Same rules as Homework 1. However, do keep in mind you are free to use technology to calculate row-reductions. There are many online resources to help you check your work. It would be wise to make use of them (Gram Schmidt has a lot of arithmetic, it's easy to make mistakes).

Problem 141 Your signature below indicates you have:

(a.) I read Section 26 of Curtis: ____________________.

(b.) I am reading Chapters 9 and 10 of Cook's Lecture Notes: ____________________.

Problem 142 Let $S = \{(1, i+2, 1), (i+1, 0, 0)\}$ be a subset of $\mathbb{C}^3$. If $W = \text{span}_{\mathbb{C}}(S)$ then find an orthonormal basis β for W.

Problem 143 Extend the basis β to γ a basis for $\mathbb{C}^3$. Find the formula for $\text{Proj}_{W^\perp}(a, b, c)$.

Problem 144 Prove that linearity $\langle cx + y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$ and the reality condition $\overline{\langle x, y\rangle} = \langle y, x\rangle$ imply the conjugate homogeneity property $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$. The identities just stated are to hold for all $x, y, z \in V$ where V is a complex inner product space. Furthermore, if $c = a + ib$ for $a, b \in \mathbb{R}$ then $\bar{c} = a - ib$.

Problem 145 Find eigenvalues and orthonormal eigenvectors for $Q(x, y) = x^2 + 4xy$. Change the formula for Q to eigencoordinates (I used $\bar{x}, \bar{y}$ for this concept in lecture). Geometrically, what is $x^2 + 4xy = 1$?

Problem 146 Write the formula for $Q(x, y, z) = 2x^2 + 4y^2 + 6z^2 + 8xy + 10xz + 12yz$ in eigencoordinates $\bar{x}, \bar{y}, \bar{z}$ to two decimal places. I want you to use technology and the theorem we proved in lecture about the diagonalization of the form. I do not want you to explicitly find the coordinate formulas relating x, y, z and $\bar{x}, \bar{y}, \bar{z}$.

Problem 147 Suppose $Q(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$. Write $Q(v) = v^T Av$ for a symmetric matrix A. Find an orthonormal eigenbasis for A and find coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$.
Hint: for this question to make sense, it must be that the matrix of Q has e-values $1, 1, 10$.

Problem 148 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3\times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 uu^T, \qquad E_2 = \lambda_2 vv^T, \qquad E_3 = \lambda_3 ww^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ and the fact you are given $Au = \lambda_1 u$ etc.

Problem 149 Notice $u = \frac{1}{\sqrt{3}}(1,-1,1)$ and $v = \frac{1}{\sqrt{2}}(0,1,1)$ and $w = \frac{1}{\sqrt{6}}(2,1,-1)$ form an orthonormal basis for $\mathbb{R}^3$. Find a matrix A with eigenvalues $12, 2, 18$ by making use of the construction of the last problem.

Problem 150 Let V be a vector space and $M, N \leq V$ and $x, y \in V$. Prove:

$$x + M \subseteq y + N \qquad \text{if and only if} \qquad M \subseteq N \ \text{ and } \ x - y \in N.$$

Problem 151 Define $\Psi : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ by $\Psi(X) = X - X^T$ for each $X \in \mathbb{F}^{n\times n}$. Show that

$$\mathbb{F}^{n\times n}/\text{Ker}(\Psi) \approx \mathcal{A}_n$$

where $\mathcal{A}_n = \{X \in \mathbb{F}^{n\times n} \mid X^T = -X\}$ are the antisymmetric $n \times n$ matrices over $\mathbb{F}$. Explain (via a theorem in Chapter 10 of my notes) why it follows that $\mathbb{F}^{n\times n} = \mathcal{S}_n \oplus \mathcal{A}_n$ where $\mathcal{S}_n$ denotes the symmetric $n \times n$ matrices over $\mathbb{F}$.
hint: use the first isomorphism theorem

Problem 152 Let $V = \text{span}_{\mathbb{R}}\{e^x, e^{2x}, \cos(x), \sin(x)\}$ and consider $T = D + 1$ where $D = d/dx : V \to V$. Show that $U = \text{span}_{\mathbb{R}}\{e^x, e^{2x}\}$ forms an invariant subspace of V with respect to T and find the matrix of $T|_U$ as well as $T_{V/U}$ using the language of §26 of Curtis (page 231-233 especially)

Problem 153 Let $W \leq V$ where V is a finite dimensional vector space over a field $\mathbb{F}$. Also, define $\text{ann}(W) = \{\alpha \in V^* \mid \forall\, x \in W, \ \ \alpha(x) = 0\}$. Prove $\dim(W) + \dim(\text{ann}(W)) = \dim(V)$.

Problem 154 Let $U \leq W \leq V$ where V is a vector space over $\mathbb{F}$ and define $\text{ann}(U) = \{\alpha \in V^* \mid \forall\, x \in U, \ \ \alpha(x) = 0\}$ and $\text{ann}(W) = \{\alpha \in V^* \mid \forall\, x \in W, \ \ \alpha(x) = 0\}$. Show $\text{ann}(W) \leq \text{ann}(U)$.

Problem 155 Suppose $U \leq W \leq V$ where V is a real inner product space. Show $W^\perp \leq U^\perp$.

I should warn, if we drop the positive definite condition and merely consider nondegenerate scalar products then the theory gets considerable more complicated. See the texts by Steve Roman (many pages in Advanced Linear Algebra) or Serge Lang (see Chapter VII §4 of Linear Algebra).

Problem 156 Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ with bases β and γ respective. Also, define dual spaces $V^* = \mathcal{L}(V, \mathbb{R})$ and $W^* = \mathcal{L}(W, \mathbb{R})$. If $T : V \to W$ is a linear transformation and $S : W^* \to V^*$ is defined by

$$(S(\alpha))(v) = \alpha(T(v))$$

for all $\alpha \in W^*$ and $v \in V$. **Then show S is a linear transformation and find $[S]_{\gamma^*,\beta^*}$.** Here, we define dual bases β^* and γ^* as follows: if $\beta = \{f_1, \dots, f_n\}$ and $\gamma = \{g_1, \dots, g_m\}$ then $f^j : V \to \mathbb{R}$ and $g^j : W \to \mathbb{R}$ are defined by linearly extending the formulas below:

$$f^j(f_i) = \delta_{ij} \qquad \& \qquad g^j(g_i) = \delta_{ij}.$$

Note, we set-aside the usual notation for exponents in this context; c^i is not the number c raised to the i-th power. A useful lemma is given by the following observation, if $x = \sum_{i=1}^{n} c^i f_i$ then $f^i(x) = c^i$. In other words, the dual vector f^i gives the i-coordinate of x upon evaluation. (your answer should relate the matrix for S to the matrix $[T]_{\beta,\gamma}$)

Problem 157 Consider S and T as in the previous problem once more. Show:

(a.) if T is surjective then S is injective

(b.) if S is injective then T is surjective

(c.) T is an isomorphism iff S is a isomorphism

Problem 158 Note that that trace : $\mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear function hence trace $\in (\mathbb{R}^{n\times n})^*$. Recall $\langle A, B\rangle = \text{trace}(AB^T)$ defines an inner product on $\mathbb{R}^{n\times n}$. Find the Riesz vector for the trace functional.

