Please follow the format which was announced in Blackboard. Thanks!

✓ **Problem 145** Find the best-fit line to the data $(1,1),(2,4),(3,0),(4,5),(10,6)$ via the method of least squares as discussed in my notes.

✓ **Problem 146** Suppose $Q(x,y) = 5x^2 + 5y^2 + 8xy$. Write $Q(v) = v^T A v$ for a symmetric matrix A. Find an orthonormal eigenbasis $\beta = \{u_1, u_2\}$ for A and find coordinates $\bar{x}, \bar{y}$ such that $v = \bar{x}u_1 + \bar{y}u_2$ gives $Q(v) = \lambda_1 \bar{x}^2 + \lambda_2 \bar{y}^2$.

✓ **Problem 147** Suppose $Q(x,y,z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz$. Write $Q(v) = v^T A v$ for a symmetric matrix A. Find an orthonormal eigenbasis for A and find coordinates $\bar{x}, \bar{y}, \bar{z}$ for which $Q(v) = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$.
Hint: for this question to make sense, it must be that the matrix of Q has e-values $1, 1, 10$.

✓ **Problem 148** Calculus of functions of several variables is best understood with the aid of linear algebra. In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function (meaning you can take as many partial derivatives as you wish) then the multivariate Taylor theorem tells us:

$$f(x) = f(p) + (\nabla f)(p) \cdot (x-p) + H(p)(x-p) + \cdots$$

where ∇f is the **gradient** of f at p and $H(p)$ is the **Hessian** of f at p. Both the gradient and Hessian are assembled from appropriate partial derivatives:

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \qquad \& \qquad [H]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Notice, if p is a **critical point** then $\nabla f(p) = 0$ and hence $f(x) = f(p) + H(p)(x-p) + \cdots$ thus the behavior of f is dominated by $H(p)$ for x near p. In fact, $H(p)(x-p) = (x-p)^T [H](x-p)$ where $[H]^T = [H]$ so the quadratic term is a quadratic form. It follows there exist eigencoordinates $y_1, \dots, y_n$ with respect to the eigenbasis $\{v_1, \dots, v_n\}$ for $[H]$ such that $y = y_1 v_1 + \cdots + y_n v_n = x - p$ and

$$H(p)(x-p) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \qquad \star.$$

It is then simple to judge whether a given critical point provides a local minimum or maximum for the function. Looking at $\star$,

(i.) if $\lambda_1, \dots, \lambda_n > 0$ then $f(p)$ is a local minimum.

(ii.) if $\lambda_1, \dots, \lambda_n < 0$ then $f(p)$ is a local maximum.

(iii.) if $\lambda_1, \dots, \lambda_n$ are all nonzero, yet differ in sign, then $f(p)$ neither local max nor min.

Unfortunately, if the spectrum of the Hessian includes eigenvalue zero then we are unable to offer a conclusion. In that case the third order terms could go either way.

Find critical points for the functions below and analyze the eigenvalues of the Hessian to classify the nature of the critical points as either max, min or saddle (case (iii.) is known as a saddle point)

(a.) let $f(x,y) = 5x^2 + 8xy - 10x + 5y^2 - 8y + 5$,

(b.) let $f(x,y,z) = x^2 + y^2 + z^2 + 4xy + 4xz + 4yz$

Remark: if you never saw partial differentiation in previous course work, I am happy to teach you in office hours. Or, ask me in class.

✓ **Problem 149** Let $A = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$.

(a.) diagonalize A,

(b.) calculate $\lim_{n\to\infty} A^n$,

(c.) let $x_o = (0.7, 0.3)$. Define $x_n = A^n x_o$ hence $x_1 = Ax_o$ and $x_2 = Ax_1 = AAx_o$ etc. Calculate x_1, x_{10} and x_{100}. What is $\lim_{n\to\infty} x_n$? How does this relate to things you found in (a.)

The vectors you find in (c.) are an example of a **Markov chain**. Notice A is a transition matrix and x_o is a probability vector.

✓ **Problem 150** A matrix $A \in \mathbb{R}^{n\times n}$ is called **normal** if $A^TA = AA^T$.

(a.) show a symmetric matrix is normal,

(b.) find an example of a 2×2 matrix which is normal, but, not symmetric,

(c.) show if $A \in \mathbb{R}^{n\times n}$ is normal then $||Ax|| = ||A^Tx||$ for all $x \in \mathbb{R}^n$,

(d.) show if $A \in \mathbb{R}^{n\times n}$ is normal then $A - cI$ is normal for all $c \in \mathbb{R}$,

(e.) show if $\lambda \in \mathbb{R}$ is e-value of normal matrix $A \in \mathbb{R}^{n\times n}$ then λ is also an e-value of A^T

(f.) show if λ_1, λ_2 are distinct real e-valued of a normal matrix $A \in \mathbb{R}^{n\times n}$ then the corresponding e-vectors are orthogonal.

✓ **Problem 151** The last problem counts double (aka Problem 149 does not exist)

✓ **Problem 152** The Riesz Representation Theorem in a finite dimensional inner product space $(V, \langle,\rangle)$ states that each linear functional $\phi : V \to \mathbb{F}$ has the form $\phi(x) = \langle x, z\rangle$ for a unique $z \in V$. We call z the **Riesz vector of** ϕ and denote $z = \sharp\phi$. Conversely, we denote $\flat z = \phi$ in this case. The maps $\sharp : V^* \to V$ and $\flat : V \to V^*$ are sometimes called the **musical morphisms** as they provide natural isomorphisms between and inner product space[1] and its dual space.

(a.) Consider $\phi(x,y,z) = 3x - y + z$. Find $\sharp\phi$ for $\mathbb{R}^3$ with the dot-product,

(b.) Consider $\phi(A) = A_{11} + A_{21}$ for $A \in \mathbb{R}^{2\times 2}$ find $\sharp\phi$.

[1]actually, this construction still makes sense for a metric space, but, I'm focusing on inner product spaces for your course

Problem 153 Calculate T^* as defined in Definition 10.6.3 in my notes. It is the unique linear tranformation on an inner product space V such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. For the given V and T find T^*:

(a.) $T(x, y) = (x + 2y, 3y)$ for $V = \mathbb{R}^2$ with the dot-product,

(b.) $T(z_1, z_2) = (3z_1 + iz_2, z_1 + (2 - 7i)z_2)$ for $V = \mathbb{C}^2$ with $\langle z, w\rangle = z^T\overline{w}$

(c.) $T(f) = f' + 3f$ where $f \in P_1(\mathbb{R})$ and $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt$

Problem 154 Let V be a complex inner product space and suppose $T : V \to V$ is a skew-hermitian map in the sense $T^* = -T$. Prove the following:

(a.) if T has eigenvalue λ then $\lambda = i\alpha$ for some $\alpha \in \mathbb{R}$ (that is to say, the eigenvalues of T are pure-imaginary)

(b.) if $W_i = \text{Ker}(T - \lambda_i Id_V)$ and $W_j = \text{Ker}(T - \lambda_j Id_V)$ where $\lambda_i \neq \lambda_j$ are distinct e-values of T then $W_i \perp W_j$.

