Please follow the format which was announced in Blackboard. Thanks!

Problem 145 Find the best-fit line to the data $(1,1), (2,4), (3,0), (4,5), (10,6)$ via the method of least squares as discussed in my notes.

Problem 146 Suppose $Q(x,y) = 5x^2 + 5y^2 + 8xy$. Write $Q(v) = v^T A v$ for a symmetric matrix A. Find an orthonormal eigenbasis $\beta = \{u_1, u_2\}$ for A and find coordinates $\bar{x}, \bar{y}$ such that $v = \bar{x}u_1 + \bar{y}u_2$ gives $Q(v) = \lambda_1 \bar{x}^2 + \lambda_2 \bar{y}^2$.

Problem 147 Suppose $Q(x,y,z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$. Write $Q(v) = v^T A v$ for a symmetric matrix A. Find an orthonormal eigenbasis for A and find coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$.
Hint: for this question to make sense, it must be that the matrix of Q has e-values $1, 1, 10$.

Problem 148 Calculus of functions of several variables is best understood with the aid of linear algebra. In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function (meaning you can take as many partial derivatives as you wish) then the multivariate Taylor theorem tells us:

$$f(x) = f(p) + (\nabla f)(p) \bullet (x - p) + H(p)(x - p) + \cdots$$

where ∇f is the **gradient** of f at p and $H(p)$ is the **Hessian** of f at p. Both the gradient and Hessian are assembled from appropriate partial derivatives:

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \qquad \& \qquad [H]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Notice, if p is a **critical point** then $\nabla f(p) = 0$ and hence $f(x) = f(p) + H(p)(x - p) + \cdots$ thus the behavior of f is dominated by $H(p)$ for x near p. In fact, $H(p)(x - p) = (x - p)^T [H](x - p)$ where $[H]^T = [H]$ so the quadratic term is a quadratic form. It follows there exist eigencoordinates $y_1, \ldots, y_n$ with respect to the eigenbasis $\{v_1, \ldots, v_n\}$ for $[H]$ such that $y = y_1 v_1 + \cdots + y_n v_n = x - p$ and

$$H(p)(x - p) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \qquad \star .$$

It is then simple to judge whether a given critical point provides a local minimum or maximum for the function. Looking at $\star$,

(**i.**) if $\lambda_1, \ldots, \lambda_n > 0$ then $f(p)$ is a local minimum.

(**ii.**) if $\lambda_1, \ldots, \lambda_n < 0$ then $f(p)$ is a local maximum.

(**iii.**) if $\lambda_1, \ldots, \lambda_n$ are all nonzero, yet differ in sign, then $f(p)$ neither local max nor min.

Unfortunately, if the spectrum of the Hessian includes eigenvalue zero then we are unable to offer a conclusion. In that case the third order terms could go either way.

Find critical points for the functions below and analyze the eigenvalues of the Hessian to classify the nature of the critical points as either max, min or saddle (case (iii.) is known as a saddle point)

(a.) let $f(x, y) = 5x^2 + 8xy - 10x + 5y^2 - 8y + 5$,

(b.) let $f(x, y, z) = x^2 + y^2 + z^2 + 4xy + 4xz + 4yz$

Remark: if you never saw partial differentiation in previous course work, I am happy to teach you in office hours. Or, ask me in class.

Problem 149 Let $A = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$.

(a.) diagonalize A,

(b.) calculate $\lim_{n \to \infty} A^n$,

(c.) let $x_o = (0.7, 0.3)$. Define $x_n = A^n x_o$ hence $x_1 = Ax_o$ and $x_2 = Ax_1 = AAx_o$ etc. Calculate x_1, x_{10} and x_{100}. What is $\lim_{n \to \infty} x_n$? How does this relate to things you found in (a.)

The vectors you find in (c.) are an example of a **Markov chain**. *Notice A is a transition matrix and x_o is a probability vector.*

Problem 150 A matrix $A \in \mathbb{R}^{n \times n}$ is called **normal** if $A^T A = AA^T$.

(a.) show a symmetric matrix is normal,

(b.) find an example of a 2×2 matrix which is normal, but, not symmetric,

(c.) show if $A \in \mathbb{R}^{n \times n}$ is normal then $||Ax|| = ||A^T x||$ for all $x \in \mathbb{R}^n$,

(d.) show if $A \in \mathbb{R}^{n \times n}$ is normal then $A - cI$ is normal for all $c \in \mathbb{R}$,

(e.) show if $\lambda \in \mathbb{R}$ is e-value of normal matrix $A \in \mathbb{R}^{n \times n}$ then λ is also an e-value of A^T

(f.) show if λ_1, λ_2 are distinct real e-valued of a normal matrix $A \in \mathbb{R}^{n \times n}$ then the corresponding e-vectors are orthogonal.