Problem 159 Let V be a complex inner product space and suppose $T : V \to V$ is a skew-hermitian map in the sense $T^\dagger = -T$. We define $T^\dagger$ to be the endomorphism implicitly given by the condition $\langle T(x), y\rangle = \langle x, T^\dagger(y)\rangle$ for all $x, y \in V$. Given this data about T, prove the following:

(a.) if T has eigenvalue λ then $\lambda = i\alpha$ for some $\alpha \in \mathbb{R}$ (that is to say, the eigenvalues of T are pure-imaginary)

(b.) if $W_i = \text{Ker}(T - \lambda_i)$ and $W_j = \text{Ker}(T - \lambda_j)$ where $\lambda_i \neq \lambda_j$ are distinct e-values of T then $W_i \perp W_j$.

Problem 160 A matrix $A \in \mathbb{R}^{n\times n}$ is called **normal** if $A^TA = AA^T$.

(a.) show a symmetric matrix is normal,

(b.) find an example of a 2×2 matrix which is normal, but, not symmetric,

(c.) show if $A \in \mathbb{R}^{n\times n}$ is normal then $||Ax|| = ||A^Tx||$ for all $x \in \mathbb{R}^n$,

(d.) show if $A \in \mathbb{R}^{n\times n}$ is normal then $A - cI$ is normal for all $c \in \mathbb{R}$,

(e.) show if $\lambda \in \mathbb{R}$ is e-value of normal matrix $A \in \mathbb{R}^{n\times n}$ then λ is also an e-value of A^T

(f.) show if λ_1, λ_2 are distinct real e-valued of a normal matrix $A \in \mathbb{R}^{n\times n}$ then the corresponding e-vectors are orthogonal.

Mission 9 Solution

P142 $S = \{ \underbrace{(1,\ i+2,\ 1)}_{V_1},\ \underbrace{(i+1,\ 0,0)}_{V_2} \}$

Set $u_1 = \frac{1}{\sqrt{2}}(i+1,\ 0,0)$ then

$$\tilde{u}_2 = V_1 - \langle V_1, u_1 \rangle u_1 = (1,\ i+2,\ 1) - \langle (1, i+2, 1), (\tfrac{i+1}{\sqrt{2}}, 0, 0) \rangle u_1$$

$$= (1,\ i+2, 1) - \frac{1-i}{\sqrt{2}} \left(\frac{1+i}{\sqrt{2}},\ 0, 0 \right)$$

$$= (1,\ i+2,\ 1) - (1,0,0)$$

$$= (0,\ i+2,\ 1)$$

$$\therefore\ u_2 = \frac{1}{\sqrt{6}}(0,\ i+2,\ 1)$$

Hence $\boxed{\beta = \{ \frac{1}{\sqrt{2}}(i+1, 0,0),\ \frac{1}{\sqrt{6}}(0,\ i+2,\ 1) \}}$ is orthonormal basis for $W = \text{span}\, S$

P143 We seek $z \in \mathbb{C}^3$ for which $\underline{\langle z, u_1 \rangle = 0}\ ^*$ and $\underline{\langle z, u_2 \rangle = 0}\ ^{**}$. If $z = (z_1, z_2, z_3)$ then we find from $*$ that $z_1 = 0$, whereas $**$ yields

$$\left\langle (z_1, z_2, z_3), \left(0, \frac{i+2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right\rangle = z_2 \left(\frac{2-i}{\sqrt{6}}\right) + z_3 \frac{1}{\sqrt{6}} = 0$$

Hence $z_2(2-i) + z_3 = 0$, set $z_2 = 1$ then $z_3 = i-2$

and $z = (0,\ 1,\ i-2) \longleftrightarrow \boxed{u_3 = \frac{1}{\sqrt{6}}(0,\ 1,\ i-2)}$

It follows $\boxed{\gamma = \{u_1, u_2, u_3\} = \{ (\frac{i+1}{\sqrt{2}}, 0, 0), (0, \frac{i+2}{\sqrt{6}}, \frac{1}{\sqrt{6}}), (0, \frac{1}{\sqrt{6}}, \frac{i-2}{\sqrt{6}}) \} \text{ is orthonormal basis for } \mathbb{C}^3}$

$$\text{Proj}_{W^\perp}(a,b,c) = \langle (a,b,c), u_3 \rangle u_3 = \frac{1}{6}\left(b - c(2+i)\right)(0,\ 1,\ i-2)$$

$$\boxed{\text{Proj}_{W^\perp}(a,b,c) = \left(0,\ \tfrac{1}{6}[b - c(2+i)],\ \tfrac{1}{6}[b(i-2) + 5c]\right)}$$

$\boxed{P144}$ Let V be complex inner product space, $x, y, z \in V$, $c \in \mathbb{C}$,

Suppose $\underbrace{\langle cx+y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle}_{\forall x,y,z \in V}$ and $\underbrace{\overline{\langle x,y\rangle} = \langle y, x\rangle}_{\forall x,y \in V}$

Consider,

$$\overline{\langle cx+y, z\rangle} = \overline{c\langle x, z\rangle + \langle y, z\rangle}$$

$$\Rightarrow \langle z, cx+y\rangle = \bar{c}\,\overline{\langle x, z\rangle} + \overline{\langle y, z\rangle} = \bar{c}\langle z, x\rangle + \langle z, y\rangle$$

Thus, $\langle z, cx\rangle = \bar{c}\langle z, x\rangle \quad \forall z, x \in V$

or, $\underline{\langle x, cy\rangle = \bar{c}\langle x, y\rangle \quad \forall x, y \in V}$.

