriction property fails. That is the point of the next problem.*

Problem 139 A **scalar product** or **metric** on $\mathbb{R}^4$ is a symmetric bilinear form which is **nondegenerate**. In particular, $g : V \times V \to \mathbb{R}$ is **nondegenerate** if $g(v, w) = 0$ for all $w \in V$ implies $v = 0$. Nondegeneracy is equivalent to the condition that the matrix of g has nonzero determinant. It can be shown that every inner product is a metric, however, the converse

[1] this problem likely makes sense over $\mathbb{C}$ as well, but, I limit our scope for your convenience.

fails. This problem intends to illustrate some of the differences. The metric given below is the so-called **Minkowski Metric** of time-space. Let $g : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be defined by:

$$g(v, w) = v^T \eta w \qquad \text{where} \qquad \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a.) show g is a symmetric bilinear form on $\mathbb{R}^4$

(b.) why is g is **not** an inner product on $\mathbb{R}^4$?

(c.) let $W = \text{span}\{e_2, e_3, e_4\}$ and show $g|_W$ is an inner product (and hence a metric).

(d.) let $C = \text{span}\{e_2 + e_1\}$ and show $g|_C$ is **not** an metric.

Problem 140 We define $\text{SO}(2, \mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} \mid A^T A = I, \ \ \det(A) = 1\}$.

(a.) show $A \in \text{SO}(2, \mathbb{R})$ has the form $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta \in \mathbb{R}$.

(b.) Given a linear isometry T of $\mathbb{R}^2$ has $\text{trace}(T) = \cancel{8}\sqrt{2}$ and $\det(T) = 1$. By what angle does T rotate?

Problem 141 Suppose $\theta \neq n\pi$ for $n \in \mathbb{Z}$. Consider $R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the complex eigenvalues and real e-vectors of R.

Problem 142 Let $\text{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I\}$. Show that: If $R \in \text{SO}(3)$ and $R \neq I$ then R has only two e-vectors of unit length for which $\lambda = 1$.

Problem 143 Let $R \in \text{SO}(3)$ with $\text{trace}(R) = 0$. By what angle does R rotate?
Hint: consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $[T] = R$ and study the basis where the third vector is the e-vector of unit-length whose existence you proved in the last problem

Problem 144 An n-parallel piped $\mathcal{P}$ with edges $v_1, \dots, v_n$ is the **convex-hull** of $v_1, \dots, v_n$. That is:

$$\mathcal{P} = \{c_1 v_1 + \cdots + c_n v_n \mid 0 \leq c_1, \dots, c_n \leq 1 \ \ \& \ \ c_1 + \cdots + c_n = 1\}.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry of the Euclidean geometry of $\mathbb{R}^n$. Show $T(\mathcal{P})$ is an n-parallel piped with the same n-volume as $\mathcal{P}$.
Reminder: the n-volume of an n-parallel piped with edges $v_1, \dots, v_n$ is given by $|det[v_1|\cdots|v_n]|$.

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Let M be a symmetric matrix and define $\Upsilon(A,B) = AB + BA$ for all $A, B \in \mathbb{R}^{n \times n}$ show Υ is a symmetric, bilinear form.

(II.) Let V be a complex vector space with inner product $\langle , \rangle$. Show $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$ for all $x, y \in V$ and $c \in \mathbb{C}$.

(III.) Let V be a real vector space with inner product $\langle , \rangle$ and let r be a positive constant. Define $g : V \times V \to \mathbb{R}$ by $g(x,y) = r\langle x, y \rangle$ for all $x, y \in V$. Show g defines an inner product on V. Comment on the geometry given by g as it relates to the geometry given by $\langle , \rangle$. In particular, compare and constrast the angles between vectors and the length of vectors as measured by $\langle , \rangle$ vs. g

(IV.) Let $\beta = \{E_{ii} \mid 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} \mid 1 \leq i < j \leq n\}$ form a basis for the symmetric matrices $S_n \leq \mathbb{R}^{n \times n}$. Show that β is an orthogonal basis with respect to the Frobenius inner product.

(V.) Suppose (V, g) forms a geometry and β is a basis for V for which G is the matrix of g. Furthermore, suppose the linear mapping $L : V \to V$ is a g-orthogonal map such that A is its matrix; $[L(x)]_\beta = A[x]_\beta$ or simply $[L]_{\beta,\beta} = A$. Show $A^T G A = G$.

(VI.) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. Find the angle between A and B as measured by the inner product $\langle A, B \rangle = \text{trace}(AB^*)$ where $B^* = \bar{B}^T$.

(VII.) Let $S = \{(1,1,1,1), (0,2,1,0), (1,2,0,1)\}$. Find an orthonormal basis β for $\text{span}(S)$. If $(a,b,c,d) \in \text{span}(S)$ then find the coordinates of (a,b,c,d) with respect to β.

(VIII.) Consider $S = \{x, e^x\}$. Find an orthonormal basis for $W = \text{span}(S)$ where the inner product is given by $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$

(IX.) Formally, we have the identity: for $-1 \leq x \leq 1$

$$\boxed{x^2 = \sum_{n=0}^{\infty} \left(\frac{\langle x^2, \cos(n\pi x) \rangle}{\langle \cos(n\pi x), \cos(n\pi x) \rangle} \right) \cos(n\pi x) + \sum_{n=1}^{\infty} \left(\frac{\langle x^2, \sin(n\pi x) \rangle}{\langle \sin(n\pi x), \sin(n\pi x) \rangle} \right) \sin(n\pi x)} \quad (\star)$$

where $\langle f(x), g(x) \rangle = \int_{-1}^1 f(x)g(x)\,dx$ in this context. I say *formally* since we've left the world of finite linear algebra here. If we truncated these sums at finite n then we'd have trigonmetric approximations of x^2 within a $2n+1$ dimensional subspace of function space. However, we consider the full infinite sum and technically we should justify that the trigonmetric series converges, and, that its limit function does reproduce x^2 on $[-1, 1]$. We leave the formalities to the analysis course. Show (formally) that $(\star)$ yields:

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$$

on $-1 \leq x \leq 1$. You can use a CAS to do the integrals which are needed. Use this result to show $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (the $p = 2$ series converges to this value).