Problem 155 There is another aspect of the real spectral theorem we should explore. For example, if $A^T = A$ for $A \in \mathbb{R}^{3\times 3}$ then there exist rank one matrices E_1, E_2, E_3 for which

$$A = E_1 + E_2 + E_3$$

and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. Suppose u, v, w form an orthonormal eigenbasis for A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respective. Define:

$$E_1 = \lambda_1 uu^T, \qquad E_2 = \lambda_2 vv^T, \qquad E_3 = \lambda_3 ww^T$$

Show: $E_1 + E_2 + E_3 = A$ and $\text{Col}(E_j) = \text{Null}(A - \lambda_j I)$ for $j = 1, 2, 3$.
Hint: use the orthonormality of $\{u, v, w\}$ *and the fact you are given* $Au = \lambda_1 u$ *etc.*

Problem 156 Notice $u = \frac{1}{\sqrt{3}}(1, -1, 1)$ and $v = \frac{1}{\sqrt{2}}(0, 1, 1)$ and $w = \frac{1}{\sqrt{6}}(2, 1, -1)$ form an orthonormal basis for $\mathbb{R}^3$. Find a matrix A with eigenvalues $12, 2, 18$ by making use of the construction of the last problem.

Problem 157 The matrix exponential is defined by

$$e^M = I + M + \frac{1}{2}M^2 + \cdots + \frac{1}{n!}M^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}M^n.$$

The following calculations make the convergence of this series for any A plausible: first, I'll give you an identity you could prove, if $AB = BA$ then $e^{A+B} = e^A e^B$. You need this for (b.).

(a.) if $D = diag(\lambda_1, \ldots, \lambda_n)$ then $e^D = diag(e^{\lambda_1}, \ldots, e^{\lambda_1})$,

(b.) let $J_k(\lambda)$ be the $k \times k$ Jordan block and write $J_k(\lambda) = \lambda I + N$ where N is strictly upper triangular. Calculate $e^{J_k(\lambda)}$, *hint:* $N^k = 0$ *so that piece is finite*

(c.) show $A \oplus B$ where $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ has $e^{A\oplus B} = e^A \oplus e^B$, (d.) show $P^{-1}e^MP = e^{P^{-1}MP}$,

(e.) For $M \in \mathbb{C}^{n\times n}$ there exists P such that $P^{-1}MP = J_{r_1}(\lambda_1) \oplus J_{r_2}(\lambda_2) \oplus \cdots \oplus J_{r_k}(\lambda_k)$. Find a formula for e^M.

Remark: I often give a proof that the matrix exponential exists for any A by an analytical argument in Math 332. The arguments above are probably better since they actually give us a path to calculate e^M provided we know the Jordan form of M.

Problem 158 The problem above is worth double.

Problem 159 One reason the matrix exponential is interesting is its role in relation to the system of ordinary differential equations $\frac{dx}{dt} = Ax$ where $A \in \mathbb{R}^{n\times n}$. In particular, it can be shown that e^{tA} is a **fundamental solution matrix** for $\frac{dx}{dt} = Ax$. This means each column $x_i = \text{col}_i(e^{tA})$ is a solution to $\frac{dx}{dt} = Ax$. The theory of differential equations then states that $x = c_1x_1 + \cdots + c_nx_n$ forms the general solution to $\frac{dx}{dt} = Ax$.

(a.) If $A = J_3(7)$ then find the general solution to $\frac{dx}{dt} = Ax$.

(b.) If $A = J_2(3) \oplus J_1(1)$ then find the general solution to $\frac{dx}{dt} = Ax$.

Problem 160 In the case A has complex eigenvalues the calculation of the matrix exponential is best accomplished with the help of the real Jordan form. I'll let you contrast the calculation in the 2×2 case for $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$ not both zero.

(a.) find complex P such that $P^{-1}AP = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix} = D$,

(b.) note $e^{tA} = e^{t(PDP^{-1})} = Pe^{tD}P^{-1}$ hence calculate e^{tA} (this ought to be real since A is real, somehow the complex quantities all reduce to a real result). *Reminder: if you didn't know already,* $e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$

(c.) Alternatively notice $A = aI + bJ$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and clearly aI commutes with bJ and $J^2 = -I$ so direct calculation of $e^{tA} = e^{t(aI+bJ)}$ goes nicely.

(d.) Find the general solution of $\frac{dx}{dt} = ax - by$ and $\frac{dy}{dt} = bx + ay$ using the spoils of battles already won.

✓**Problem 161** Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ with bases β and γ respective. Also, define dual spaces $V^* = \mathcal{L}(V,\mathbb{R})$ and $W^* = \mathcal{L}(W,\mathbb{R})$. If $T: V \to W$ is a linear transformation and $S: W^* \to V^*$ is defined by

$$(S(\alpha))(v) = \alpha(T(v))$$

for all $\alpha \in W^*$ and $v \in V$. **Then show S is a linear transformation and find $[S]_{\gamma^*,\beta^*}$.** Here, we define dual bases β^* and γ^* as follows: if $\beta = \{f_1, \dots, f_n\}$ and $\gamma = \{g_1, \dots, g_m\}$ then $f^j : V \to \mathbb{R}$ and $g^j : W \to \mathbb{R}$ are defined by linearly extending the formulas below:

$$f^j(f_i) = \delta_{ij} \qquad \& \qquad g^j(g_i) = \delta_{ij}.$$

Note, we set-aside the usual notation for exponents in this context; c^i is not the number c raised to the i-th power. A useful lemma is given by the following observation, if $x = \sum_{i=1}^{n} c^i f_i$ then $f^i(x) = c^i$. In other words, the dual vector f^i gives the i-coordinate of x upon evaluation. (your answer should relate the matrix for S to the matrix $[T]_{\beta,\gamma}$)

✓**Problem 162** Consider S and T as in the previous problem once more. Show:

(a.) if T is surjective then S is injective

(b.) if S is injective then T is surjective

(c.) T is an isomorphism iff S is a isomorphism

Remark: the problems below are not handed in, but, I almost assigned them. If you need further practice, perhaps it would be wise to work these. I am happy to discuss them in the Help Session.

(I.) Let $A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$. Find an complex eigenbasis for A. Also, construct a real basis β for which $[\beta]^{-1}A[\beta]$ is in real Jordan form.

(II.) Solve $\frac{dx}{dt} = Ax$ where A is the matrix in the previous problem.

(III.) Consider $A = J_4(3)$. Find diagonalizable matrix D and a nilpotent matrix N for which $A = D + N$ and $DN = ND$. Calculate e^{tA} with the help of the $A = D + N$ decomposition.

(IV.) Once more consider $A = J_4(3)$. Let $B = A^2$. What is the Jordan form of B? How is it related to A ?

(V.) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Calculate e^{tA} and write the general solution to $\frac{d\vec{r}}{dt} = A\vec{r}$

(VI.) Suppose $T : V \to V$ has characteristic polynomial $p(x) = (x^2+9)^2(x-3)^3$. Make a table which lists the possible real Jordan forms for T. For each case, determine the minimal polynomial. For which case(s) is $T_{\mathbb{C}}$ diagonalizable? For which case(s) is T diagonalizable?