Remark: I probably forgot to emphasize, a complex inner product space has a almost linear conjugation

$$\overline{z+w} \longmapsto \bar{z} + \bar{w} \quad \text{(additive)}$$

$$\overline{cz} \longmapsto \bar{c}\,\bar{z} \quad \text{(conjugate homogeneous)}$$

$\boxed{P145}$ $Q(x,y) = x^2 + 4xy$ explicitly diagonalize into eigen coordinates and comment on geometry of $x^2 + 4xy = 1$

$$\det\begin{bmatrix} 1-\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda(\lambda - 1) - 4 = \lambda^2 - \lambda - 4$$

$$= \left(\lambda - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - \tfrac{16}{4}$$

$$= \left(\lambda - \tfrac{1}{2}\right)^2 - \left(\tfrac{\sqrt{17}}{2}\right)^2$$

$$= \left(\lambda - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{17}\right)\left(\lambda - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{17}\right)$$

$$\therefore \quad \underline{\lambda_1 = \frac{1+\sqrt{17}}{2}} \quad \text{and} \quad \underline{\lambda_2 = \frac{1-\sqrt{17}}{2}}.$$

continued

$\boxed{P145}$ Continued

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \left(\frac{1+\sqrt{17}}{2}\right)\begin{bmatrix} u \\ v \end{bmatrix}$$

$$2u = \left(\frac{1+\sqrt{17}}{2}\right)v \quad \therefore \quad v = \frac{4}{1+\sqrt{17}}u$$

Choose $u = 1+\sqrt{17}$ then $v = 4$

and $v_1 = (1+\sqrt{17}, 4)$, $\|v_1\| = \sqrt{(1+\sqrt{17})^2+16}$

$\|v_1\| = \sqrt{18+2\sqrt{17}+16}$

$\|v_1\| = \underbrace{\sqrt{34+\sqrt{68}}}_{c_1}$

$$\underline{u_1 = \frac{1}{\sqrt{34+\sqrt{68}}}(1+\sqrt{17}, 4)}.$$

Likewise,

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \left(\frac{1-\sqrt{17}}{2}\right)\begin{bmatrix} u \\ v \end{bmatrix}$$

$$2u = \left(\frac{1-\sqrt{17}}{2}\right)v \longrightarrow v = \frac{4}{1-\sqrt{17}}u, \text{ set } u = 1-\sqrt{17}$$

$\therefore v_2 = (1-\sqrt{17}, 4)$, $\|v_2\| = \sqrt{(1-\sqrt{17})^2+16}$

$= \sqrt{1+17-2\sqrt{17}+16}$

$= \underbrace{\sqrt{34-\sqrt{68}}}_{c_2}$

$$\underline{u_2 = \frac{1}{\sqrt{34-\sqrt{68}}}(1-\sqrt{17}, 4)}.$$

$\beta = \{u_1, u_2\}$ and $\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = [\beta]^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = [\beta]^T\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{c_1}[1+\sqrt{17}, 4] \\ \frac{1}{c_2}[1-\sqrt{17}, 4] \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{c_1}(1+\sqrt{17})x + \frac{4}{c_1}y \\ \frac{1}{c_2}(1-\sqrt{17})x + \frac{4}{c_2}y \end{bmatrix}$$

$$\therefore \boxed{\bar{x} = \frac{1}{\sqrt{34+\sqrt{68}}}\left((1+\sqrt{17})x + 4y\right)} \;\&\; \boxed{\bar{y} = \frac{(1-\sqrt{17})x + 4y}{\sqrt{34-\sqrt{68}}}}$$

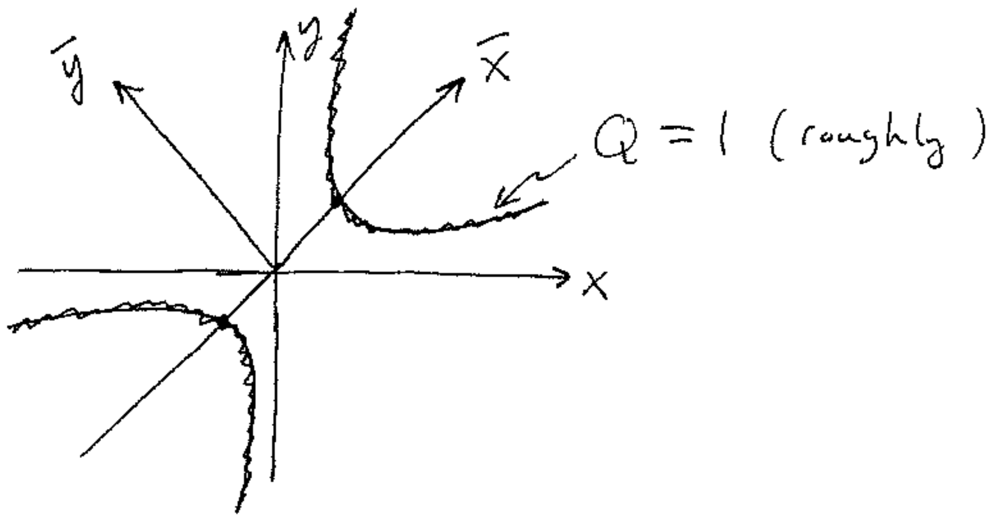
P145 continued

forgive me, I'll skip explicit calculation of

$\begin{bmatrix} x \\ y \end{bmatrix} = [\beta]\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$ to change $Q(x,y) = x^2 + 4xy$

into $Q(\bar{x}u_1 + \bar{y}u_2) = \boxed{\left(\frac{1+\sqrt{17}}{2}\right)\bar{x}^2 + \left(\frac{1-\sqrt{17}}{2}\right)\bar{y}^2}$

Thus $x^2 + 4xy = \underbrace{\left(\frac{1+\sqrt{17}}{2}\right)\bar{x}^2 + \left(\frac{1-\sqrt{17}}{2}\right)\bar{y}^2 = 1}_{\text{hyperbola}}$



Remark: on your test, if problem explicit in nature, e-values will belong to $\mathbb{Z}$. This promise I make to you.

P146 $Q(x,y,z) = 2x^2 + 4y^2 + 6z^2 + 8xy + 10xz + 12yz$

$Q(v) = v^T \underbrace{\begin{bmatrix} 2 & 4 & 5 \\ 4 & 4 & 6 \\ 5 & 6 & 6 \end{bmatrix}}_{A} v$

(old version calculator) used www.arndt-bruenner.de (linked at course page)

A has $\lambda_1 \approx -1.39$, $\lambda_2 = -1$, $\lambda_3 \approx 14.39$

Thus $\boxed{Q(\bar{x}u_1 + \bar{y}u_2 + \bar{z}u_3) \cong -1.39\bar{x}^2 - \bar{y}^2 + 14.39\bar{z}^2}$

where u_1, u_2, u_3 are unit e-vectors of $\lambda_1, \lambda_2, \lambda_3$ respective

Bonus Solution:

Let $Q(x,y,z) = 5x^2+5y^2+2z^2+8xy+4xz+4yz$

write $Q(v) = v^TAv$ for some symmetric matrix A

$$Q(x,y,z) = [x,\ y,\ z]\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ (I invite the reader to check)

Hence, $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

P147

find P with $P^TP = I$ such that $P^{-1}AP$ is diagonal. That is diagonalize A. Also if $(\bar{x},\bar{y},\bar{z}) = P^{-1}(x,y,z)$ find formula for Q in terms of $\bar{x},\bar{y},\bar{z}$.