Mission 8 : Solution (Math 321)

$\boxed{\text{P127}}$ Consider $V_1 = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\1\\3\\1\end{bmatrix}$, $V_2 = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\-3\\1\\-1\end{bmatrix}$, $V_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\0\\2\end{bmatrix}$

Notice $V_i \cdot V_j = \delta_{ij}$ for $1 \leq i,j \leq 3$. Thus $\beta = \{V_1, V_2, V_3, V_4\}$ will be orthonormal if we find $V_4 \in \{V_1, V_2, V_3\}^\perp$ and normalize $\|V_4\| = 1$. Thus calculate $V_4 \in \text{Null}\,[V_1|V_2|V_3]^T$,

$$\begin{bmatrix}1 & 1 & 3 & 1\\ 1 & -3 & 1 & -1\\ -1 & -1 & 0 & 2\end{bmatrix}\underbrace{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}}_{V_4'} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{array}{l} V_4' \cdot V_1 = 0\\ V_4' \cdot V_2 = 0 \\ V_4' \cdot V_3 = 0\end{array}$$

Scale factors $\sqrt{12}, \sqrt{12}, \sqrt{6}$ not needed

By a standard calculation, $\text{rref}\begin{bmatrix}1 & 1 & 3 & 1\\ 1 & -3 & 1 & -1\\ -1 & -1 & 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$

thus $a = 2d$, $b = 0$, $c = -d \Rightarrow V_4' = d(2, 0, -1, 1)$

hence $\|V_4\| = 1 \Rightarrow \boxed{V_4 = \frac{1}{\sqrt{6}}(2, 0, -1, 1)}$

"x" or any $x \notin \text{span}\{V_i\}_{i=1}^{3}$ will work here.

Alternatively: run G.S.A. on $\tilde{\beta} = \{V_1, V_2, V_3, e_3\}$

obtain $V_1'' = V_1$, $V_2'' = V_2$, $V_3'' = V_3$ and

$$V_4' = e_3 - (e_3 \cdot V_1)V_1 - (e_3 \cdot V_2)V_2 - (e_3 \cdot V_3)V_3$$
$$= (0,0,1,0) - \tfrac{3}{12}(1,1,3,1) - \tfrac{1}{12}(1,-3,1,-1) + \tfrac{0}{6}(1,1,0,-2)$$
$$= (-1/4 - 1/12,\ -3/12 + 3/12,\ 1 - 9/12 - 1/12,\ -3/12 + 1/12)$$
$$= \tfrac{1}{12}(-4, 0, 2, -2)$$
$$= \tfrac{-1}{6}(2, 0, -1, 1) \Rightarrow \boxed{V_4 = \frac{1}{\sqrt{6}}(2, 0, -1, 1)}$$

Finally, $[(0,0,1,0)]_\beta = (V_1 \cdot e_3,\ V_2 \cdot e_3,\ V_3 \cdot e_3,\ V_4 \cdot e_3)$

$$\boxed{[(0,0,1,0)]_\beta = \left(\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}, 0, \frac{-1}{\sqrt{6}}\right)}$$

$\boxed{P128}$ $\quad W = \text{span}\{\underbrace{\overset{u_1}{(2,2,1,0)}, \overset{u_2}{(1,1,1,0)}}_{\gamma}\}$

(a.) Run G.S.A. on γ to find orthonormal β_W with span $\beta_W = \text{span}\,\gamma = W$,

$$u_1'' = \frac{1}{\|u_1\|} u_1 = \frac{1}{3}(2,2,1,0)$$

$$u_2' = u_2 - (u_2 \cdot u_1'')u_1'' = (1,1,1,0) - \frac{1}{9}(5)(2,2,1,0)$$

$$\Rightarrow u_2' = \left(1 - \frac{10}{9},\ 1 - \frac{10}{9},\ 1 - \frac{5}{9},\ 0\right)$$

$$\Rightarrow u_2' = \frac{1}{9}(-1,-1,4,0)$$

$$\therefore\ u_2'' = \frac{1}{\|u_2'\|} u_2' = \frac{1}{\|\frac{1}{9}(-1,-1,4,0)\|} \frac{1}{9}(-1,-1,4,0)$$

notice the $\frac{1}{9}$ cancels.

$$\Rightarrow u_2'' = \frac{1}{\sqrt{18}}(-1,-1,4,0)$$

$$\therefore\ \beta_W = \left\{ \frac{1}{3}(2,2,1,0),\ \frac{1}{\sqrt{18}}(-1,-1,4,0) \right\}.$$

(b.) $\text{Proj}_W(a,b,c,d) = \frac{1}{9}\left[(2,2,1,0)\cdot(a,b,c,d)\right](2,2,1,0) + \frac{1}{18}\left[(-1,-1,4,0)\cdot(a,b,c,d)\right](-1,-1,4,0)$

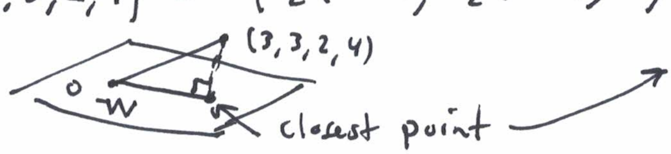
$$= \boxed{\frac{1}{9}(2a+2b+c)(2,2,1,0) + \frac{1}{18}(-a-b+4c)(-1,-1,4,0)}$$

$$= \frac{1}{2}(a+b,\ a+b,\ 2c,\ 0)$$

$$= \boxed{\left(\tfrac{1}{2}(a+b),\ \tfrac{1}{2}(a+b),\ c,\ 0\right)}$$

either of these has its merits.