Problem 151 The last problem counts double (aka Problem 149 does not exist)

Problem 152 The Riesz Representation Theorem in a finite dimensional inner product space $(V, \langle, \rangle)$ states that each linear functional $\phi : V \to \mathbb{F}$ has the form $\phi(x) = \langle x, z \rangle$ for a unique $z \in V$. We call z the **Riesz vector of** ϕ and denote $z = \sharp\phi$. Conversely, we denote $\flat z = \phi$ in this case. The maps $\sharp : V^* \to V$ and $\flat : V \to V^*$ are sometimes called the **musical morphisms** as they provide natural isomorphisms between and inner product space[1] and its dual space.

(a.) Consider $\phi(x, y, z) = 3x - y + z$. Find $\sharp\phi$ for $\mathbb{R}^3$ with the dot-product,

(b.) Consider $\phi(A) = A_{11} + A_{21}$ for $A \in \mathbb{R}^{2 \times 2}$ find $\sharp\phi$.

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[1]actually, this construction still makes sense for a metric space, but, I'm focusing on inner product spaces for your course

Problem 153 Calculate T^* as defined in Definition 10.6.3 in my notes. It is the unique linear tranformation on an inner product space V such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$. For the given V and T find T^*:

(a.) $T(x, y) = (x + 2y, 3y)$ for $V = \mathbb{R}^2$ with the dot-product,

(b.) $T(z_1, z_2) = (3z_1 + iz_2, z_1 + (2 - 7i)z_2)$ for $V = \mathbb{C}^2$ with $\langle z, w \rangle = z^T \overline{w}$

(c.) $T(f) = f' + 3f$ where $f \in P_1(\mathbb{R})$ and $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$

Problem 154 Let V be a complex inner product space and suppose $T : V \to V$ is a skew-hermitian map in the sense $T^* = -T$. Prove the following:

(a.) if T has eigenvalue λ then $\lambda = i\alpha$ for some $\alpha \in \mathbb{R}$ (that is to say, the eigenvalues of T are pure-imaginary)

(b.) if $W_i = \mathrm{Ker}(T - \lambda_i Id_V)$ and $W_j = \mathrm{Ker}(T - \lambda_j Id_V)$ where $\lambda_i \neq \lambda_j$ are distinct e-values of T then $W_i \perp W_j$.

Problem 155 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3 \times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\mathrm{Col}(E_j) = \mathrm{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 u u^T, \qquad E_2 = \lambda_2 v v^T, \qquad E_3 = \lambda_3 w w^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\mathrm{Col}(E_j) = \mathrm{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ and the fact you are given $Au = \lambda_1 u$ etc.

Problem 156 Notice $u = \frac{1}{\sqrt{3}}(1, -1, 1)$ and $v = \frac{1}{\sqrt{2}}(0, 1, 1)$ and $w = \frac{1}{\sqrt{6}}(2, 1, -1)$ form an orthonormal basis for $\mathbb{R}^3$. Find a matrix A with eigenvalues $12, 2, 18$ by making use of the construction of the last problem.

Problem 157 The matrix exponential is defined by

$$e^M = I + M + \frac{1}{2}M^2 + \cdots + \frac{1}{n!}M^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}M^n.$$

The following calculations make the convergence of this series for any A plausible: first, I'll give you an identity you could prove, if $AB = BA$ then $e^{A+B} = e^A e^B$. You need this for (b.).

(a.) if $D = diag(\lambda_1, \ldots, \lambda_n)$ then $e^D = diag(e^{\lambda_1}, \ldots, e^{\lambda_1})$,

(b.) let $J_k(\lambda)$ be the $k \times k$ Jordan block and write $J_k(\lambda) = \lambda I + N$ where N is strictly upper triangular. Calculate $e^{J_k(\lambda)}$, hint: $N^k = 0$ so that piece is finite

(c.) show $A \oplus B$ where $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ has $e^{A \oplus B} = e^A \oplus e^B$,

(d.) show $P^{-1}e^M P = e^{P^{-1}MP}$,

(e.) For $M \in \mathbb{C}^{n \times n}$ there exists P such that $P^{-1}MP = J_{r_1}(\lambda_1) \oplus J_{r_2}(\lambda_2) \oplus \cdots \oplus J_{r_k}(\lambda_k)$. Find a formula for e^M.