We know the sol^n is to find e-basis which is orthonormal for the symmetric matrix A (spectral Th^m says this is possible)

$$\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$= (5-\lambda)\left[(\lambda-5)(\lambda-2)-4\right] - 4\left[4(2-\lambda)-4\right] + 2\left[8-2(5-\lambda)\right]$$

$$= (5-\lambda)\left[\lambda^2-7\lambda+6\right] - 4\left[4-4\lambda\right] + 2\left[-2+2\lambda\right]$$

$$= (5-\lambda)\left[(\lambda-1)(\lambda-6)\right] + 16(\lambda-1) + 4(\lambda-1)$$

$$= (\lambda-1)\left[(5-\lambda)(\lambda-6)+16+4\right]$$

$$= (\lambda-1)\left[\lambda^2+11\lambda-30+20\right]$$

$$= (\lambda-1)\left[-\lambda^2+11\lambda-10\right]$$

$$= -(\lambda-1)\left[\lambda^2-11\lambda+10\right]$$

$$= -(\lambda-1)^2(\lambda-10) \quad \therefore \quad \lambda_1 = 1 \text{ with algebraic multiplicity } 2, \quad \lambda_2 = 10.$$

$\underline{\lambda_1 = 1}$ $\quad A - I = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence $(u,v,w) \in \text{Null}(A-I)$ has $2u + 2v + w = 0$

$\therefore \quad (u,v,w) = (u, v, -2u-2v) = u(1,0,-2) + v(0,1,-2)$

$\therefore \quad \text{Null}(A-I) = \text{span}\{\underbrace{\tfrac{1}{\sqrt{5}}(1,0,-2), \ \tfrac{1}{\sqrt{5}}(0,1,-2)}_{\text{orthonormal basis for } E_1}\}$.

$\underline{\lambda_2 = 10}$

$$A - 10I = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $(u,v,w) \in \text{Null}(A-10I) = E_2$ has $u - 2w = 0$ and $v - 2w = 0$ $\therefore$ $(u,v,w) = (2w, 2w, w) = (2,2,1)w$

$\therefore \quad E_2 = \text{span}\{\tfrac{1}{3}(2,2,1)\}$

Thus $\beta = \left\{ \overset{\vec{u}_1}{\tfrac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}}, \overset{\vec{u}_2}{\tfrac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\-2\end{bmatrix}}, \overset{\vec{u}_3}{\tfrac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}} \right\}$ is orthonormal e-basis for A and as $[\beta]^{-1} = [\beta]^T$ and $[v]_\beta = [\beta]^{-1}v$ for $\mathbb{R}^n$,

$$\begin{bmatrix}\bar{x}\\\bar{y}\\\bar{z}\end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/3 \\ 0 & 1/\sqrt{5} & 2/3 \\ -2/\sqrt{5} & -2/\sqrt{5} & 1/3 \end{bmatrix}^T \begin{bmatrix}x\\y\\z\end{bmatrix} \quad \boxed{\begin{aligned} x &= \tfrac{1}{\sqrt{5}}\bar{x} + \tfrac{2}{3}\bar{z} \\ y &= \tfrac{1}{\sqrt{5}}\bar{y} + \tfrac{2}{3}\bar{z} \\ z &= \tfrac{-2}{\sqrt{5}}\bar{x} - \tfrac{2}{\sqrt{5}}\bar{y} + \tfrac{1}{3}\bar{z} \end{aligned}}$$

$$= \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \\ 2/3 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix} \tfrac{1}{\sqrt{5}}x - \tfrac{2}{\sqrt{5}}z \\ \tfrac{1}{\sqrt{5}}y - \tfrac{2}{\sqrt{5}}z \\ \tfrac{2}{3}x + \tfrac{2}{3}y + \tfrac{1}{3}z \end{bmatrix} = \begin{bmatrix}\bar{x}\\\bar{y}\\\bar{z}\end{bmatrix}$$

Thus $\boxed{\bar{x} = \tfrac{1}{\sqrt{5}}(x - 2z), \quad \bar{y} = \tfrac{1}{\sqrt{5}}(y - 2z), \quad \bar{z} = \tfrac{2}{3}(x + y + \tfrac{1}{2}z)}$

And, by our general derivation, (from Lecture)

$$\boxed{Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2}$$

where I expressed $v = \bar{x}\vec{u}_1 + \bar{y}\vec{u}_2 + \bar{z}\vec{u}_3$.

* I noticed $[v]_\beta = [\beta]^{-1}v \Rightarrow v = [\beta][v]_\beta$.

P148 Suppose $\{u, v, w\}$ forms an orthonormal e-basis for $A \in \mathbb{R}^{3\times 3}$ with e-values $\lambda_1, \lambda_2, \lambda_3$ respective ($Au = \lambda_1 u$, $Av = \lambda_2 v$, $Aw = \lambda_3 w$)

Define: $E_1 = \lambda_1 uu^T$, $E_2 = \lambda_2 vv^T$, $E_3 = \lambda_3 ww^T$

SHow: $E_1 + E_2 + E_3 = A$ and $Col(E_j) = Null(A - \lambda_j I)$ for $j = 1, 2, 3$

Consider $L(x) = (E_1 + E_2 + E_3)x$ and $L_A(x) = Ax$.
If L and L_A agree on a basis for $\mathbb{R}^3$ then $L = L_A$ and thus $E_1 + E_2 + E_3 = A$. Consider,

① $L_A(u) = Au = \lambda_1 u$

$$L(u) = (E_1 + E_2 + E_3)u = (\lambda_1 uu^T + \lambda_2 vv^T + \lambda_3 ww^T)u$$
$$= \lambda_1 u \underbrace{u^T u}_{0} + \lambda_2 v \underbrace{v^T u}_{0} + \lambda_3 w \underbrace{w^T u}_{0}$$

thus $L(u) = \lambda_1 u$ Recall $x \cdot y = x^T y$.

and we find $L_A(u) = L(u) = \lambda_1 u$.

