Same rules as Homework 1. However, do keep in mind you are free to use technology to calculate row-reductions. There are many online resources to help you check your work. It would be wise to make use of them (Gram Schmidt has a lot of arithmetic, it's easy to make mistakes).

Problem 141 Your signature below indicates you have:

(a.) I read Section 26 of Curtis: ____________________.

(b.) I am reading Chapters 9 and 10 of Cook's Lecture Notes: ____________________.

Problem 142 Let $S = \{(1, i+2, 1), (i+1, 0, 0)\}$ be a subset of $\mathbb{C}^3$. If $W = \text{span}_{\mathbb{C}}(S)$ then find an orthonormal basis β for W.

Problem 143 Extend the basis β to γ a basis for $\mathbb{C}^3$. Find the formula for $\text{Proj}_{W^\perp}(a, b, c)$.

Problem 144 Prove that linearity $\langle cx + y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$ and the reality condition $\overline{\langle x, y\rangle} = \langle y, x\rangle$ imply the conjugate homogeneity property $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$. The identities just stated are to hold for all $x, y, z \in V$ where V is a complex inner product space. Furthermore, if $c = a + ib$ for $a, b \in \mathbb{R}$ then $\bar{c} = a - ib$.

Problem 145 Find eigenvalues and orthonormal eigenvectors for $Q(x, y) = x^2 + 4xy$. Change the formula for Q to eigencoordinates (I used $\bar{x}, \bar{y}$ for this concept in lecture). Geometrically, what is $x^2 + 4xy = 1$?

Problem 146 Write the formula for $Q(x, y, z) = 2x^2 + 4y^2 + 6z^2 + 8xy + 10xz + 12yz$ in eigencoordinates $\bar{x}, \bar{y}, \bar{z}$ to two decimal places. I want you to use technology and the theorem we proved in lecture about the diagonalization of the form. I do not want you to explicitly find the coordinate formulas relating x, y, z and $\bar{x}, \bar{y}, \bar{z}$.

Problem 147 Suppose $Q(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$. Write $Q(v) = v^T Av$ for a symmetric matrix A. Find an orthonormal eigenbasis for A and find coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$.
Hint: for this question to make sense, it must be that the matrix of Q has e-values $1, 1, 10$.

Problem 148 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3\times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 uu^T, \qquad E_2 = \lambda_2 vv^T, \qquad E_3 = \lambda_3 ww^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ and the fact you are given $Au = \lambda_1 u$ etc.

Problem 149 Notice $u = \frac{1}{\sqrt{3}}(1,-1,1)$ and $v = \frac{1}{\sqrt{2}}(0,1,1)$ and $w = \frac{1}{\sqrt{6}}(2,1,-1)$ form an orthonormal basis for $\mathbb{R}^3$. Find a matrix A with eigenvalues $12, 2, 18$ by making use of the construction of the last problem.

Problem 150 Let V be a vector space and $M, N \leq V$ and $x, y \in V$. Prove:

$$x + M \subseteq y + N \qquad \text{if and only if} \qquad M \subseteq N \;\text{ and }\; x - y \in N.$$

Problem 151 Define $\Psi : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $\Psi(X) = X - X^T$ for each $X \in \mathbb{F}^{n\times n}$. Show that

$$\mathbb{F}^{n\times n}/\text{Ker}(\Psi) \approx \mathcal{A}_n$$

where $\mathcal{A}_n = \{X \in \mathbb{F}^{n\times n} \mid X^T = -X\}$ are the antisymmetric $n \times n$ matrices over $\mathbb{F}$. Explain (via a theorem in Chapter 10 of my notes) why it follows that $\mathbb{F}^{n\times n} = \mathcal{S}_n \oplus \mathcal{A}_n$ where $\mathcal{S}_n$ denotes the symmetric $n \times n$ matrices over $\mathbb{F}$.
hint: use the first isomorphism theorem

Problem 152 Let $V = \text{span}_{\mathbb{R}}\{e^x, e^{2x}, \cos(x), \sin(x)\}$ and consider $T = D + 1$ where $D = d/dx : V \to V$. Show that $U = \text{span}_{\mathbb{R}}\{e^x, e^{2x}\}$ forms an invariant subspace of V with respect to T and find the matrix of $T|_U$ as well as $T_{V/U}$ using the language of §26 of Curtis (page 231-233 especially)

Problem 153 Let $W \leq V$ where V is a finite dimensional vector space over a field $\mathbb{F}$. Also, define $\text{ann}(W) = \{\alpha \in V^* \mid \forall\, x \in W,\;\; \alpha(x) = 0\}$. Prove $\dim(W) + \dim(\text{ann}(W)) = \dim(V)$.