(VII.) Let V be finite dimensional over $\mathbb{F}$. Two linear operators $T, U : V \to V$ are **simultaneously diagonalizable** if there exists a basis β for V such that both $[U]_{\beta,\beta}$ and $[T]_{\beta,\beta}$ are diagonal.

(a.) Prove that simultaneously diagonalizable linear transformations commute; $UT = TU$

(b.) Suppose $T, U : V \to V$ are commuting diagonalizable linear transformations on the finite dimensional vector space V over $\mathbb{F}$. **Prove:** T and U are simultaneously diagonalizable.

Hint (for part (b.) which is considerably more difficult than (a.)): for any eigenvalue λ of T show that $\mathcal{E}_\lambda(T)$ is U-invariant, then notice that the restriction of a diagonalizable linear operator to an of its invariant subspaces is once more diagonalizable. These observations are useful towards the desired argument here.

(VIII.) A **stochastic** or **transition matrix** is a matrix $A \in \mathbb{R}^{n\times n}$ such that $A_{ij} \geq 0$ and

$$A_{1j} + A_{2j} + \cdots + A_{nj} = 1$$

for each $j = 1, 2, \ldots, n$. In words, a transition matrix is a non-negative matrix where each column's entries sum to 1. A vector with non-negative entries which sum to 1 is called a **probabilty vector**. Thus, a transtion matrix is a square matrix formed by concatenating probability vectors. With the above terminology in mind:

(a.) show the product of transition matrices is a transition matrix,

(b.) show the product of a transition matrix and a probablility vector is a probability vector,

(IX.) A **sequence of matrices** is a matrix-valued of $\mathbb{N}$; $n \mapsto A_n$. As with real or complex sequences, we can calculate the $\lim_{n\to\infty} A_n$. It turns out that such a limit exists iff the limit of each component sequence $n \mapsto (A_n)_{ij}$ exist. In particular, $\lim_{n\to\infty} A_n = L$ if and only if $\lim_{n\to\infty}(A_n)_{ij} = L_{ij}$ for all $1 \leq i, j \leq n$. If $\lim_{n\to\infty} A_n = L$ and P, Q are square matrices then it is known:

$$\lim_{n\to\infty}(PA_n) = PL \qquad \& \qquad \lim_{n\to\infty}(A_nQ) = LQ$$

Limits of complex matrices have a few simple guidelines. Let

$$S = \{\lambda \in \mathbb{C} \mid |\lambda| < 1 \text{ or } \lambda = 1\}.$$

In complex analysis you learn $\lim_{n\to\infty} z^n$ exists if and only if $z \in S$. Given some time, you can show: for $A \in \mathbb{C}^{n\times n}$ the $\lim_{n\to\infty} A_n$ exists if and only if the following two conditions hold

(i.) every eigenvalue of A is contained in S,

(ii.) if 1 is an eigenvalue of A then the geometric and algebraic multiplicity of $\lambda = 1$ agree.

Given the discussion above, complete the following:

(a.) if Q is invertible and $\lim_{n\to\infty} B_n = L$ then $\lim_{n\to\infty} Q^{-1}B_nQ = Q^{-1}LQ$,

(b.) if A is diagonalizable and each eigenvalue of A is contained in S then $\lim_{n\to\infty} A^n$ exists.

(c.) Show $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ gives a divergent sequence A^n. Comment on the meaning of this calculation as it relates to (i.) and (ii.).

The Hokage-Level Problem: Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ be the projection defined by $\pi(x) = x - (x \bullet e_j)e_j$ for each $x \in \mathbb{R}^n$ for $j = 1, \ldots, n$. Suppose $\mathcal{P}$ is an $(n-1)$-dimensional paralell-piped which is formed by the convex-hull of $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ suspended at base-point $p \in (0, \infty)^n$;

$$\mathcal{P} = \left\{ p + \sum_{j=1}^{n-1} \alpha_j v_j \;\middle|\; \alpha_j \in [0,1] \;\&\; \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

Let $n \in \mathbb{R}^n$ be a unit-vector in $\{v_1, \cdots, v_{n-1}\}^{\perp}$. The $(n-1)$-**area** of $\mathcal{P}$ is given by **area**$(\mathcal{P}) = |\det[v_1|\cdots|v_{n-1}|n]|$. We can study the area of the **shadows** formed by $\mathcal{P}$ on the coordinate hyperplanes. Let $\mathcal{P}_j = \pi_j(\mathcal{P})$ define the shadow of $\mathcal{P}$ on the $x_j = 0$ coordinate plane. Notice,

$$\mathcal{P}_j = \left\{ \pi_j(p) + \sum_{i=1}^{n-1} \alpha_i \pi_j(v_i) \;\middle|\; \alpha_j \in [0,1] \;\&\; \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

which shows $\mathcal{P}_j$ is formed by the convex-hull $\pi_j(v_1), \dots, \pi_j(v_n)$ of attached at basepoint $\pi_j(p)$. It follows that the $(n-1)$-area of the $\mathcal{P}_j$ can be calculated as follows:

$$\textbf{area}(\mathcal{P}_j) = |\det[\pi_j(v_1)|\cdots|\pi_j(v_{n-1})|e_j]|.$$

since e_j is perpendicular to $\mathcal{P}_j$. In the case $n = 2$ the 1-dimensional paralell-piped is just a line-segment. For example, if $v_1 = (1, 1)$ then $(1/\sqrt{2}, -1/\sqrt{2})$ is perpendicular to v_1 and

$$\det \begin{bmatrix} 1 & 1/\sqrt{2} \\ 1 & -1/\sqrt{2} \end{bmatrix} = -2/\sqrt{2} = -\sqrt{2} \quad \Rightarrow \quad \textbf{area}(\mathcal{P}) = \sqrt{2}.$$

Of course, this is actually the length of the line-segment. Also, notice

$$\textbf{area}(\mathcal{P}_1)^2 + \textbf{area}(\mathcal{P}_2)^2 = 1^2 + 1^2 = \sqrt{2}^2 = \textbf{area}(\mathcal{P})^2.$$

This is not suprising. However, perhaps the fact this generalizes to n-dimensions in the following sense is not already known to you:

$$\textbf{area}(\mathcal{P}_1)^2 + \textbf{area}(\mathcal{P}_2)^2 + \cdots \textbf{area}(\mathcal{P}_n)^2 = \textbf{area}(\mathcal{P})^2$$

Prove it. You might call this the generalized Pythagorean identity, I'm not sure its history or formal name. That said, the formula I give for generalized area could just as well be termed generalized volume. Also, you could **define**

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \left[\begin{array}{c|c|c|c|c} & & & & e_1 \\ & & & & e_2 \\ v_1 & v_2 & \cdots & v_{n-1} & \vdots \\ & & & & e_n \end{array} \right] \in \mathbb{R}^n$$

where we insist the determinant is calculated via the Laplace expansion by minors along the last column. You can show $v_1 \times v_2 \times \cdots \times v_{n-1} \in \{v_1, \dots, v_{n-1}\}^\perp$. But, if n is a unit-vector which spans $\{v_1, \dots, v_{n-1}\}^\perp$ then the $(n-1)$-ry cross-product must be a vector parallel to n and thus:

$$v_1 \times v_2 \times \cdots \times v_{n-1} = [(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n]\, n$$