② $L_A(v) = Av = \lambda_2 v$

$$L(v) = \lambda_1 u \underbrace{u^T v}_{0} + \lambda_2 v \underbrace{v^T v}_{1} + \lambda_3 w \underbrace{w^T v}_{0} = \lambda_2 v = L_A(v).$$

③ $L_A(w) = Aw = \lambda_3 w$

$$L(w) = \lambda_1 u \underbrace{u^T w}_{0} + \lambda_2 v \underbrace{v^T w}_{0} + \lambda_3 w \underbrace{w^T w}_{1} = \lambda_3 w = L_A(w)$$

Therefore, $\boxed{A = E_1 + E_2 + E_3}$

scalars based on λ_1 & components of u

Observe $E_1 = \lambda_1 uu^T = [\#u \,|\, *u \,|\, @u] \therefore Col(E_1) = span\{u\}$

But, $Null(A - \lambda_1 I) = span\{u\}$ since λ_1 has geometric multiplicity 1. (oops! You need $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$)

Thus $Col(E_1) = Null(A - \lambda_1 I)$. Same argument works for 2 & 3.

P149

$$u = \frac{1}{\sqrt{3}}(1,-1,1)$$
$$v = \frac{1}{\sqrt{2}}(0,1,1)$$
$$w = \frac{1}{\sqrt{6}}(2,1,-1)$$

Construct A for which u, v, w are e-vector with e-values 12, 2, 18

$$A = 12uu^T + 2vv^T + 18ww^T$$

$$= \frac{12}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{2}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \frac{18}{6}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 12 & 6 & -6 \\ 6 & 3 & -3 \\ -6 & -3 & 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 16 & 2 & -2 \\ 2 & 8 & -6 \\ -2 & -6 & 8 \end{bmatrix}}$$

as a check on my calculation,
trace $(A) = 16 + 8 + 8 = 32$
and $\lambda_1 + \lambda_2 + \lambda_3 = 12 + 2 + 18 = 32$ ✓

$\boxed{P150}$ Let $M, N \leq V$ and $x, y \in V$.

Suppose $x + M \subseteq y + N$. Let $m \in M$ then $x+m \in x+M \subseteq y+N$ and thus $x+m \in y+N \Rightarrow \exists\, n \in N$ such that $x+m = y+n$. Observe $m = y-x+n$. If $M \subseteq N$ then $m \in M \Rightarrow m \in N$, thus $y-x+n \in N$ and as $N \leq V$ we find $y-x \in N$.//

Conversely suppose $M \subseteq N$ and $x-y \in N$. Let $z \in x+M$ hence $\exists\, m \in M$ for which $z = x+m$. Now $x-y \in N$ implies $\exists\, n \in N$ such that $\underbrace{x-y=n}_{x=y+n}$ and we find:

$$z = x+m = y+n+m$$

However, $M \subseteq N$ thus $n, m \in N$ and $n+m = n_2 \in N$ as $N \leq V$. Thus, $z = y+n_2 \Rightarrow z \in y+N$. This proves $x+M \subseteq y+N$.//

In summary,

$$\left(x+y \subseteq y+N\right) \Longleftrightarrow \left(M \subseteq N \text{ and } x-y \in N\right)$$

$\boxed{P151}$ $\Psi : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}^{n \times n}$ s.t. $\Psi(X) = X - X^T$ then observe $\Psi(cX+Y) = cX+Y-(cX+Y) = c(X-X^T)+Y-Y^T = c\Psi(X)+\Psi(Y)$

thus Ψ is linear transformation and

$$\text{Ker}(\Psi) = \{X \in \mathbb{F}^{n\times n} \mid X - X^T = 0\} = S_n \leftarrow \text{symmetric } n\times n \text{ matrices.}$$

Also if $A \in A_n$ $(A^T = -A)$ then notice

$$\Psi(A/2) = \frac{A}{2} - \left(\frac{A}{2}\right)^T = \frac{A}{2} + \frac{A}{2} = A \quad \therefore \Psi(\mathbb{F}^{n\times n}) = A_n$$

thus, by 1st isomorphism thm; $\mathbb{F}^{n\times n}/\text{Ker}\,\Psi \approx A_n$

Then as $\text{Ker}\,\Psi = \cancel{S_n}$ ~~it follows~~ $\cancel{\mathbb{F}^{n\times n}} = A_n \oplus S_n$.

Sorry folks wrong direction

P151 Continued, we've shown $\mathbb{F}^{n\times n}/\text{Ker}\,\Psi \approx A_n$ and it is true that $A\oplus B/B \approx A$ but, I don't see how that gets us to $S_n \oplus A_n = \mathbb{F}^{n\times n}$.

I can show it, If $A \in \mathbb{F}^{n\times n}$ then

$$A = \underbrace{\tfrac{1}{2}(A+A^T)}_{\in S_n} + \underbrace{\tfrac{1}{2}(A-A^T)}_{\in A_n} \Rightarrow \underline{S_n + A_n = \mathbb{F}^{n\times n}}.$$

$*$ then $A \in S_n \cap A_n \Rightarrow A^T = A$ and $A^T = -A \therefore A = -A \Rightarrow A = 0$ thus $S_n \cap A_n = \{0\}$ and we conclude $\underline{\mathbb{F}^{n\times n} = S_n \oplus A_n}$.

Remark: perhaps $\exists$ a th$^{\underline{m}}$ in my notes that avoids $*$ but, I'm not sure which one today ☺.

P152 $V = \text{span}_{\mathbb{R}}\underbrace{\{e^x, e^{2x}, \cos x, \sin x\}}_{\beta}$, $T = d/dx + 1$

$$[T]_{\beta\beta} = \left[[\underbrace{T(e^x)}_{2e^x}]_\beta \,\Big|\, [\underbrace{T(e^{2x})}_{3e^{2x}}]_\beta \,\Big|\, [\underbrace{T(\cos x)}_{-\sin x+\cos x}]_\beta \,\Big|\, [\underbrace{T(\sin x)}_{\cos x+\sin x}]_\beta \right] = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$U = \text{span}_{\mathbb{R}}\{e^x, e^{2x}\}$ has $T(U) = \{T(u) \mid u \in U\} = \text{span}\{2e^x, 3e^{2x}\} = U$.

then I can read from $[T]_{\beta\beta}$,

$$\boxed{[T|_U] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \quad \& \quad \boxed{[T_{V/U}] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$

A bit of extra detail,

$$\begin{aligned} (T_{V/U})(\cos x + U) &= T(\cos x) + U \\ &= \cos(x) - \sin x + U \\ &= (\cos x + U) + (-\sin x + U) \end{aligned}$$

$\boxed{P153}$ Let $W \leq V$ and consider $ann(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \ \forall w \in W\}$.

Let $\beta = \{w_1, w_2, \ldots, w_k\}$ form a basis for W

then extend β to $\gamma = \{w_1, \ldots, w_k, v_1, \ldots, v_{n-k}\}$ a basis for V.