(c.) $\text{Proj}_W(3,3,2,4) = \left(\frac{1}{2}(3+3),\ \frac{1}{2}(3+3),\ 2,\ 0\right) = \boxed{(3,3,2,0)}$



$\boxed{\text{P129}}$ Let $W = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} = A_2 \leq \mathbb{R}^{2\times 2}$ with $\langle A, B\rangle = tr(AB^T)$

(a.) $$W^\perp = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \bar{c}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = 0 \right\}$$

Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W^\perp$ iff $\underbrace{\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle}_{b - c} = 0$

$b - c = 0$

thus $$W^\perp = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$$
$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$$

Notice $\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle = 0$
and $\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle = 0$ hence $\{E_{11}, E_{12} + E_{21}, E_{22}\} = \gamma$
is already orthogonal. Normalizing γ yields

$$\boxed{\beta = \left\{ E_{11}, \tfrac{1}{\sqrt{2}}(E_{12} + E_{21}), E_{22} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}$$

(orthonormal basis for $W^\perp$)

(b.) $$Proj_{W^\perp}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(Note $\left\{\tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ is basis for $W^\perp$ which is O.N.)

$$= \tfrac{1}{2}(b - c)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= \underline{\begin{bmatrix} 0 & \frac{1}{2}(b-c) \\ -\frac{1}{2}(b-c) & 0 \end{bmatrix}}.$$

aka $\underline{Proj_{W^\perp}(A) = \dfrac{A - A^T}{2}}$.

(c.) $$Proj_W\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - Proj_{W^\perp}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underline{\begin{bmatrix} a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & d \end{bmatrix}}.$$

or, could calculate $\underbrace{Proj_W\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} = \langle A, E_{11}\rangle E_{11} + \frac{1}{2}\langle A, E_{12} + E_{21}\rangle (E_{12} + E_{21}) + \langle A, E_{22}\rangle E_{22}$

and I know from our discussion this semester and previous problems we can also express $\underline{Proj_W(A) = \dfrac{A + A^T}{2}}$

(d.) $$Proj_W\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \tfrac{1}{2}\left(\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\right) = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$ ← closest matrix in W to $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$.

<u>Remark</u>: do you understand <u>why</u> $Proj_W(A)$ is closest matrix in W to A? Do you have a picture in mind to help guide this thought? (I stumbled in Lecture on this pt., but fixed it in Help Session)

$\boxed{P130}$ ~~$\mathcal{P} = \{(x,y,z,w) | \ldots\}$~~ ← also reasonable, but I'm solving

$$\mathcal{P} = \{(w,x,y,z) \mid w+x+y+z=0 \ \& \ w-x-y-z=0\}$$

$$\mathcal{P} = \text{Null}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} = \text{Null}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} w=0 \\ x=-y-z \end{array}$$

Hence $\mathcal{P} = \text{span}\{\underbrace{(0,-1,1,0)}_{u_1}, \underbrace{(0,-1,0,1)}_{u_2}\}$

$$u_1'' = \frac{1}{\sqrt{2}}(0,-1,1,0)$$

$$u_2' = u_2 - (u_2 \cdot u_1'')u_1'' = (0,-1,0,1) - \tfrac{1}{2}(1)(0,-1,1,0)$$

$$\Rightarrow u_2' = (0,-\tfrac{1}{2},-\tfrac{1}{2},1) = \tfrac{1}{2}(0,-1,-1,2) \Rightarrow \underline{u_2'' = \frac{1}{\sqrt{6}}(0,-1,-1,2)}$$

Thus $\{u_1'', u_2''\}$ serve as orthonormal basis for $\mathcal{P}$ and we can calculate $Proj_{\mathcal{P}}(1,1,1,1)$ to find pt. on $\mathcal{P}$ which is closest to $(1,1,1,1)$,

$$\begin{aligned} Proj_{\mathcal{P}}(1,1,1,1) &= [(1,1,1,1)\cdot u_1'']u_1'' + [(1,1,1,1)\cdot u_2'']u_2'' \\ &= \tfrac{1}{2}(0)u_1'' + \tfrac{1}{6}(0)u_2'' \\ &= \boxed{(0,0,0,0)} \end{aligned}$$

Apparently, $(1,1,1,1) \in \mathcal{P}^\perp$ which we could have seen from $(1,1,1,1)\cdot u_1 = 0$ and $(1,1,1,1)\cdot u_2 = 0$ and then concluded that $(1,1,1,1) = \underbrace{(1,1,1,1)}_{\in \mathcal{P}^\perp} + \underbrace{(0,0,0,0)}_{\mathcal{P}}$ to save ourselves trouble of G.S.A. etc.
I was not lazy enough here!

$\boxed{P131}$ Let $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$ for $f,g:\mathbb{R}\to\mathbb{R}$ continuous (say $f,g\in C^0(\mathbb{R})$). Notice the def$^{\underline{n}}$ of inner product in the case of a real vector space requires,

1.) $\langle cf_1+f_2, g\rangle = c\langle f_1,g\rangle + \langle f_2,g\rangle$

2.) $\langle f,g\rangle = \langle g,f\rangle$

3.) $\langle f,f\rangle \geq 0$ and $\langle f,f\rangle = 0 \Leftrightarrow f=0$.

You can prove these are equivalent to our def$^{\underline{n}}$ and I encourage you to use (1.), (2.), (3.) if such a question arose on an exam ~~or~~ final.

For $f,g\in C^0(\mathbb{R})$ both 1.) and 2.) follow quickly from known properties of the definite integral, or function mult.

$$\int f(x)g(x)\,dx = \int g(x)f(x)\,dx \longleftrightarrow \langle f,g\rangle = \langle g,f\rangle$$

$$\underbrace{\int \big(cf_1(x)+f_2(x)\big)g(x)\,dx}_{\langle cf_1+f_2,\,g\rangle} = c\int f_1(x)g(x)\,dx + \int f_2(x)g(x)\,dx$$

$$\langle cf_1+f_2, g\rangle = c\langle f_1,g\rangle + \langle f_2,g\rangle$$

However, at (3.) we find trouble with $C^0(\mathbb{R})$, consider

$y=f(x)$ $\qquad \langle f,f\rangle = \int_0^1 0\,dx = 0$

YET.... $\underline{f\neq 0}$.