Problem 154 Let $U \leq W \leq V$ where V is a vector space over $\mathbb{F}$ and define
$\text{ann}(U) = \{\alpha \in V^* \mid \forall\, x \in U,\;\; \alpha(x) = 0\}$ and $\text{ann}(W) = \{\alpha \in V^* \mid \forall\, x \in W,\;\; \alpha(x) = 0\}$. Show $\text{ann}(W) \leq \text{ann}(U)$.

Problem 155 Suppose $U \leq W \leq V$ where V is a real inner product space. Show $W^\perp \leq U^\perp$.

I should warn, if we drop the positive definite condition and merely consider nondegenerate scalar products then the theory gets considerable more complicated. See the texts by Steve Roman (many pages in Advanced Linear Algebra) or Serge Lang (see Chapter VII §4 of Linear Algebra).

Problem 156 Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ with bases β and γ respective. Also, define dual spaces $V^* = \mathcal{L}(V, \mathbb{R})$ and $W^* = \mathcal{L}(W, \mathbb{R})$. If $T : V \to W$ is a linear transformation and $S : W^* \to V^*$ is defined by

$$(S(\alpha))(v) = \alpha(T(v))$$

for all $\alpha \in W^*$ and $v \in V$. **Then show S is a linear transformation and find $[S]_{\gamma^*,\beta^*}$.** Here, we define dual bases β^* and γ^* as follows: if $\beta = \{f_1, \dots, f_n\}$ and $\gamma = \{g_1, \dots, g_m\}$ then $f^j : V \to \mathbb{R}$ and $g^j : W \to \mathbb{R}$ are defined by linearly extending the formulas below:

$$f^j(f_i) = \delta_{ij} \qquad \& \qquad g^j(g_i) = \delta_{ij}.$$

Note, we set-aside the usual notation for exponents in this context; c^i is not the number c raised to the i-th power. A useful lemma is given by the following observation, if $x = \sum_{i=1}^n c^i f_i$ then $f^i(x) = c^i$. In other words, the dual vector f^i gives the i-coordinate of x upon evaluation. (your answer should relate the matrix for S to the matrix $[T]_{\beta,\gamma}$)

Problem 157 Consider S and T as in the previous problem once more. Show:

(a.) if T is surjective then S is injective

(b.) if S is injective then T is surjective

(c.) T is an isomorphism iff S is a isomorphism

Problem 158 Note that that trace : $\mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear function hence trace $\in (\mathbb{R}^{n\times n})^*$. Recall $\langle A, B\rangle = \text{trace}(AB^T)$ defines an inner product on $\mathbb{R}^{n\times n}$. Find the Riesz vector for the trace functional.

Problem 159 Let V be a complex inner product space and suppose $T : V \to V$ is a skew-hermitian map in the sense $T^\dagger = -T$. We define $T^\dagger$ to be the endomorphism implicitly given by the condition $\langle T(x), y\rangle = \langle x, T^\dagger(y)\rangle$ for all $x, y \in V$. Given this data about T, prove the following:

(a.) if T has eigenvalue λ then $\lambda = i\alpha$ for some $\alpha \in \mathbb{R}$ (that is to say, the eigenvalues of T are pure-imaginary)

(b.) if $W_i = \text{Ker}(T - \lambda_i)$ and $W_j = \text{Ker}(T - \lambda_j)$ where $\lambda_i \neq \lambda_j$ are distinct e-values of T then $W_i \perp W_j$.

Problem 160 A matrix $A \in \mathbb{R}^{n\times n}$ is called **normal** if $A^T A = AA^T$.

(a.) show a symmetric matrix is normal,

(b.) find an example of a 2×2 matrix which is normal, but, not symmetric,

(c.) show if $A \in \mathbb{R}^{n\times n}$ is normal then $||Ax|| = ||A^T x||$ for all $x \in \mathbb{R}^n$,

(d.) show if $A \in \mathbb{R}^{n\times n}$ is normal then $A - cI$ is normal for all $c \in \mathbb{R}$,

(e.) show if $\lambda \in \mathbb{R}$ is e-value of normal matrix $A \in \mathbb{R}^{n\times n}$ then λ is also an e-value of A^T

(f.) show if λ_1, λ_2 are distinct real e-valued of a normal matrix $A \in \mathbb{R}^{n\times n}$ then the corresponding e-vectors are orthogonal.