Note, $n \bullet n = 1$ as we assumed n is unit-vector and we can show

$$(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n = \det[v_1|v_2|\dots|v_{n-1}|n]$$

Notice this generalized cross-product is just an extension of the heurstic determinant commonly used in multivariate calculus to define the standard cross-product. In particular, the following is equivalent to the column-based definition

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_n \\ & v_1^T & & \\ & v_2^T & & \\ & \vdots & & \\ & v_{n-1}^T & & \end{bmatrix}$$

where we insist the determinant is calculated via the Laplace expansion by minors along the first row. In any event, my point in this discussion is merely that we can calculate higher-dimensional volumes with determinants and these go hand-in-hand with generalized tertiary cross-products. In particular,

$$||v_1 \times v_2 \times \cdots \times v_{n-1}|| = \textbf{vol}(\mathcal{P})$$

where $\mathcal{P}$ is formed by the convex hull of $v_1, \dots, v_{n-1}$. When $n = 2$ this gives vector length, when $n = 3$ this is the familar result that the area of the parallelogram with sides $\vec{A}, \vec{B}$ is just $||\vec{A} \times \vec{B}||$.

P145 Find best-fit line through data $(1,1), (2,4), (3,0), (4,5), (10,6)$ via least squares method.

We have model $y = c_1 x + c_2$ hence data suggests,

$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + c_2 &= 4 \\ 3c_1 + c_2 &= 0 \\ 4c_1 + c_2 &= 5 \\ 10c_1 + c_2 &= 6 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 10 & 1 \end{bmatrix}}_{M} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 0 \\ 5 \\ 6 \end{bmatrix}}_{b}$$

$Mc = b$ has best approximate sol$^{\underline{n}}$ via $M^T M c = M^T b$ sol$^{\underline{n}}$,

$$M^T M = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 10 & 1 \end{bmatrix} = \begin{bmatrix} 130 & 20 \\ 20 & 5 \end{bmatrix}$$

Hence $(M^T M)^{-1} = \dfrac{1}{250} \begin{bmatrix} 5 & -20 \\ -20 & 130 \end{bmatrix}$ thus,

$$c = \frac{1}{250} \begin{bmatrix} 5 & -20 \\ -20 & 130 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 0 \\ 5 \\ 6 \end{bmatrix}$$

$$c = \frac{1}{250} \begin{bmatrix} 5 & -20 \\ -20 & 130 \end{bmatrix} \begin{bmatrix} 89 \\ 16 \end{bmatrix}$$

$$c = \frac{1}{250} \begin{bmatrix} 125 \\ 300 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 6/5 \end{bmatrix}$$

Thus $\boxed{y = \frac{1}{2}x + \frac{6}{5} = 0.5x + 1.2 \text{ is best fit line}}$

- (we use Th$^{\underline{m}}$ that $Ax = b$ has best approximate sol$^{\underline{n}}$ from the normal eq$^{\underline{n}}$s $A^T A x = A^T b$.) -
(Here $A = M$)

$\boxed{P146}$ $Q(x,y) = 5x^2 + 5y^2 + 8xy$

$$Q(v) = v^T \underbrace{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}_{A} v$$

1.) Calculate Eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = (\lambda-5)^2 - 4^2$$
$$= (\lambda-9)(\lambda-1)$$
$$\Rightarrow \underline{\lambda_1 = 1} \ \ \& \ \ \underline{\lambda_2 = 9}$$

2.) Find orthonormal e-basis for A,

$$A - I = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \Rightarrow u_1 = \frac{1}{\sqrt{2}}(1,-1)$$

$$A - 9I = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \Rightarrow u_2 = \frac{1}{\sqrt{2}}(1,1)$$

Remark: in principle 2×2 symmetric is extra nice because spectral Thm implies $\mathcal{E}_1 \perp \mathcal{E}_2$ so you know $\mathcal{E}_1 = \text{span}\{u_1\}$ & $\mathcal{E}_2 = \text{span}\{u_2\}$ have $u_1 \perp u_2$. Once we find u_1 we can simply rotate $90°$ to find u_2!

$$\beta = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \therefore \quad [\beta] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in SO(2).$$

3.) construct eigencoordinates $\bar{x}, \bar{y}$ s.t. $(x,y) = \bar{x}u_1 + \bar{y}u_2 = v$
$$= [u_1|u_2]\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = [\beta]^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = [\beta]^T\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \boxed{\begin{aligned} \bar{x} &= \tfrac{1}{\sqrt{2}}(x-y) \\ \bar{y} &= \tfrac{1}{\sqrt{2}}(x+y) \end{aligned}}$$

these provide $\underline{Q(v) = \bar{x}^2 + 9\bar{y}^2}$.

$\boxed{P147}$ Writing $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we find,

$$Q(x,y,z) = 5x^2 + 5y^2 + 2z^2 + 8xy + 4xz + 4yz = v^T \underbrace{\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}}_{A} v$$

It was given A has $\lambda_1 = 1$, $\lambda_2 = 10$.
Hence work towards finding orthonormal bases for $\mathcal{E}_1 = \text{Null}(A - I)$ and $\mathcal{E}_2 = \text{Null}(A - 10I)$

1.) $A - I = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $(u,v,w) \in \mathcal{E}_1$ has $w = -2u - 2v$

$\therefore \mathcal{E}_1 = \text{span}\{\underbrace{(1,0,-2)}_{u_1}, \underbrace{(0,1,-2)}_{u_2}\}$

Run G.S.A. on $\{u_1, u_2\}$,

$$u_1'' = \frac{1}{\sqrt{5}}(1,0,-2)$$

$$u_2' = u_2 - (u_2 \cdot u_1'')u_1'' = (0,1,-2) - (4/5)(1,0,-2)$$

$$u_2' = (-4/5, 1, -2 + 8/5) = \left(\frac{-4}{5}, \frac{5}{5}, \frac{-2}{5}\right) = \frac{1}{5}(-4,5,-2)$$

$$\therefore \ u_2'' = \frac{1}{\sqrt{45}}(-4,5,-2)$$

Then $\mathcal{E}_1 = \text{span}\{u_1'', u_2''\}$ as above.

2.) $A - 10I = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} -20 & 16 & 8 \\ 20 & -25 & 10 \\ 20 & 20 & -80 \end{bmatrix} \sim \begin{bmatrix} -20 & 16 & 8 \\ 0 & -9 & 18 \\ 0 & 36 & -72 \end{bmatrix}$

$\sim \begin{bmatrix} -20 & 16 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -20 & 0 & 40 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $(u,v,w) \in \mathcal{E}_2$ has $u = 2w$, $v = 2w$

Hence $\mathcal{E}_2 = \text{span}\{\underbrace{\tfrac{1}{3}(2,2,1)}_{u_3''}\}$.