In contrast, if $f(x) = a_0+a_1x+\cdots+a_nx^n \in \mathbb{R}[x]$ then $a(x)a(x) = b_0+b_1x+\cdots+b_{2n}x^{2n}$ supposing $a_n\neq 0$ we also have $b_{2n} = (a_n)^2\neq 0$. And, for $f(x)=a(x)$,

$$\langle f,f\rangle = \int_0^1 a(x)a(x)\,dx = \int_0^1 (b_0+b_1x+\cdots+b_{2n}x^{2n})\,dx$$

$$\langle a(x),a(x)\rangle = b_0 + \frac{b_1}{2}+\cdots+\frac{b_{2n}}{2n+1} \quad (*)$$

where $b_0 = a_0^2$ and $b_1 = a_0a_1+a_1a_0$ etc... $b_{2n}=a_n^2$.

we need to argue $(*)$ forces $a_0=0,\dots,a_n=0$ iff $a(x)=0$ and $*\geq 0$.

$\boxed{P131}$ Direct proof that $*$ behaves well is not obvious to me at the moment. I'll use G.S.A. on $\{1, x, x^2, \dots, x^n\}$ to produce orthonormal $\{P_0, P_1, P_2, \dots, P_n\}$ w.r.t. $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$

Then if $a(x) \in \mathbb{R}[x]$ with degree n then

$$a(x) = \sum_{i=0}^{n} \langle a(x), P_i\rangle P_i = \sum_{i=0}^{n} c_i P_i$$

and so,

$$\begin{aligned}\langle a(x), a(x)\rangle &= \Big\langle \sum_{i=0}^{n} c_i P_i, \sum_{j=0}^{n} c_j P_j \Big\rangle \\ &= \sum_{i,j=0}^{n} c_i c_j \langle P_i, P_j\rangle \\ &= \sum_{i,j=0}^{n} c_i c_j \delta_{ij} \\ &= \sum_{i=0}^{n} c_i^2 \\ &= c_0^2 + c_1^2 + \cdots + c_n^2 \geq 0.\end{aligned}$$

Thus $\langle a(x), a(x)\rangle = 0 \iff \underbrace{c_0^2 + c_1^2 + \cdots + c_n^2 = 0}_{\text{only has}}$

only has $c_0 = 0, c_1 = 0, \dots, c_n = 0$ Sol$^{\underline{n}}$ hence $a(x) = 0$.

__Remark__: sorry I said this was "easy". I think I was wrong. Of course, there may be an easier path to this result...

$\boxed{P131}$ working on showing $\langle f(x), g(x)\rangle$ satisfies $\langle f(x), f(x)\rangle \geq 0$ and $\langle f(x), f(x)\rangle = 0 \iff f(x) = 0$ in case $f(x) \in \mathbb{R}[x]$. Since the formula $*$ is a bit formiddeble, seems like induction on $\deg(f(x))$ might be wise here.

<u>Induction Hypothesis</u>: $\langle f,f\rangle \geq 0$ and $\langle f,f\rangle = 0$ iff $f=0$ for all f of $\deg(f) \leq n$.

<u>$n=0$</u> $\langle c,c\rangle = \int_0^1 c^2\,dx = c^2 \geq 0 \quad \forall c \in \mathbb{R}$ and $c^2 = 0 \iff c = 0$ thus I.H. true for $n=0$.

Suppose I.H. true for n. Let $f(x) \in \mathbb{R}[x]$ be of $\deg(f(x)) = n+1$ then $f(x) = a_{n+1}x^{n+1} + \underbrace{a_n x^n + \cdots + a_0}_{g(x)}$ where $a_{n+1} \neq 0$. Consider,

$$\begin{aligned}
\langle f(x), f(x)\rangle &= \langle a_{n+1}x^{n+1} + g(x),\ a_{n+1}x^{n+1} + g(x)\rangle \\
&= a_{n+1}^2 \langle x^{n+1}, x^{n+1}\rangle + 2a_{n+1}\langle x^{n+1}, g(x)\rangle + \langle g(x), g(x)\rangle \\
&= \underbrace{a_{n+1}^2 \int_0^1 x^{2n+2}\,dx}_{\substack{\text{clearly} \\ \text{non-negative}}} + \underbrace{2a_{n+1}\int_0^1 x^{n+1}g(x)\,dx}_{?} + \underbrace{\int_0^1 g(x)g(x)\,dx}_{\substack{\|g(x)\|^2 \geq 0 \\ \text{by induction.}}}
\end{aligned}$$

Well, ...

—(I gave up on this approach in favor of the G.S.A.-based calculation on previous page...
I leave this to make you feel better ☺)—

$\boxed{P132}$ $\{u_1, u_2, u_3\} = \{1, x, x^2\}$. Run GSA. to orthonormalize.

$$\langle u_1, u_1\rangle = \int_0^1 1\,dx = 1 \quad \therefore \quad \underline{\underline{u_1'' = 1}}.$$

$$u_2' = u_2 - \langle u_2, u_1''\rangle u_1'' = x - \left(\int_0^1 x\,dx\right) 1 = x - \tfrac{1}{2} = u_2'$$

$$\|u_2'\|^2 = \int_0^1 (x-\tfrac{1}{2})(x-\tfrac{1}{2})dx = \int_0^1 (x^2 - x + \tfrac{1}{4})dx = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4}$$

Thus $\|u_2'\| = \sqrt{\frac{1}{12}} \Rightarrow \underline{\underline{u_2'' = \sqrt{12}\,(x - \tfrac{1}{2})}}.$

Calculate,

$$\langle u_3, u_1''\rangle = \int_0^1 x^2 dx = \tfrac{1}{3}$$

$$\langle u_3, u_2''\rangle = \int_0^1 x^2 \sqrt{12}\,(x - \tfrac{1}{2})dx = \sqrt{12}\int_0^1 (x^3 - \tfrac{1}{2}x^2)dx$$

$$\Rightarrow \langle u_3, u_2''\rangle = \sqrt{12}\left(\tfrac{1}{4} - \tfrac{1}{6}\right) = \frac{\sqrt{12}}{12}$$

Consequently,

$$\begin{aligned} u_3' &= x^2 - \langle u_3, u_1''\rangle u_1'' - \langle u_3, u_2''\rangle u_2'' \\ &= x^2 - \tfrac{1}{3} - \tfrac{\sqrt{12}}{12}\sqrt{12}\,(x - \tfrac{1}{2}) \\ &= x^2 - \tfrac{1}{3} - x + \tfrac{1}{2} \\ &= x^2 - x + 1/6 \end{aligned}$$