Remark: I often give a proof that the matrix exponential exists for any A by an analytical argument in Math 332. The arguments above are probably better since they actually give us a path to calculate e^M provided we know the Jordan form of M.

Problem 158 The problem above is worth double.

Problem 159 One reason the matrix exponential is interesting is its role in relation to the system of ordinary differential equations $\frac{dx}{dt} = Ax$ where $A \in \mathbb{R}^{n \times n}$. In particular, it can be shown that e^{tA} is a **fundamental solution matrix** for $\frac{dx}{dt} = Ax$. This means each column $x_i = \mathrm{col}_i(e^{tA})$ is a solution to $\frac{dx}{dt} = Ax$. The theory of differential equations then states that $x = c_1 x_1 + \cdots + c_n x_n$ forms the general solution to $\frac{dx}{dt} = Ax$.

(a.) If $A = J_3(7)$ then find the general solution to $\frac{dx}{dt} = Ax$.

(b.) If $A = J_2(3) \oplus J_1(1)$ then find the general solution to $\frac{dx}{dt} = Ax$.

Problem 160 In the case A has complex eigenvalues the calculation of the matrix exponential is best accomplished with the help of the real Jordan form. I'll let you contrast the calculation in the 2×2 case for $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$ not both zero.

(a.) find complex P such that $P^{-1}AP = \begin{bmatrix} a + ib & 0 \\ 0 & a - ib \end{bmatrix} = D$,

(b.) note $e^{tA} = e^{t(PDP^{-1})} = Pe^{tD}P^{-1}$ hence calculate e^{tA} (this ought to be real since A is real, somehow the complex quantities all reduce to a real result). *Reminder: if you didn't know already, $e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$*

(c.) Alternatively notice $A = aI + bJ$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and clearly aI commutes with bJ and $J^2 = -I$ so direct calculation of $e^{tA} = e^{t(aI+bJ)}$ goes nicely.

(d.) Find the general solution of $\frac{dx}{dt} = ax - by$ and $\frac{dy}{dt} = bx + ay$ using the spoils of battles already won.

Problem 161 Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ with bases β and γ respective. Also, define dual spaces $V^* = \mathcal{L}(V, \mathbb{R})$ and $W^* = \mathcal{L}(W, \mathbb{R})$. If $T : V \to W$ is a linear transformation and $S : W^* \to V^*$ is defined by

$$(S(\alpha))(v) = \alpha(T(v))$$

for all $\alpha \in W^*$ and $v \in V$. **Then show S is a linear transformation and find $[S]_{\gamma^*, \beta^*}$.** Here, we define dual bases β^* and γ^* as follows: if $\beta = \{f_1, \ldots, f_n\}$ and $\gamma = \{g_1, \ldots, g_m\}$ then $f^j : V \to \mathbb{R}$ and $g^j : W \to \mathbb{R}$ are defined by linearly extending the formulas below:

$$f^j(f_i) = \delta_{ij} \qquad \& \qquad g^j(g_i) = \delta_{ij}.$$

Note, we set-aside the usual notation for exponents in this context; c^i is not the number c raised to the i-th power. A useful lemma is given by the following observation, if $x = \sum_{i=1}^{n} c^i f_i$ then $f^i(x) = c^i$. In other words, the dual vector f^i gives the i-coordinate of x upon evaluation. (your answer should relate the matrix for S to the matrix $[T]_{\beta,\gamma}$)

Problem 162 Consider S and T as in the previous problem once more. Show:

(a.) if T is surjective then S is injective

(b.) if S is injective then T is surjective

(c.) T is an isomorphism iff S is a isomorphism

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Let $A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$. Find an complex eigenbasis for A. Also, construct a real basis β for which $[\beta]^{-1}A[\beta]$ is in real Jordan form.

(II.) Solve $\frac{dx}{dt} = Ax$ where A is the matrix in the previous problem.

(III.) Consider $A = J_4(3)$. Find diagonalizable matrix D and a nilpotent matrix N for which $A = D + N$ and $DN = ND$. Calculate e^{tA} with the help of the $A = D + N$ decomposition.

(IV.) Once more consider $A = J_4(3)$. Let $B = A^2$. What is the Jordan form of B? How is it related to A ?