Notice $v_1, \ldots, v_{n-k} \notin W$ by the LI of γ. Next

form the dual basis $\gamma^* = \{w^1, \ldots, w^k, v^1, \ldots, v^{n-k}\}$

where, by construction, $w^i(w_j) = \delta_{ij}$ for $1 \leq i,j \leq k$

and $v^i(v_j) = \delta_{ij}$ for $1 \leq i,j \leq n-k$ and $w^i(v_l) = 0$, $v^q(w_p) = 0$

for all $i,p = 1,2,\ldots,k$ and $l,q = 1,2,\ldots,n-k$. Suppose

$\alpha \in ann(W)$ and $x = \sum_{i=1}^{k} x^i w_i + \sum_{j=1}^{n-k} y^j v_j \in V$ then

$$\alpha(x) = \sum_{i=1}^{k} x^i \underbrace{\alpha(w_i)}_{0} + \sum_{j=1}^{n-k} y^j \alpha(v_j) = \sum_{j=1}^{n-k} \alpha(v_j) v^j(x)$$

However, this shows $\alpha = \sum_{j=1}^{n} \alpha(v_j) v^j \in span\{v^1, \ldots, v^{n-k}\}$

$\therefore$ $\beta_\perp^* = \{v^1, \ldots, v^{n-k}\}$ is a LI spanning set for $ann(W)$.

Thus $\dim(ann(W)) = n-k = \dim(V) - \dim(W)$

and we deduce, $\underline{\dim(W) + \dim(ann(W)) = \dim(V)}$ //

P154 Let $U \leq W \leq V$ where V is V-space over $\mathbb{F}$ and
$\text{ann}(U) = \{\alpha \in V^* \mid \alpha(u) = 0 \ \forall u \in U\}$
$\text{ann}(W) = \{\alpha \in V^* \mid \alpha(w) = 0 \ \forall w \in W\}$.

Observe $0 \in \text{ann}(W)$ as $0(v) = 0 \ \forall v \in V$ defines $0 \in V^*$ and clearly $0 \in \text{ann}(W)$ $\therefore$ $\text{ann}(W) \neq \emptyset$. If $\alpha_1, \alpha_2 \in \text{ann}(W)$ then $(\alpha_1 + c\alpha_2)(w) = \alpha_1(w) + c\alpha_2(w) = 0 \ \forall w \in W \therefore \alpha_1 + c\alpha_2 \in \text{ann}(W)$ thus $\text{ann}(W) \leq V^*$ by the subspace test Th$^{\underline{m}}$. Likewise $\text{ann}(U) \leq V^*$.

Hence, to show $\text{ann}(W) \leq \text{ann}(U)$ it suffices to show $\text{ann}(W) \subseteq \text{ann}(U)$.

Let $\alpha \in \text{ann}(W)$ and consider $u \in U \Rightarrow u \in W$ as $U \leq W$ thus $\alpha(u) = 0$ as $u \in W$. But, $u \in U$ was arbitrary and we've shown $\alpha \in \text{ann}(U)$ $\therefore$ $\text{ann}(W) \subseteq \text{ann}(U)$ (subset)

$\Rightarrow$ $\underline{\text{ann}(W) \leq \text{ann}(U)}$. (subspace)

P155 Let $(V, \langle , \rangle)$ form a real inner product space and $U \leq W \leq V$. Suppose $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}$ and $U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \ \forall u \in U\}$. We've shown in Lecture $W^\perp \leq V$ so I'll skip proof that $W^\perp$ & $U^\perp$ form subspaces. Let $x \in W^\perp$ hence $\langle x, w \rangle = 0 \ \forall w \in W$. Let $u \in U$ then as $U \leq W$ we have $u \in W$ thus $\langle x, u \rangle = 0$ and we find $x \in U^\perp$ $\therefore$ $W^\perp \subseteq U^\perp$ (subset) hence $W^\perp \leq U^\perp$.

(Extra Sol^n, wasn't assigned, but... nice)

$\boxed{}$ If $\alpha(V_1, V_2, V_3) = V_1 + 3V_3$ then find $[\alpha]_{\beta^*}$ in terms of $\beta^* = \{e^1, e^2, e^3\}$ as defined in P118

$$\alpha(v) = v^1 + 3v^3 \quad \text{for} \quad v = (v^1, v^2, v^3)$$
$$= e^1(v) + 3e^3(v)$$
$$= (1 \cdot e^1 + 0 \cdot e^2 + 3 \cdot e^3)(v) \hookrightarrow \alpha = e^1 + 0 \cdot e^2 + 3 \cdot e^3$$
$$\therefore \boxed{[\alpha]_{\beta^*} = (1, 0, 3)}$$

$\boxed{P156}$

Given $V = \text{span}\,\beta$, $V^* = \text{span}\,\beta^*$ & $W = \text{span}\,\gamma$, $W^* = \text{span}\,\gamma^*$. If $T : V \longrightarrow W$ a linear tranformation and we define $S^*: W^* \to V^*$ by $(S(\alpha))(v) = \alpha(T(v))$ $\forall v \in V$ and $\alpha \in W^*$. Then show $S \in \mathcal{L}(W^*, V^*)$ and calculate $[S]_{\gamma^*, \beta^*}$

Consider, for $\alpha, \beta \in W^*$ and $c \in \mathbb{R}$,
$$(S(c\alpha + \beta))(v) = (c\alpha + \beta)(T(v))$$
$$= c\,\alpha(T(v)) + \beta(T(v))$$
$$= (c\,S(\alpha) + S(\beta))(v) \quad \forall v \in V \Rightarrow S \in \mathcal{L}(W^*, V^*).$$

Next, consider, $\gamma^* = \{g^1, g^2, \dots, g^m\}$ so,

$$[S]_{\gamma^*, \beta^*} = \Big[[S(g^1)]_{\beta^*} \Big| [S(g^2)]_{\beta^*} \Big| \cdots \Big| [S(g^m)]_{\beta^*} \Big] \quad *$$

$$= \left[\begin{bmatrix} S(g^1)(f_1) \\ S(g^1)(f_2) \\ \vdots \\ S(g^1)(f_n) \end{bmatrix} \middle| \begin{bmatrix} S(g^2)(f_1) \\ S(g^2)(f_2) \\ \vdots \\ S(g^2)(f_n) \end{bmatrix} \middle| \cdots \middle| \begin{bmatrix} S(g^m)(f_1) \\ S(g^m)(f_2) \\ \vdots \\ S(g^m)(f_n) \end{bmatrix} \right]$$

$$= \begin{bmatrix} g^1(T(f_1)) & g^2(T(f_1)) & \cdots & g^m(T(f_1)) \\ \vdots & & & \\ g^1(T(f_n)) & g^2(T(f_n)) & \cdots & g^m(T(f_n)) \end{bmatrix} \quad **$$

$$= \begin{bmatrix} ([T(f_1)]_\gamma)^T \\ \vdots \\ ([T(f_n)]_\gamma)^T \end{bmatrix}$$

$$= \Big[[T(f_1)]_\gamma \Big| \cdots \, [T(f_n)]_\gamma \Big]^T$$

$$= ([T]_{\beta, \gamma})^T$$

I'll expand on * & ** next

P156 continued

$*$: $[S(g^1)]_{\beta^*} = (c_1, c_2, \dots, c_n)$ means that

$S(g^1) = c_1 f^1 + c_2 f^2 + \dots + c_n f^n$ and we can

select the values of $c_1, c_2, \dots, c_n$ by using $f^i(f_j) = \delta_{ij}$

$$(S(g^1))(f_i) = c_1 \underbrace{f^1(f_i)}_{0} + \dots + c_i \underbrace{f^i(f_i)}_{1} + \dots + c_n \underbrace{f^n(f_i)}_{0} = c_i$$