Normalize,

$$\begin{aligned} \langle u_3', u_3'\rangle = \|u_3'\|^2 &= \int_0^1 (x^2 - x + \tfrac{1}{6})(x^2 - x + \tfrac{1}{6})\,dx \\ &= \int_0^1 (x^4 - 2x^3 + \tfrac{1}{3}x^2 + x^2 - \tfrac{1}{3}x + \tfrac{1}{36})dx \\ &= \int_0^1 (x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{1}{3}x + \tfrac{1}{36})dx \\ &= \tfrac{1}{5} - \tfrac{2}{4} + \tfrac{4}{9} - \tfrac{1}{6} + \tfrac{1}{36} \\ &= \tfrac{1}{180} \quad \therefore \quad \frac{1}{\|u_3'\|} = \sqrt{180} \end{aligned}$$

$$\therefore \quad \boxed{\left\{1,\ \sqrt{12}\,(x - \tfrac{1}{2}),\ \sqrt{180}\,(x^2 - x + \tfrac{1}{6})\right\}}$$

$\boxed{P133}$ $V = P_2(\mathbb{R}) \cup \{ce^x \,/\, c \in \mathbb{R}\} \leq C^0(\mathbb{R})$

$V = \text{span}\{1, x, x^2, e^x\}$. Let $W = P_2(\mathbb{R})$

then w.r.t. $\langle f, g \rangle = \int_0^1 fg\,dx$ we found

$\beta = \{1, \sqrt{12}(x - 1/2), \sqrt{180}(x^2 - x + 1/6)\} = \{u_1'', u_2'', u_3''\}$

is orthonormal basis for W w.r.t. $\langle , \rangle$. Thus

$$\text{Proj}_W(e^x) = \langle e^x, 1 \rangle 1 + \langle e^x, \sqrt{12}(x - 1/2) \rangle u_2'' + \langle e^x, u_3'' \rangle u_3''$$

Calculate,

$$\langle e^x, 1 \rangle = \int_0^1 e^x dx = e^x \Big|_0^1 = e - 1 \cong 1.718$$

$$\langle e^x, \sqrt{12}(x - 1/2) \rangle = \int_0^1 \sqrt{12}(xe^x - \tfrac{1}{2}e^x)dx \cong 0.488$$

$$\langle e^x, \sqrt{180}(x^2 - x + \tfrac{1}{6}) \rangle = \int_0^1 \sqrt{180}(x^2 - x + \tfrac{1}{6})e^x dx \cong 0.0625$$

Then

$$\text{Proj}_W(e^x) \cong 1.718 + 0.488\sqrt{12}(x - \tfrac{1}{2}) + 0.0625\sqrt{180}(x^2 - x + \tfrac{1}{6})$$

$$\boxed{\text{Proj}_W(e^x) = 0.8385x^2 + 0.852x + 1.01}$$

vs.

Ex. 9.4.19 based on $\langle f, g \rangle = \int_{-1}^1 fg\,dx$ where

we found $\text{Proj}_{P_2(\mathbb{R})}(e^x) \cong 0.017x^2 + 1.103x + 1.03$

(see end of sol^{n} for graph comparing e^x and these two quadratic approximations.)

$\boxed{P134}$ Prove $|\langle x,y\rangle| \leq \|x\|\cdot\|y\| \ \forall x,y \in V(\mathbb{F})$ where $\|x\| = \sqrt{\langle x,x\rangle}$.

(a.) By construction $0 \leq \|x - cy\|^2$ since $z = x - cy$ has $\|z\|^2 = \langle z,z\rangle = \overline{\langle z,z\rangle} \Rightarrow \langle z,z\rangle \in \mathbb{R}$ hence $\|z\| = \sqrt{\langle z,z\rangle} \geq 0$. Consider then, $\forall x,y \in V$ and $c \in \mathbb{F}$ we calculate:

$$\begin{aligned}
\|x - cy\|^2 &= \langle x - cy, x - cy\rangle \\
&= \langle x, x - cy\rangle - c\langle y, x - cy\rangle \\
&= \langle x,x\rangle - \bar{c}\langle x,y\rangle - c\langle y,x\rangle + c\bar{c}\langle y,y\rangle \\
&= \underline{\|x\|^2 - c\langle y,x\rangle - \bar{c}\langle x,y\rangle + |c|^2\|y\|^2} \ *
\end{aligned}$$

(b.) Setting $c = \dfrac{\langle x,y\rangle}{\langle y,y\rangle}$ makes sense for $y \neq 0$. Substitute into $*$,

$$0 \leq \|x - cy\|^2 = \|x\|^2 - \frac{\langle x,y\rangle}{\langle y,y\rangle}\langle y,x\rangle - \overline{\left(\frac{\langle x,y\rangle}{\langle y,y\rangle}\right)}\langle x,y\rangle + |c|^2\|y\|^2$$

Since $\langle y,y\rangle = \|y\|^2$,

$$0 \leq \|x\|^2 - 2|\langle x,y\rangle|^2\frac{1}{\|y\|^2} + \frac{|\langle x,y\rangle|^2}{(\|y\|^2)^2}\|y\|^2$$

Multiply by $\|y\|^2$,

$$\Rightarrow \ 0 \leq \|x\|^2\|y\|^2 - 2|\langle x,y\rangle|^2 + |\langle x,y\rangle|^2$$

$$\Rightarrow \ |\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2$$

(c.) Thus $|\langle x,y\rangle| \leq \|x\|\,\|y\|$. when $x,y \in V$ and $y \neq 0$.

Incidentally, $y = 0$ gives $\langle x,0\rangle = 0$ and $\|x\|\cdot\|0\| = 0$ so the Cauchy-Schwarz inequality is true $\forall x,y \in V$.