3.) $\underbrace{\beta = \left\{\frac{1}{\sqrt{5}}(1,0,-2), \frac{1}{\sqrt{45}}(-4,5,-2), \frac{1}{3}(2,2,1)\right\}}_{\text{orthonormal eigenbasis for } A}$

P 147 continued

$$\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix} = [\beta]^{-1}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = [\beta]^{T}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$ as $[\beta]^T = [\beta]^{-1}$ for orthonormal basis.

Thus

$$\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{5}}[1, 0, -2]\\ \frac{1}{3\sqrt{5}}[-4, 5, -2]\\ \frac{1}{3}[2, 2, 1]\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{\sqrt{5}}(x - 2z)\\ \frac{1}{3\sqrt{5}}(-4x + 5y - 2z)\\ \frac{1}{3}(2x + 2y + z)\end{bmatrix}$$

That is,

$$\boxed{\begin{aligned}\bar{x} &= \frac{1}{\sqrt{5}}(x - 2z)\\ \bar{y} &= \frac{1}{3\sqrt{5}}(-4x + 5y - 2z)\\ \bar{z} &= \frac{1}{3}(2x + 2y + z)\end{aligned}}$$

these provide,

$$Q(\bar{x}u_1'' + \bar{y}u_2'' + \bar{z}u_3'') = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2$$

P148 Find critical points and analyze e-values of the Hessian to classify each ~~extrea~~ critical point as min/max or saddle.

(a.) $f(x,y) = 5x^2 + 8xy - 10x + 5y^2 - 8y + 5$

(b.) $f(x,y,z) = x^2 + y^2 + z^2 + 4xy + 4xz + 4yz$

(a.) $\nabla f = \langle f_x, f_y \rangle = \langle 10x + 8y - 10,\ 8x + 10y - 8 \rangle$

$$\underbrace{\nabla f = \langle 0, 0 \rangle}_{\text{critical pt. condition}} \quad \longleftrightarrow \quad \left.\begin{array}{l} 10x + 8y = 10 \\ 8x + 10y = 8 \end{array}\right\} \text{ has sol}^{\underline{n}} \ (1,0).$$

Hence $(1,0)$ is only critical point. Generally,

Remark: generally evaluating at $(1,0)$ is more interesting.

$$[Q] = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \underbrace{\begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}}_{A}$$

Then,

$$\det[A - \lambda I] = \det\begin{bmatrix} 10-\lambda & 8 \\ 8 & 10-\lambda \end{bmatrix} = (\lambda - 10)^2 - 8^2$$
$$= (\lambda - 10 - 8)(\lambda - 10 + 8)$$
$$= (\lambda - 18)(\lambda - 2)$$

Hence $\lambda_1 = 18$, $\lambda_2 = 2 \Rightarrow \boxed{f(1,0) = 0 \text{ is local min.}}$

(b.) $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x + 4y + 4z,\ 2y + 4x + 4z,\ 2z + 4x + 4y \rangle$

$$\nabla f = 0 \quad \longleftrightarrow \quad \underbrace{\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}}_{B} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\det B = 2(4-16) - 4(8-16) + 4(16-8)$
$= -24 + 32 + 32 \neq 0$

Thus $(0,0,0)$ is only critical pt.

$$\text{Consider } [Q] = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}}_{A}$$

funny.

P148 continued

we need to calculate the spectrum of A,

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-x & 4 & 4 \\ 4 & 2-x & 4 \\ 4 & 4 & 2-x \end{bmatrix}$$

decided to use $\lambda = x$ instead for change of pace ☺

$r_3 - r_2$

$$= \det\begin{bmatrix} 2-x & 4 & 4 \\ 4 & 2-x & 4 \\ 0 & x+2 & -2-x \end{bmatrix}$$

: since $4-(2-x) = x+2$ and $2-x-4 = -(x+2)$

$r_2 + r_3$

$$= \det\begin{bmatrix} 2-x & 4 & 4 \\ 4 & 0 & -x+2 \\ 0 & x+2 & -2-x \end{bmatrix}$$

$$= (2-x)(x+2)(-x+2)(-1) - 4(4)(-2-x) + 4(4(x+2))$$

$$= (x+2)\left[(2-x)(2+x) + 16 + 16\right]$$

$$= (x+2)\left[-x^2 + 4 + 32\right]$$

$$= (x+2)(36-x^2)$$

$$= (x+2)(6-x)(6+x) \quad \therefore \quad \underline{\lambda_1 = -2,\ \lambda_2 = 6,\ \lambda_3 = -6}.$$

Thus $\underline{f(0,0,0) = 0 \text{ is at a saddle point.}}$ (neither local min nor max)

Remark: I'm not certain my row-operations were super helpful. But, it would be wise to review to see what can help with the $\det(A-\lambda I) = 0$ calculation. Certainly row-swaps are permitted and row-add. & rescalings do not change the sol$^{\underline{n}}$ set for $\det(A-\lambda I) = 0$. Think about it...

$\boxed{P149}$ $A = \frac{1}{100}\begin{bmatrix} 90 & 2 \\ 10 & 98 \end{bmatrix}$

(a.) $\det(A - \lambda I) = \det\begin{bmatrix} 0.9-\lambda & 0.02 \\ 0.1 & 0.98-\lambda \end{bmatrix}$

$= (\lambda - 0.9)(\lambda - 0.98) - 0.1(0.02)$

$= \lambda^2 - 1.88\lambda + 0.882 - 0.002$

$= \lambda^2 - 1.88\lambda + 0.880$

$= (\lambda - 1)(\lambda - 0.88) \quad \therefore \underline{\lambda_1 = 1, \ \lambda_2 = 0.88}.$

$A - I = \begin{bmatrix} -0.1 & 0.02 \\ 0.1 & -0.02 \end{bmatrix} \sim \begin{bmatrix} 10 & -2 \\ 0 & 0 \end{bmatrix} \quad \therefore \ \mathcal{E}_1 = \text{span}\{(1,5)\}$

$A - 0.88I = \begin{bmatrix} 0.02 & 0.02 \\ 0.1 & 0.1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \therefore \ \mathcal{E}_2 = \text{span}\{(1,-1)\}$

Hence $\beta = \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is e-basis for A and

we can calculate $[\beta]^{-1} A [\beta] = \begin{bmatrix} 1 & 0 \\ 0 & 0.88 \end{bmatrix} = D$.

(b.) So, $A^n = [\beta] D^n [\beta]^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.88 \end{bmatrix}^n \frac{1}{-6} \begin{bmatrix} -1 & -1 \\ -5 & 1 \end{bmatrix}$

$= \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.88^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & -1 \end{bmatrix}$

$= \frac{1}{6} \begin{bmatrix} 1 & (0.88)^n \\ 5 & -(0.88)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & -1 \end{bmatrix}$

$= \frac{1}{6} \left[\begin{array}{c|c} 1 + 5(0.88)^n & 1 - (0.88)^n \\ \hline 5 - 5(0.88)^n & 5 + (0.88)^n \end{array} \right]$

Thus $\lim_{n \to \infty} (A^n) = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$.