Therefore,

$$[S(g^1)]_{\beta^*} = \big((S(g^1))(f_1),\ (S(g^1))(f_2), \dots, (S(g^1))(f_n) \big)$$

Of course this calculation holds for $\alpha \in V^*$ just the same

$$[\alpha]_{\beta^*} = (\alpha(f_1), \alpha(f_2), \dots, \alpha(f_n)).$$

$**$: $[T(f_1)]_\gamma^T = [g^1(T(f_1)), g^2(T(f_1)), \dots, g^m(T(f_1))]$

Observe $[T(f_1)]_\gamma = (c_1, c_2, \dots, c_m)$ indicates that

$T(f_1) = c_1 g_1 + c_2 g_2 + \dots + c_m g_m$. However, as

$g^j(g_i) = \delta_{ij}$ we deduce $g^j(T(f_1)) = c_j$

hence $[T(f_1)]_\gamma^T = [g^1(T(f_1)), \dots, g^m(T(f_1))]$.

Likewise, for any $w \in W$ we have

$[w]_\gamma = (g^1(w), g^2(w), \dots, g^m(w))$ where $\gamma^* = \{g^1, \dots, g^m\}$ is the dual-basis to $\gamma = \{g_1, \dots, g_m\}$.

<u>Remark</u>: you'll notice the notation for $[T]_{\beta,\gamma}$ is replaced with $[T]_\beta^\gamma$ in other texts such as Damiano & Little (see Prop. 2.2.15, page 78, $\underline{[T(v)]_\beta = [T]_\alpha^\beta [V]_\alpha}$ translate $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$)

continued ↙

P 156 continued

It is instructive to explore how the matrix of $T: V \to W$ depends explicitly on the basis β for V and γ^* for W^*.

Assume $\dim V = n$ and $\dim W = m$,

$$T(v) = \sum_{i=1}^{m} g^i[T(v)]\, g_i \quad (\text{by } **)$$

$$= \sum_{i=1}^{m} g^i\Big[T\Big(\sum_{j=1}^{n} f^j(v) f_j\Big)\Big] g_i \quad (\text{by } ** \text{ for } V)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f^j(v)\, g^i[T(f_j)]\, g_i$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \Big(\underbrace{g^i[T(f_j)]}_{A^i{}_j}\, f^j(v) \Big) g_i$$

$$\left([T(v)]_\gamma\right)^i = \sum_{i=1}^{m} \sum_{j=1}^{n} A^i{}_j\, V^j$$

Our notation has been $\left([T]_{\beta,\gamma}\right)_{ij} = g^i(T(f_j))$

But, you can see that $\left([T]_\beta^\gamma\right)^i{}_j = g^i(T(f_j))$

is more reflective of the role β and γ^* play in defining the matrix of T. In particular, to use other bases $[T]_{\bar\beta}^{\bar\gamma} = \bar g^i(T(\bar f_j))$

we see immediately the transformed matrix transforms the same way as $\bar\beta$ but, instead of γ, as $\bar\gamma^*$ relates to γ^*. It turns out $\bar\beta$ & $\bar\beta^*$ are inversely related to β and β^* respective. This must occur as $f^i(f_j) = \delta_{ij}$ and $\bar f^i(\bar f_j) = \delta_{ij}$ as well...

$\boxed{P157}$ Let $T: V \longrightarrow W$ and $S: W^* \longrightarrow V^*$ be defined, where as in $\boxed{P156}$, $(S(\alpha))(v) = \alpha(T(v))$ $\forall v \in V, \alpha \in W^*$

(a.) if T is surjective then $T(V) = W$. Let $\dim(V) = n$, $\dim(W) = m$ thus $[T]_{\beta\gamma}$ is $m \times n$ and $T(V) = W \Rightarrow$ $\text{Col}\,[T]_{\beta\gamma}$ has dimension m or, if you prefer, $\exists$ m-LI columns of $[T]_{\beta\gamma}$. We found $[S]_{\gamma^*\beta^*} = ([T]_{\beta,\gamma})^T$ ← has m-LI rows

Thus $[S]_{\gamma^*,\beta^*}$ is an $n \times m$ matrix with m-LI rows which indicates each column in $[S]_{\gamma^*\beta^*}$ is a pivot column $\therefore$ $\text{Null}\,[S]_{\gamma^*\beta^*} = \{0\} \Rightarrow \text{Ker}(S) = \{0\}$

$\therefore$ <u>S is injective</u>.

(b.) if S is injective then $[S]_{\gamma^*\beta^*}$ is an $n \times m$ matrix with m-LI columns since $\text{Null}\,[S]_{\gamma^*\beta^*} = \{0\}$. But, $[T]_{\beta,\gamma} = ([S]_{\gamma^*\beta^*})^T$ is an $m \times n$ matrix with m-LI rows and hence m-LI columns

$\Rightarrow \dim(\text{Col}\,[T]_{\beta,\gamma}) = m \Rightarrow T(V) = W$

$\therefore$ <u>T is surjective</u>.

(c.) If T is an isomorphism then T is injective and surjective $\therefore$ S is injective by (a.). Since $\dim V = \dim(\underbrace{\cancel{\text{Ker}}\,T}_{0}) + \dim(T(V)) \Rightarrow \dim V = \dim W$.

Then $\dim W^* = \dim(\underbrace{\cancel{\text{Ker}}\,S}_{0}) + \dim(S(W^*)) \Rightarrow$ <u>$\dim(S(W^*)) = \dim V^*$</u> as $\dim V = \dim V^*$ and $\dim W = \dim W^*$. Hence <u>$S(W^*) = V^*$</u>. Thus S is an isomorphism. (I leave converse to reader)

P158

$\text{trace} : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ $\therefore$ $\text{trace} \in (\mathbb{R}^{n\times n})^*$ as trace is linear.

Consider $\langle A, B \rangle = \text{trace}(AB^T)$ as an inner product on $\mathbb{R}^{n\times n}$

Find Riesz vector for trace functional.