$\boxed{P135}$ Consider $x, y \in V$ with $\langle , \rangle$ and $\|x\| = \sqrt{\langle x, x\rangle}$,

$$
\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2 \\
&= \|x\|^2 + 2\,\mathrm{Re}\,\langle x,y\rangle + \|y\|^2 \quad : \text{see Remark 1} \\
&\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \quad : \text{see Remark 2} \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad : \text{using } \boxed{P134} \\
&= (\|x\| + \|y\|)^2
\end{aligned}
$$

Since $\|x+y\|, \|x\|+\|y\| > 0$ we simply take $\sqrt{\;}$ of inequality above to find the <u>triangle</u> inequality,

$$\|x+y\| \leq \|x\| + \|y\|.$$

<u>Remark 1</u>: If $z = a+ib$ where $a, b \in \mathbb{R}$ then $\bar{z} = a - ib$ thus $z + \bar{z} = (a+ib)+(a-ib) = 2a$. We <u>define</u> $\mathrm{Re}(z) = a$ and observe $\mathrm{Re}(z) = \frac{1}{2}(z+\bar{z})$ apply this to $z = \langle x,y\rangle$ to see $2\,\mathrm{Re}\langle x,y\rangle = \langle x,y\rangle + \overline{\langle x,y\rangle}$.

<u>Remark 2</u>: if $z = a+ib$ then $|z| = \sqrt{a^2+b^2}$ notice $-a \leq |a| \leq \sqrt{a^2} \leq \sqrt{a^2+b^2}$ so in particular as $|a| = \pm a$ we find $a \leq \sqrt{a^2+b^2}$. Applying this to $a = \mathrm{Re}\langle x,y\rangle$ we have $\mathrm{Re}\langle x,y\rangle \leq |\langle x,y\rangle|$.

$\boxed{P136}$ $\cos\tilde{\theta} = \dfrac{|\langle x,y\rangle|}{\|x\|\,\|y\|}$ vs. $\cos\theta = \dfrac{Re\langle x,y\rangle}{\|x\|\cdot\|y\|}$

(a.) $x = (1,\ 1+i)$ & $y = (1-i,\ 2)$

$$\|x\| = \sqrt{1^2 + |1+i|^2} = \sqrt{1+2} = \sqrt{3}$$

$$\|y\| = \sqrt{(1-i)(1+i)+4} = \sqrt{2+4} = \sqrt{6}$$

$$\langle x, y\rangle = 1(1+i) + (1+i)(2) = 3+3i$$

Thus,

$$\cos\tilde{\theta} = \frac{|3+3i|}{\sqrt{3}\,\sqrt{6}} = \frac{\sqrt{18}}{\sqrt{18}} = 1 \Rightarrow \boxed{\tilde{\theta} = 0}$$

$$\cos\theta = \frac{Re\,(3+3i)}{\sqrt{3}\,\sqrt{6}} = \frac{3}{3\sqrt{2}} \Rightarrow \boxed{\theta = \pi/4}$$

(b.) $A = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} i & 4i \\ 4i & 4i \end{bmatrix}$

$$\|A\| = \sqrt{1+16+16+16} = \sqrt{49} = 7.$$

$$\|B\| = \sqrt{1+16+16+16} = 7.$$

$$\langle A, B\rangle = -i - 16i - 16i - 16i = -49i$$

Thus,

$$\cos\tilde{\theta} = \frac{|-49i|}{7\cdot 7} = \frac{49}{49} = 1 \Rightarrow \boxed{\tilde{\theta} = 0}$$

$$\cos\theta = \frac{Re\,(-49i)}{49} = \frac{0}{49} = 0 \Rightarrow \boxed{\theta = \pi/2}$$

<u>Remark</u>: $i = e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$

whereas $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{i\pi/4}.$

$\boxed{P137}$ $x = (1, 1+i)$, $y = (1-i, 2)$ in $\mathbb{C}^2$
and $A = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$, $B = \begin{bmatrix} i & 4i \\ 4i & 4i \end{bmatrix}$.

(a.) $\{x, y\}$ is linearly dependent over $\mathbb{C}$ in $\mathbb{C}^2$ since $(1-i)x = y$.

(b.) $\{x, y\}$ is LI in $\mathbb{C}^2$ over $\mathbb{R}$. Since $x, y \neq 0$ if we had linear dep. then $\exists r \in \mathbb{R}$ s.t. $y = rx$. But, $(1-i, 2) = r(1, 1+i)$ implies $1-i = r$ which $\rightarrow\leftarrow$ $r \in \mathbb{R}$. Hence $\{x, y\}$ is LI over $\mathbb{R}$.

(c.) $\{A, B\}$ is lin dep. over $\mathbb{C}$ since $B = iA$.

(d.) if $B = rA$ for $r \in \mathbb{R}$ then $\begin{bmatrix} i & 4i \\ 4i & 4i \end{bmatrix} = \begin{bmatrix} r & 4r \\ 4r & 4r \end{bmatrix}$ provides $r = i$ $\rightarrow\leftarrow$ $r \in \mathbb{R}$. Thus $\{A, B\}$ is LI over $\mathbb{R}$.

$\boxed{P138}$ Let $g : V \times V \longrightarrow \mathbb{R}$ denote inner-product. Let $W \subseteq V$ be a non-trivial subspace. Let $g|_W : W \times W \to \mathbb{R}$ be defined by $g|_W(x, y) = g(x, y)$ $\forall x, y \in W$. Since $x, y \in W \subseteq V \Rightarrow x, y \in V$ we find all the axioms of inner-product follow to $g|_W$,

$$g|_W(x, y) = g(x, y) = g(y, x) = g|_W(y, x)$$

$$g|_W(x_1 + x_2, y) = g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y)$$
$$= g|_W(x_1, y) + g|_W(x_2, y).$$

and so forth. Similar arguments prove

$$g|_W(x, cy) = c\, g|_W(x, y) \ \& \ g|_W(x, x) \geq 0.$$

$\boxed{P139}$ Let $\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ and $g(v,w) = v^T \eta w$

(a.) To see $g : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is symmetric notice

$$\begin{aligned} g(v,w) = v^T \eta w &= -v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4 \\ &= -w_1 v_1 + w_2 v_2 + w_3 v_3 + w_4 v_4 \\ &= w^T \eta v \\ &= g(w,v) \end{aligned}$$

Then linearity in 1^{st} slot is easily checked,

$$\begin{aligned} g(cv_1 + v_2, w) &= (cv_1 + v_2)^T \eta w \\ &= c v_1^T \eta w + v_2^T \eta w \\ &= c\, g(v_1, w) + g(v_2, w). \end{aligned}$$

By symmetry we obtain linearity in 2^{nd} slot,

$$\begin{aligned} g(w, cv_1 + v_2) &= g(cv_1 + v_2, w) \\ &= c\, g(v_1, w) + g(v_2, w) \\ &= c\, g(w, v_1) + g(w, v_2). \end{aligned}$$

(b.) Notice $(1,1,0,0) = v \neq 0$ yet $g(v,v) = -1+1+0+0 = 0$. (for inner product recall, $\langle x,x\rangle = 0 \Leftrightarrow x = 0$)

There are infinitely many such examples. Any sol^{n} to $v = (a,b,c,d)$ with $g(v,v) = 0 \rightarrow \underbrace{a^2 = b^2 + c^2 + d^2}_{\text{hyper cone}}$.