P149 continued

(c.) Let $X_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

Then $X_n = A^n X_0 \Rightarrow X_1 = AX_0, \quad X_2 = AX_1 = A^2 X_0$ etc.

$$X_1 = AX_0 = \frac{1}{100}\begin{bmatrix} 90 & 2 \\ 10 & 98 \end{bmatrix}\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$= \frac{1}{1000}\begin{bmatrix} 90 & 2 \\ 10 & 98 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$= \frac{1}{1000}\begin{bmatrix} 636 \\ 364 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 0.636 \\ 0.364 \end{bmatrix} = X_1}$$

$(0.88)^{10} \cong 0.279$

$$X_{10} = \frac{1}{6}\left[\begin{array}{c|c} 1 + 5(0.88)^{10} & 1 - (0.88)^{10} \\ \hline 5 - 5(0.88)^{10} & 5 + (0.88)^{10} \end{array}\right]\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$\cong \frac{1}{60}\left[\begin{array}{c|c} 2.393 & 0.721 \\ \hline 3.607 & 5.279 \end{array}\right]\begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\cong \frac{1}{60}\begin{bmatrix} 18.91 \\ 41.086 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 0.32 \\ 0.68 \end{bmatrix} = X_{10}}.$$

$(0.88)^{100} \cong 2.807 \times 10^{-6}$.

Likewise, $X_{100} = \frac{1}{6}\left[\begin{array}{c|c} 1 + 5(0.88)^{100} & 1 - (0.88)^{100} \\ \hline 5 - 5(0.88)^{100} & 5 + (0.88)^{100} \end{array}\right]\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

$$\approx \frac{1}{6}\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.166.. \\ 0.833.. \end{bmatrix} = \underline{\begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix} = X_{100}}$$

We find $\lim_{n \to \infty}(X_n) = \frac{1}{6}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ it is no accident that this is the scalar multiple of $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$. The $1/6$ maintains probability.

P150 & P151 (counts for 2 problems) Damiano & Little

exercise 7 of §4.5 of pg. 205 of D & L

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be normal if $AA^T = A^TA$

(a.) Show if A is symmetric then A is normal:

$A^T = A \Rightarrow AA^T = A^TA$ applying symmetry twice.

(b.) Find an example of 2x2 normal matrix that is not symmetric:

Many examples can be given. I like the text's answer.
Consult a rotation matrix $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for $\theta \neq k\pi$

Also $R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ & $R^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ ← not a rotation, actually a dilation & rotation.

$$\left.\begin{aligned} RR^T &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ R^TR &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}\right\} \quad RR^T = R^TR$$

(c.) Show if A is normal then $\|Ax\| = \|A^Tx\| \ \forall x \in \mathbb{R}^n$

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= (Ax)^T Ax \\ &= x^T A^T A x \\ &= x^T A A^T x \\ &= (A^Tx)^T A^T x \\ &= \langle A^Tx, A^Tx \rangle \\ &= \|A^Tx\|^2 \end{aligned}$$

[this conclusion is valid as $\|v\| \geq 0 \ \forall v \in \mathbb{R}^n$.

$\therefore$ $\|Ax\| = \|A^Tx\| \ \forall x \in \mathbb{R}^n$

(d.) show if A is normal then $A - cI$ is normal $\forall c \in \mathbb{R}$

$$\begin{aligned} (A - cI)(A - cI)^T &= (A - cI)(A^T - cI) \\ &= AA^T - cIA^T - cAI + c^2 II \\ &= A^TA - cA^TI - cIA + c^2 II \\ &= A^T(A - cI) - cI^T(A - cI) \\ &= (A^T - cI^T)(A - cI) \\ &= (A - cI)^T(A - cI) \end{aligned}$$

normal A and I commute with everything

$\therefore$ $A - cI$ normal //

continued

(e.) show if $\lambda \in \mathbb{R}$ is an e-value of normal matrix A then λ is also an e-value of A^T

$\lambda \in \mathbb{R}$ an e-value of $A \Rightarrow \text{Null}(A - \lambda I) \neq \{0\}$

Hence $\exists x \in \mathbb{R}^n$ s.t. $(A - \lambda I)x = 0$ for $x \neq 0$.

Notice A normal $\Rightarrow A - \lambda I$ also normal by part d.

Also, $\|(A - \lambda I)x\| = \|(A - \lambda I)^T x\|$ by part c.

Hence, $\|(A - \lambda I)x\| = \|(A^T - \lambda I)x\| = 0$ which implies, by property $\|v\| = 0 \Leftrightarrow v = 0$ for $\|\cdot\|$, that $(A^T - \lambda I)x = 0$. Thus $x \neq 0$ is also e-vector for A^T with e-value λ.

(f.) Show that if λ_1 and λ_2 are distinct real e-values of a normal matrix A then corresponding e-vectors are orthogonal

Let $x \in E_{\lambda_1}$ and $y \in E_{\lambda_2}$ both non zero (since I'm looking at eigenvectors) then $Ax = \lambda_1 x$ & $Ay = \lambda_2 y$

Observe $\langle x, y \rangle$ is shown to be zero as follows:

As AA^T is symmetric for A normal,

$$\langle A^T A x, y \rangle = \langle x, A^T A y \rangle$$

$$\Rightarrow \langle A^T \lambda_1 x, y \rangle = \langle x, A^T \lambda_2 y \rangle \quad \text{property of } \langle , \rangle$$

$$\Rightarrow \lambda_1 \langle A^T x, y \rangle = \lambda_2 \langle x, A^T y \rangle \quad \text{by part e.}$$

$$\Rightarrow \lambda_1 \langle \lambda_1 x, y \rangle = \lambda_2 \langle x, \lambda_2 y \rangle$$

$$\Rightarrow (\lambda_1^2 - \lambda_2^2) \langle x, y \rangle = 0$$

However, as $\lambda_1 \neq \lambda_2 \Rightarrow \lambda_1^2 - \lambda_2^2 \neq 0$ so $\langle x, y \rangle = 0$ which goes to show $x \perp y$ as desired. //

$\boxed{\text{PROBLEM 152}}$ To each $\phi \in V^* = \mathcal{L}(V, \mathbb{F})$ there exists $\#\phi = z$ s.t. $\phi(x) = \langle x, z \rangle \ \forall x \in V$. In other words, we define $\#\phi$ via $\phi(x) = \langle x, \#\phi \rangle$ or in yet other words $\# : V^* \longrightarrow V$ is the musical morphism $\#$. Incidentally, $\#^{-1} = \flat : V \to V^*$.

(a.) Given $\phi(x, y, z) = 3x - y + z \quad \forall (x, y, z) \in \mathbb{R}^3$ where $\langle , \rangle$ is given by dot-product.