We want to find X for which

$$\underbrace{\text{trace}(A) = \langle A, X \rangle}_{\text{trace}(A) = \text{trace}(AX^T)} \quad \text{for all } A \in \mathbb{R}^{n\times n}$$

Well, apparently, we may use $\boxed{X = I}$ ← Riesz' vector

(Riesz' vector is unique, so my guess is inescapable.)

In our musical notation, $\flat\,\text{trace} = I$.

159 #7 of §5.3 pg. 248 of Damiano & Little

A linear transformation $T: V \to V$ is skew-adjoint or skew-Hermitian if $T = -T^*$

a.) show e-values of T are pure imaginary

b.) show e-vectors of distinct e-values are orthogonal

(a.) Suppose $T(v) = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v \neq 0$ in V over $\mathbb{C}$.

Notice $\langle v, v \rangle = \|v\|^2 \neq 0$. Moreover,

$$\begin{aligned} \langle T(v), v \rangle &= \langle v, T^*(v) \rangle \quad : \text{(by def}^{\underline{n}} \text{ of } T^*) \\ &= \langle v, -T(v) \rangle \quad : \text{as } T = -T^* \text{ hence } T^* = -T. \\ &= \langle v, -\lambda v \rangle \end{aligned}$$

But, $\langle T(v), v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2$

whereas $\langle v, -\lambda v \rangle = \overline{-\lambda} \langle v, v \rangle = -\bar{\lambda} \|v\|^2$ hence $\lambda \|v\|^2 = -\bar{\lambda} \|v\|^2$

from which $\boxed{\lambda = -\bar{\lambda}}$ follows as $\|v\|^2 \neq 0$. Hence $\underline{\lambda \in i\mathbb{R}}$.

(b.) Suppose $\mu \neq \lambda$ and let $x \in E_\mu$ and $y \in E_\lambda$ and consider,

① $\langle T(x), y \rangle = \langle \mu x, y \rangle = \mu \langle x, y \rangle$

② $\langle T(x), y \rangle = \langle x, T^*(y) \rangle = \langle x, -T(y) \rangle = \langle x, -\lambda y \rangle = -\bar{\lambda} \langle x, y \rangle$

But, from (a.) we know $-\bar{\lambda} = \lambda$ hence, as ① = ②,

$$\mu \langle x, y \rangle = \lambda \langle x, y \rangle \Rightarrow \underbrace{(\mu - \lambda)}_{\neq 0} \langle x, y \rangle = 0$$

$$\Rightarrow \langle x, y \rangle = 0$$

Hence $E_\mu \perp E_\lambda$ where $\mu \neq \lambda$ and $T = -T^*$. //

P160 exercise 7 of §4.5 of pg. 205 of D&L

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be normal if $AA^T = A^TA$

(a.) Show if A is symmetric then A is normal:

$A^T = A \Rightarrow AA^T = A^TA$ applying symmetry twice.

(b.) Find an example of 2x2 normal matrix that is not symmetric:

Many examples can be given. I like the text's answer.
Consult a rotation matrix $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for $\theta \neq k\pi$

Also $R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ & $R^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ ← not a rotation, actually a dilation & rotation.

$$\left.\begin{aligned} RR^T &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ R^TR &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}\right\} \quad RR^T = R^TR$$

(c.) Show if A is normal then $\|Ax\| = \|A^Tx\| \ \forall x \in \mathbb{R}^n$

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= (Ax)^T Ax \\ &= x^T A^T A x \\ &= x^T A A^T x \\ &= (A^Tx)^T A^T x \\ &= \langle A^Tx, A^Tx \rangle \\ &= \|A^Tx\|^2 \end{aligned}$$

$\therefore \ \|Ax\| = \|A^Tx\| \ \forall x \in \mathbb{R}^n$.

{this conclusion is valid as $\|v\| \geq 0 \ \forall v \in \mathbb{R}^n$.

(d.) Show if A is normal then $A - cI$ is normal $\forall c \in \mathbb{R}$

$$\begin{aligned} (A-cI)(A-cI)^T &= (A-cI)(A^T - cI) \\ &= AA^T - cIA^T - cAI + c^2 II \\ &= A^TA - cA^TI - cIA + c^2 II \\ &= A^T(A-cI) - cI^T(A-cI) \\ &= (A^T - cI^T)(A-cI) \\ &= (A-cI)^T(A-cI) \end{aligned}$$

normal A and I commutes with everything

$\therefore$ $A - cI$ normal //

(e.) show if $\lambda \in \mathbb{R}$ is an e-value of normal matrix A then λ is also an e-value of A^T

$\lambda \in \mathbb{R}$ an e-value of $A \Rightarrow \text{Null}(A - \lambda I) \neq \{0\}$

Hence $\exists x \in \mathbb{R}^n$ s.t. $(A - \lambda I)x = 0$ for $x \neq 0$.

Notice A normal $\Rightarrow A - \lambda I$ also normal by part d.

Also, $\|(A - \lambda I)x\| = \|(A - \lambda I)^T x\|$ by part c.

Hence, $\|(A - \lambda I)x\| = \|(A^T - \lambda I)x\| = 0$ which implies, by property $\|v\| = 0 \Leftrightarrow v = 0$ for $\|\cdot\|$, that $(A^T - \lambda I)x = 0$. Thus $x \neq 0$ is also e-vector for A^T with e-value λ.

(f.) Show that if λ_1 and λ_2 are distinct real e-values of a normal matrix A then corresponding e-vectors are orthogonal

---

Let $x \in E_{\lambda_1}$ and $y \in E_{\lambda_2}$ both non zero (since I'm looking at eigenvectors) then $Ax = \lambda_1 x$ & $Ay = \lambda_2 y$

Observe $\langle x, y \rangle$ is shown to be zero as follows:

As AA^T is symmetric for A normal,

$$\langle A^T A x, y \rangle = \langle x, A^T A y \rangle$$

$$\Rightarrow \langle A^T \lambda_1 x, y \rangle = \langle x, A^T \lambda_2 y \rangle$$

property of $\langle , \rangle$

$$\Rightarrow \lambda_1 \langle A^T x, y \rangle = \lambda_2 \langle x, A^T y \rangle$$

by part e.

$$\Rightarrow \lambda_1 \langle \lambda_1 x, y \rangle = \lambda_2 \langle x, \lambda_2 y \rangle$$

$$\Rightarrow (\lambda_1^2 - \lambda_2^2) \langle x, y \rangle = 0$$

However, as $\lambda_1 \neq \lambda_2 \Rightarrow \lambda_1^2 - \lambda_2^2 \neq 0 \therefore \langle x, y \rangle = 0$ which goes to show $x \perp y$ as desired. //