(c.) $W = \text{span}\{e_2, e_3, e_4\}$ then

$$\begin{aligned} g|_W (x_2 e_2 + x_3 e_3 + x_4 e_4,\ y_2 e_2 + y_3 e_3 + y_4 e_4) &= g((0, x_2, x_3, x_4), (0, y_2, y_3, y_4)) \\ &= x_2 y_2 + x_3 y_3 + x_4 y_4 \end{aligned}$$

In other words, $w \in W$ has $w = (0, \vec{w})$ and $g|_W(v,w) = \vec{v} \cdot \vec{w}$ thus $g|_W$ is clearly an inner-product as we know dot-product forms inner-product.

(d.) $C = \text{span}\{\underbrace{(1,1,0,0)}_{v_0}\} \Rightarrow g|_C(\alpha v_0, \beta v_0) = g(\alpha v_0, \beta v_0) = \alpha\beta\, g(v_0, v_0) = 0.$

Thus $g|_C = 0$ which is <u>not</u> nondegenerate $\therefore g|_C$ not a metric on C.

P140 $SO(2,\mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} \mid A^TA = I \text{ and } \det(A) = 1\}$

(a.) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2,\mathbb{R})$ then $A^TA = I$ implies $A^{-1} = A^T$ thus, using 2×2 inverse f-la,

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad (*)$$

But, $\det(A) = 1 = ad - bc$ hence we find from $(*)$ that $a = d$ and $b = -c$. Moreover,

$$\det(A) = ad - bc = a^2 + b^2 = 1$$

Hence set $a = \cos\theta$ and $b = \sin\theta$ for some $\theta \in \mathbb{R}$ and we find $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ as requested.

(b.) a linear isometry fixes zero and is thus an orthogonal transformation on $\mathbb{R}^2$. Moreover $T(x) = Rx$

$\langle T(x), T(y)\rangle = T(x)\cdot T(y) = (Rx)\cdot(Ry) = x^TR^TRy = x\cdot y$

implies $R^TR = I$ and as $\det T = \det R = 1$ we find $[T] = R \in SO(2,\mathbb{R})$ thus $[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **

and $\text{trace}\,[T] = \text{trace}(T) = \sqrt{2} = 2\cos\theta \quad \therefore \cos\theta = \frac{\sqrt{2}}{2}$

Thus $\boxed{\theta = \pi/4}$

Remark: to grader: if they did not go through ** do not penalize. This was done in class etc.

$\boxed{P141}$ $\theta \neq n\pi$ for $n \in \mathbb{Z}$,

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(R - \lambda I) = \det\begin{bmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = \left[(\lambda - \cos\theta)^2 + \sin^2\theta\right](\lambda - 1)$$

Thus $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$ (Euler's Formula)
or $\lambda = 1$. Then

$$R - I = \begin{bmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathcal{E}_{\lambda=1} = \text{span}\{(0,0,1)\}$. (Can see $Re_3 = e_3$ w/o all my calculation!)

Notice we can be sure our description of $\mathcal{E}_{\lambda=1}$ is complete since the <u>geometric multiplicity</u> is at <u>most</u> the $\underbrace{\text{alg. multiplicity}}_{\text{(which is one for } \lambda = 1)}$

<u>Remark</u>: while I did not ask for it, you should be aware $R(e_1 \pm ie_2) = e^{\pm i\theta}(e_1 \pm ie_2)$ by our discussion of the real Jordan form and the structure of complex e-vectors for real matrices. Here $\underbrace{\text{span}\{e_1, e_2\}}_{\text{plane of rotation.}} = \{\underbrace{e_3}_{\text{axis of rotation.}}\}^\perp$

$\boxed{P142}$ $SO(3,\mathbb{R}) = \{R \in \mathbb{R}^{3\times 3} \,/\, R^TR = I \;\&\; \det R = 1\}$

Suppose $R \neq I$ and $R \in SO(3,\mathbb{R})$. Then notice

$R^TR = I \Rightarrow R^{-1} = R^T$. Furthermore, recall

$Av = \lambda v \Rightarrow A^{-1}Av = \lambda A^{-1}v \Rightarrow \underline{A^{-1}v = \frac{1}{\lambda}v}$.

Likewise $\det(A - \lambda I) = \det(A^T - \lambda I)$ thus

both A and A^T share same e-values while

A and A^{-1} have reciprocal e-values. Then

$R^{-1} = R^T$ implies the e-values $\lambda_1, \lambda_2, \lambda_3$ of R

have $\underline{\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right\} = \{\lambda_1, \lambda_2, \lambda_3\}}$ $*$

where not all of $\lambda_1, \lambda_2, \lambda_3$ are one since $R \neq I$.

We also know $\det(R) = \lambda_1\lambda_2\lambda_3 = 1$. Consider $*$

we do <u>not</u> know the order matches,

If two e-values were one, wlog say $\lambda_1 = \lambda_2 = 1$

then $*$ provides $\frac{1}{\lambda_3} = \lambda_3 \Rightarrow \lambda_3^2 = +1 \therefore \lambda_3 = \pm 1$

since complex sol$^{\underline{n}}$ is not possible as $\lambda_1, \lambda_2 \in \mathbb{R}$ already assumed.

Thus either $\lambda_3 = 1$ (which means $R = I$) or $\lambda_3 = -1$

which is impossible as $\lambda_1\lambda_2\lambda_3 = \det(R) = 1 \neq (1)(1)(-1)$.

Thus at most <u>one</u> e-value, say $\lambda_1 = 1$, can be one.