We seek $\#\phi$ such that

$$\phi(v) = v \cdot (\#\phi)$$

observe,

$$\phi(x, y, z) = 3x - y + z = (x, y, z) \cdot (3, -1, 1)$$

Thus $\boxed{\#\phi = (3, -1, 1)}$

(b.) Given $\phi(A) = A_{11} + A_{21}$ find $\#\phi$ w.r.t. Frob. inn. product on $\mathbb{R}^{n \times n}$, $\langle A, B \rangle = \text{trace}(AB^T) = A_{11}B_{11} + A_{12}B_{12} + A_{21}B_{21} + A_{22}B_{22}$

We want $\#\phi = B$ s.t.

$$\phi(A) = \langle A, \#\phi \rangle = \langle A, B \rangle$$

$$A_{11} + A_{21} = A_{11}B_{11} + A_{12}B_{12} + A_{21}B_{21} + A_{22}B_{22}$$

by inspection it is clear that $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ does nicely, $\boxed{\#\phi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}$. You can check:

$$\phi(A) = \left\langle A, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

$\boxed{P153}$ Calculate T^*. Ignoring problem statement $\exists$ three natural methods: for orthonormal basis β,

$\beta = \{v_1, \dots, v_n\}$

1.) $\cancel{T^*(x) = \sum_{i=1}^{n} \langle v_i, T(x) \rangle v_i}$

2.) $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$

3.) $[T^*]_{\beta\beta} = \left(\overline{[T]_{\beta\beta}}\right)^T$

No!

$T^*(x) = \sum_{i=1}^{n} \langle x, T(v_i) \rangle v_i$

SORRY IF I TOLD YOU OTHERWISE!

Probably (3.) is easiest.

(a.) $T(x,y) = (x+2y, 3y) = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}_{[T] = [T]_{\beta\beta} \text{ for } \beta = \{e_1, e_2\}} \begin{bmatrix} x \\ y \end{bmatrix}$

thus $[T^*]_{\beta\beta} = \left(\overline{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}\right)^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

Hence, $\boxed{T^*(x,y) = (x, 2x+3y)}$

But, (1.) is also easy enough.

$$\cancel{\begin{aligned} T^*(x,y) &= [e_1 \cdot T(x,y)]e_1 + [e_2 \cdot T(x,y)]e_2 \\ &= (x+2y)e_1 + (3y)e_2 \\ &= (x+2y, 3y) \end{aligned}}$$

<u>oh noes.</u>

Actually, $T^*(\vec{x}) = \sum_{i=1}^{n} \langle \vec{x}, T(v_i) \rangle v_i$

$T(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T(e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\begin{aligned} T^*(x,y) &= ((x,y) \cdot (1,0))e_1 + ((x,y) \cdot (2,3))e_2 \\ &= xe_1 + (2x+3y)e_2 \\ &= \underline{(x, 2x+3y)}. \end{aligned}$$

Phew, that's better.

P153 If $V = \text{span}\{\beta\}$ where $\beta = \{v_1, .., v_n\}$ is orthonormal then we defined $T^*: V \to V$ for given $T \in \text{End}(V)$ as follows:

$$T^*(x) = \sum_{i=1}^{n} \langle x, T(v_i) \rangle v_i$$

we proved $\langle T(x), y \rangle = \langle x, T^*(y) \rangle \ \forall x, y \in V$ and $[T^*]_{\beta\beta} = \left(\overline{[T]_{\beta\beta}}\right)^T$. In the case $\mathbb{F} = \mathbb{R}$ the relation simplifies,

$$[T^*]_{\beta\beta} = ([T]_{\beta\beta})^T$$

(a.) $T(x,y) = (x+2y, 3y) = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}_{[T]} \begin{bmatrix} x \\ y \end{bmatrix}$ where $\beta = \{e_1, e_2\}$

orthonormal w.r.t. dot-product.

Then $[T^*] = [T]^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

hence $\boxed{T^*(x,y) = (x, 2x+3y).}$

<u>OR</u>

$$\begin{aligned} T^*(x,y) &= \langle (x,y), T(e_1) \rangle e_1 + \langle (x,y), T(e_2) \rangle e_2 \\ &= (x,y) \cdot (1,0)\, e_1 + (x,y) \cdot (2,3)\, e_2 \\ &= x e_1 + (2x+3y) e_2 \\ &= \underline{(x, 2x+3y)}. \end{aligned}$$

yep.

(16.) $T(z,w) = (3z + iw,\ z + (2-7i)w) \quad \forall (z,w) \in \mathbb{C}^2$

$$= \underbrace{\begin{bmatrix} 3 & i \\ 1 & 2-7i \end{bmatrix}}_{[T] = [T]_{\beta\beta}} \begin{pmatrix} z \\ w \end{pmatrix}$$

w.r.t. $\beta = \{e_1, e_2\}$ which is $\langle , \rangle$ orthonormal for $\langle v, w\rangle = v^T \overline{w}$

Thus,

$$[T^*] = \overline{[T]}^T$$

$$= \left(\overline{\begin{bmatrix} 3 & i \\ 1 & 2-7i \end{bmatrix}}\right)^T$$

$$= \begin{bmatrix} 3 & -i \\ 1 & 2+7i \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 1 \\ -i & 2+7i \end{bmatrix}$$

Therefore, $\boxed{T^*(z,w) = (3z + w,\ -iz + (2+7i)w)}$

<u>OR</u>:

$$T^*(z,w) = \langle (z,w), T(1,0)\rangle (1,0) + \langle (z,w), T(0,1)\rangle (0,1)$$

$$= \langle (z,w), (3,1)\rangle (1,0) + \langle (z,w), (i, 2-7i)\rangle (0,1)$$

$$= (3z + w)(1,0) + (-iz + (2+7i)w)(0,1)$$

$$= \underline{(3z + w,\ -iz + (2+7i)w)}.$$

yep.

(c.) another way,

$$\begin{tikzcd} \square \arrow[r, "T"] \arrow[d, "\Phi_\beta"'] & \square \arrow[d, "\Phi_\beta"] \\ \square \arrow[r, "L(T)_{\beta\beta}"] & \square \end{tikzcd}$$

$$\gamma = \{1, x\}$$

$$T(a+bx) = \frac{d}{dx}(a+bx) + 3(a+bx) = 3a+b+3bx$$

Consequently, $[T]_{\gamma\gamma} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

But $\beta = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x \right\}$ is different

basis and so

$$[T]_{\beta\beta} = [\Phi_\beta \circ T \circ \Phi_\beta^{-1}]$$
$$= [\Phi_\beta \circ \Phi_\gamma^{-1} \circ \Phi_\gamma \circ T \circ \Phi_\gamma^{-1} \circ \Phi_\gamma \circ \Phi_\beta^{-1}]$$
$$= [\Phi_\beta \circ \Phi_\gamma^{-1}][T]_{\gamma\gamma}\underbrace{[\Phi_\gamma \circ \Phi_\beta^{-1}]}_{P}.$$

$$P = \left[[\Phi_\beta^{-1}(e_1)]_\gamma \,\middle|\, [\Phi_\beta^{-1}(e_2)]_\gamma \right]$$
$$= \left[[1/\sqrt{2}]_\gamma \,\middle|\, [\sqrt{\tfrac{3}{2}}\,x]_\gamma \right]$$
$$= \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{3/2} \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}}_{P}$$

$$[T]_{\beta\beta} = P^{-1}[T]_{\gamma\gamma}P$$
$$= \cancel{\left(\frac{1}{\sqrt{2}}\right)^{-1}}\begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\cancel{\frac{1}{\sqrt{2}}}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & \sqrt{3} \\ 0 & 3 \end{bmatrix} \Rightarrow [T^*]_{\beta\beta} = \begin{bmatrix} 3 & 0 \\ \sqrt{3} & 3 \end{bmatrix}$$

$a = \bar{a}/\sqrt{2}$
$b = \bar{b}\sqrt{3/2}$
to compare with **

$$\therefore \boxed{T^*\left(\bar{a}\frac{1}{\sqrt{2}} + \bar{b}\sqrt{\frac{3}{2}}\,x\right) = \frac{3\bar{a}}{\sqrt{2}} + \left(\bar{a}\sqrt{3} + 3\bar{b}\right)\sqrt{\frac{3}{2}}\,x}$$

P153 continued

(c.) $T(f) = f' + 3f$ where $f \in P_1(\mathbb{R})$ and $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Recall we showed $\beta = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x \right\}$ is orthonormal in Ex. 9.2.26.