(V.) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Calculate e^{tA} and write the general solution to $\frac{d\vec{r}}{dt} = A\vec{r}$

(VI.) Suppose $T : V \to V$ has characteristic polynomial $p(x) = (x^2 + 9)^2(x-3)^3$. Make a table which lists the possible real Jordan forms for T. For each case, determine the minimal polynomial. For which case(s) is $T_{\mathbb{C}}$ diagonalizable? For which case(s) is T diagonalizable?

(VII.) Let V be finite dimensional over $\mathbb{F}$. Two linear operators $T, U : V \to V$ are **simultaneously diagonalizable** if there exists a basis β for V such that both $[U]_{\beta,\beta}$ and $[T]_{\beta,\beta}$ are diagonal.

(a.) Prove that simultaneously diagonalizable linear transformations commute; $UT = TU$

(b.) Suppose $T, U : V \to V$ are commuting diagonalizable linear transformations on the finite dimensional vector space V over $\mathbb{F}$. **Prove:** T and U are simultaneously diagonalizable.

Hint (for part (b.) which is considerably more difficult than (a.)): for any eigenvalue λ of T show that $\mathcal{E}_\lambda(T)$ is U-invariant, then notice that the restriction of a diagonalizable linear operator to an of its invariant subspaces is once more diagonalizable. These observations are useful towards the desired argument here.

(VIII.) A **stochastic** or **transition matrix** is a matrix $A \in \mathbb{R}^{n \times n}$ such that $A_{ij} \geq 0$ and

$$A_{1j} + A_{2j} + \cdots + A_{nj} = 1$$

for each $j = 1, 2, \ldots, n$. In words, a transition matrix is a non-negative matrix where each column's entries sum to 1. A vector with non-negative entries which sum to 1 is called a **probabilty vector**. Thus, a transtion matrix is a square matrix formed by concatenating probability vectors. With the above terminology in mind:

(a.) show the product of transition matrices is a transition matrix,

(b.) show the product of a transition matrix and a probablility vector is a probability vector,

(IX.) A **sequence of matrices** is a matrix-valued of $\mathbb{N}$; $n \mapsto A_n$. As with real or complex sequences, we can calculate the $\lim_{n \to \infty} A_n$. It turns out that such a limit exists iff the limit of each component sequence $n \mapsto (A_n)_{ij}$ exist. In particular, $\lim_{n \to \infty} A_n = L$ if and only if $\lim_{n \to \infty} (A_n)_{ij} = L_{ij}$ for all $1 \leq i, j \leq n$. If $\lim_{n \to \infty} A_n = L$ and P, Q are square matrices then it is known:

$$\lim_{n \to \infty} (PA_n) = PL \qquad \& \qquad \lim_{n \to \infty} (A_n Q) = LQ$$

Limits of complex matrices have a few simple guidelines. Let

$$S = \{\lambda \in \mathbb{C} \mid |\lambda| < 1 \text{ or } \lambda = 1\}.$$

In complex analysis you learn $\lim_{n \to \infty} z^n$ exists if and only if $z \in S$. Given some time, you can show: for $A \in \mathbb{C}^{n \times n}$ the $\lim_{n \to \infty} A_n$ exists if and only if the following two conditions hold

(i.) every eigenvalue of A is contained in S,

(ii.) if 1 is an eigenvalue of A then the geometric and algebraic multiplicity of $\lambda = 1$ agree.

Given the discussion above, complete the following:

(a.) if Q is invertible and $\lim_{n \to \infty} B_n = L$ then $\lim_{n \to \infty} Q^{-1} B_n Q = Q^{-1} L Q$,

(b.) if A is diagonalizable and each eigenvalue of A is contained in S then $\lim_{n \to \infty} A^n$ exists.

(c.) Show $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ gives a divergent sequence A^n. Comment on the meaning of this calculation as it relates to (i.) and (ii.).