Hence algebraic mult. of $\lambda_1 = 1$ is one $\Rightarrow$ geom. mult.

is just one. Hence $E_{\lambda=1} = \text{span}\{\pm u\}$ where $||u|| = 1$.

P142 continued:

There is a gap in the previous page.

Q: how do we know $\exists \lambda_1$ with $\lambda_1 = 1$?

I've argued why there can be at most one e-value $\lambda = 1$. But, why is there even one?

Again, we know some things,

1.) $\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \} = \{ \lambda_1, \lambda_2, \lambda_3 \}$

2.) $\lambda_1, \lambda_2, \lambda_3$ are zeros of $\underbrace{\det(A - \lambda I) = 0}_{\text{real cubic eq}^{\underline{n}}}$

3.) either all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$
or wlog $\lambda_1 \in \mathbb{R}$ and $\lambda_2 = \overline{\lambda_3}$

4.) $\lambda_1 \lambda_2 \lambda_3 = 1$.

I'll attempt a proof that at least one of $\lambda_1, \lambda_2, \lambda_3$ must be one.

- If $\frac{1}{\lambda_1} = \lambda_1$ then $\lambda_1^2 = 1 \Rightarrow \lambda_1 = 1$ or $\lambda_1 = -1$
  if $\lambda_1 = -1$ then $\lambda_2 \lambda_3 = -1$ and $\{ \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \} = \{ \lambda_2, \lambda_3 \}$
  thus $\frac{1}{\lambda_2} = \lambda_2$ or λ_3 and $\frac{1}{\lambda_2} = \lambda_2 \Rightarrow \lambda_2^2 = 1 \Rightarrow \lambda_2 = \pm 1$
  so if $\lambda_2 = -1$ then $\lambda_3 = 1$ as $(-1)(-1)\lambda_3 = 1$. If $\lambda_2 = 1$ then great.
- If $\frac{1}{\lambda_1} = \lambda_2$ (wlog) then $\frac{1}{\lambda_2} = \lambda_1$ and so $\lambda_1 \lambda_2 \lambda_3 = \underline{\lambda_3 = 1}$.

Anyway you slice it, we're stuck with one as a sol$^{\underline{n}}$.

[illegible]

$\boxed{P143}$ Let $R \in SO(3,\mathbb{R})$ and $\text{trace}(R) = 0$

Let $\beta = \{v_1, v_2, v_3\}$ where $Rv_3 = v_3$ and $\text{span}\{v_1, v_2\} = \{v_3\}^\perp$. We know $v_3 \neq 0$ with $\|v_3\| = 1$ exists by P142 and since $\{v_3\}^\perp \oplus \text{span}\{v_3\} = \mathbb{R}^3$ we know $\{v_3\}^\perp$ is two-dim'l. Furthermore, we may choose v_1, v_2 orthonormal so β is orthonormal.

Notice $\{v_3\}^\perp$ and $\text{span}\{v_3\}$ are T-invariant spaces. If $av_1 + bv_2 \in \{v_3\}^\perp$ then

$$\begin{aligned} T(av_1 + bv_2) \cdot v_3 &= T(av_1 + bv_2) \cdot T(v_3) \\ &= (av_1 + bv_2) \cdot v_3 \\ &= 0 \end{aligned}$$

Thus $T(av_1 + bv_2) \in \{v_3\}^\perp$ and certainly $\text{span}\{v_3\}$ is T-invariant. Consequently, setting $\{v_3\}^\perp = P$

$$[T]_{\beta\beta} = \left[\begin{array}{c|c} [T_P]_{\beta_P\beta_P} & 0 \\ \hline 0 & [T_{\{v_3\}}]_{\{v_3\},\{v_3\}} \end{array}\right] \qquad \beta_P = \{v_1, v_2\}$$

$$= \left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

But, as $[T]_{\beta\beta}^T = [T]_{\beta\beta}^{-1}$ and $\det [T]_{\beta\beta} = 1$ we deduce hence $\text{trace}(R) = 2\cos\theta + 1 = 0 \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{-1}{2}\right) = 2\pi/3}$

$\boxed{P144}$ $\mathcal{P} = \{ c_1 v_1 + \cdots + c_n v_n \mid 0 \leq c_1, \dots, c_n \leq 1 \text{ \& } \underbrace{\cancel{c_1 + \cdots + c_n = 1}}_{\text{don't want this!}} \}$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an LINEAR isometry of Euclidean Geometry (SHOULD HAVE ADDED!)

then $\underbrace{T(x) = Rx + P}_{\text{rigid motion is composite of rotation/reflection \& translation.}}$ where $R \in O(n, \mathbb{R})$ and $P \in \mathbb{R}^n$

Observe, T is a bijection and,

$$\begin{aligned} T(c_1 v_1 + \cdots + c_n v_n) &= \cancel{c_1 T(v_1) + \cdots + c_n T(v_n)} \quad \text{FALSE!} \\ &= R(c_1 v_1 + \cdots + c_n v_n) + P \\ &= c_1 R v_1 + \cdots + c_n R v_n + P \end{aligned}$$

$$\begin{aligned} \text{Thus } T(\mathcal{P}) &= \{ P + c_1 R v_1 + \cdots + c_n R v_n \mid 0 \leq c_1, \dots, c_n \leq 1 \} \\ &= P + \underbrace{R\mathcal{P}}_{\substack{\text{is parallel n-piped with} \\ \text{edges } Rv_1, Rv_2, \dots, Rv_n}} \end{aligned}$$

Sorry folks I should have forced $P = 0$ for sake of this semester's discussion. Then,

$$\begin{aligned} \det(Rv_1 | Rv_2 | \cdots | Rv_n) &= \det(R[v_1 | \cdots | v_n]) \\ &= \det(R) \det(v_1 | \cdots | v_n) \end{aligned}$$

Then $\underbrace{|\det[Rv_1 | \cdots | Rv_n]|}_{\text{Vol}(R\mathcal{P})} = |\underbrace{\det(R)}_{\pm 1}| \; \underbrace{|\det[v_1 | \cdots | v_n]|}_{\text{Vol}(\mathcal{P})}$

Remark: grader, be very generous here ☺.