The Hokage-Level Problem: Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ be the projection defined by $\pi(x) = x - (x \bullet e_j) e_j$ for each $x \in \mathbb{R}^n$ for $j = 1, \ldots, n$. Suppose $\mathcal{P}$ is an $(n-1)$-dimensional paralell-piped which is formed by the convex-hull of $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ suspended at base-point $p \in (0, \infty)^n$;

$$\mathcal{P} = \left\{ p + \sum_{j=1}^{n-1} \alpha_j v_j \;\middle|\; \alpha_j \in [0,1] \;\&\; \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

Let $n \in \mathbb{R}^n$ be a unit-vector in $\{v_1, \cdots, v_{n-1}\}^{\perp}$. The $(n-1)$-**area** of $\mathcal{P}$ is given by $\mathbf{area}(\mathcal{P}) = |\det[v_1| \cdots |v_{n-1}|n]|$. We can study the area of the **shadows** formed by $\mathcal{P}$ on the coordinate hyperplanes. Let $\mathcal{P}_j = \pi_j(\mathcal{P})$ define the shadow of $\mathcal{P}$ on the $x_j = 0$ coordinate plane. Notice,

$$\mathcal{P}_j = \left\{ \pi_j(p) + \sum_{i=1}^{n-1} \alpha_i \pi_j(v_i) \;\middle|\; \alpha_j \in [0,1] \;\&\; \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

which shows $\mathcal{P}_j$ is formed by the convex-hull $\pi_j(v_1), \ldots, \pi_j(v_n)$ of attached at basepoint $\pi_j(p)$. It follows that the $(n-1)$-area of the $\mathcal{P}_j$ can be calculated as follows:

$$\mathbf{area}(\mathcal{P}_j) = |\det[\pi_j(v_1)| \cdots |\pi_j(v_{n-1})|e_j]|.$$

since e_j is perpendicular to $\mathcal{P}_j$. In the case $n = 2$ the 1-dimensional paralell-piped is just a line-segment. For example, if $v_1 = (1,1)$ then $(1/\sqrt{2}, -1/\sqrt{2})$ is perpendicular to v_1 and

$$\det \begin{bmatrix} 1 & 1/\sqrt{2} \\ 1 & -1/\sqrt{2} \end{bmatrix} = -2/\sqrt{2} = -\sqrt{2} \quad \Rightarrow \quad \mathbf{area}(\mathcal{P}) = \sqrt{2}.$$

Of course, this is actually the length of the line-segment. Also, notice

$$\mathbf{area}(\mathcal{P}_1)^2 + \mathbf{area}(\mathcal{P}_2)^2 = 1^2 + 1^2 = \sqrt{2}^2 = \mathbf{area}(\mathcal{P})^2.$$

This is not suprising. However, perhaps the fact this generalizes to n-dimensions in the following sense is not already known to you:

$$\mathbf{area}(\mathcal{P}_1)^2 + \mathbf{area}(\mathcal{P}_2)^2 + \cdots \mathbf{area}(\mathcal{P}_n)^2 = \mathbf{area}(\mathcal{P})^2$$

Prove it. You might call this the generalized Pythagorean identity, I'm not sure its history or formal name. That said, the formula I give for generalized area could just as well be termed generalized volume. Also, you could **define**

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{bmatrix} v_1 \mid v_2 \mid \cdots \mid v_{n-1} & \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{matrix} \end{bmatrix} \in \mathbb{R}^n$$

where we insist the determinant is calculated via the Laplace expansion by minors along the last column. You can show $v_1 \times v_2 \times \cdots \times v_{n-1} \in \{v_1, \ldots, v_{n-1}\}^\perp$. But, if n is a unit-vector which spans $\{v_1, \ldots, v_{n-1}\}^\perp$ then the $(n-1)$-ry cross-product must be a vector parallel to n and thus:

$$v_1 \times v_2 \times \cdots \times v_{n-1} = [(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n]\, n$$

Note, $n \bullet n = 1$ as we assumed n is unit-vector and we can show

$$(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n = \det[v_1|v_2| \ldots |v_{n-1}|n]$$

Notice this generalized cross-product is just an extension of the heurstic determinant commonly used in multivariate calculus to define the standard cross-product. In particular, the following is equivalent to the column-based definition

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_n \\ & & v_1^T & \\ & & v_2^T & \\ & & \vdots & \\ & & v_{n-1}^T & \end{bmatrix}$$

where we insist the determinant is calculated via the Laplace expansion by minors along the first row. In any event, my point in this discussion is merely that we can calculate higher-dimensional volumes with determinants and these go hand-in-hand with generalized tertiary cross-products. In particular,

$$||v_1 \times v_2 \times \cdots \times v_{n-1}|| = \mathbf{vol}(\mathcal{P})$$

where $\mathcal{P}$ is formed by the convex hull of $v_1, \ldots, v_{n-1}$. When $n = 2$ this gives vector length, when $n = 3$ this is the familar result that the area of the parallelogram with sides $\vec{A}, \vec{B}$ is just $||\vec{A} \times \vec{B}||$.