Hence,

$$T^*(f) = \left\langle f, T\left(\tfrac{1}{\sqrt{2}}\right)\right\rangle \frac{1}{\sqrt{2}} + \left\langle f, T\left(\sqrt{\tfrac{3}{2}}\,x\right)\right\rangle \sqrt{\tfrac{3}{2}}\,x$$

easiest to calculate for $f = 1$ and $f = x$ then piece together $T^*(a+bx) = aT^*(1) + bT^*(x)$,

$$T^*(1) = \langle 1, \underbrace{3/\sqrt{2}}_{\substack{f'=0 \\ 3f = 3/\sqrt{2}}} \rangle \frac{1}{\sqrt{2}} + \langle 1, \underbrace{\sqrt{\tfrac{3}{2}} + 3\sqrt{\tfrac{3}{2}}\,x}_{f'+3f} \rangle \sqrt{\tfrac{3}{2}}\,x$$

$$\langle 1, 3/\sqrt{2}\rangle = \int_{-1}^{1} \frac{3}{\sqrt{2}}\,dx = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

$$\sqrt{\tfrac{3}{2}}\,\langle 1, 1+3x\rangle = \sqrt{\tfrac{3}{2}}\left(\int_{-1}^{1} (1 + \underbrace{3x}_{0})\,dx \right) = 2\sqrt{\tfrac{3}{2}}$$

Thus,

$$T^*(1) = 3\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\right) + 2\sqrt{\tfrac{3}{2}}\sqrt{\tfrac{3}{2}}\,x \Rightarrow \underline{\underline{T^*(1) = 3 + 3x}} \ *$$

Likewise, as

$\langle x, x\rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}.$

$$\langle x, 3/\sqrt{2}\rangle = 0$$

$$\langle x, \sqrt{\tfrac{3}{2}} + 3\sqrt{\tfrac{3}{2}}\,x\rangle = \sqrt{3/2}\,\cancel{\langle x, 1\rangle}^{\,0} + 3\sqrt{\tfrac{3}{2}}\,\langle x, x\rangle = 3\sqrt{\tfrac{3}{2}}\,\tfrac{2}{3} = ? = 2\sqrt{\tfrac{3}{2}}$$

we find $T^*(x) = 2\left(\sqrt{\frac{3}{2}}\right)^2 x = 3x$. Thus,

$$T^*(a+bx) = a(3+3x) + b(3x)$$

$$\boxed{T^*(a+bx) = 3a + (3a+3b)x} \quad \text{— } (**)$$

P153 continued

$$T^*(a+bx) = 3a + (3a+3b)x$$

vs.

$$T^*\left(\bar{a}\left(\tfrac{1}{\sqrt{2}}\right) + \bar{b}\left(\sqrt{\tfrac{3}{2}}x\right)\right) = \underline{3\frac{\bar{a}}{\sqrt{2}} + \left(\bar{a}\sqrt{3} + 3\bar{b}\right)\sqrt{\tfrac{3}{2}}x} \;\star$$

$$=$$

Setting $a = \bar{a}/\sqrt{2}$ and $b = \bar{b}\sqrt{\frac{3}{2}}$

we find, $\bar{a} = a\sqrt{2}$ and $\bar{b} = \sqrt{\frac{2}{3}}\,b$

plug into $\star$

$$T^*\left(\bar{a}\left(\tfrac{1}{\sqrt{2}}\right) + \bar{b}\left(\sqrt{\tfrac{3}{2}}x\right)\right) = \frac{3}{\sqrt{2}}(a\sqrt{2}) +$$

$$+ \left(a\sqrt{2}\sqrt{3} + 3\sqrt{\tfrac{2}{3}}b\right)\sqrt{\tfrac{3}{2}}x$$

$$= 3a + (3a+3b)x.$$

(just checking equivalence of my answers)

P154 Suppose $T: V \to V$ has $T^* = -T$ where V is inner product space over $\mathbb{C}$.

(a.) Suppose $T(x) = \lambda x$ where $x \neq 0$ and $\lambda \in \mathbb{C}$.

Then $(T - \lambda)(x) = 0$ and

$$
\begin{aligned}
0 = \langle (T-\lambda)(x), x \rangle &= \langle x, (T-\lambda)^*(x) \rangle \\
&= \langle x, (T^* - \bar{\lambda})(x) \rangle \qquad T^* = -T \\
&= \langle x, -T(x) - \bar{\lambda} x \rangle \\
&= \langle x, -(\lambda + \bar{\lambda}) x \rangle \\
&= -(\overline{\lambda + \bar{\lambda}}) \langle x, x \rangle
\end{aligned}
$$

Thus $\bar{\lambda} + \lambda = 0$ so if $\lambda = \alpha + i\beta$ then

$\bar{\lambda} + \lambda = (\alpha - i\beta) + (\alpha + i\beta) = 2\alpha = 0 \Rightarrow \alpha = 0$.

Thus $\boxed{\lambda = i\beta \text{ for some } \beta \in \mathbb{R}}$ - (logically equivalent to $\lambda = i\alpha$ for $\alpha \in \mathbb{R}$) -

(b.) Let $x \in \overbrace{Ker(T - \lambda_i)}^{E_i}$ and $y \in \overbrace{Ker(T - \lambda_j)}^{E_j}$

Assume $\lambda_i \neq \lambda_j$ and note $(T - \lambda_i)(x) = T(x) - \lambda_i x = \lambda_i x - \lambda_i x = 0$.

Thus consider,

$$
\begin{aligned}
0 &= \langle (T - \lambda_i)(x), y \rangle \qquad (T - \lambda_i)^* = T^* - \bar{\lambda}_i \\
&= \langle x, (T^* - \bar{\lambda}_i)(y) \rangle \qquad \left(\text{technically, } T - \lambda_i = T - \lambda_i Id_V\right) \\
&= \langle x, -T(y) - \bar{\lambda}_i y \rangle \\
&= \langle x, -\lambda_j y - \bar{\lambda}_i y \rangle \\
&= -(\overline{\lambda_j + \bar{\lambda}_i}) \langle x, y \rangle \quad \star
\end{aligned}
$$

But, $\overline{\lambda_j + \bar{\lambda}_i} = \bar{\lambda}_j + \lambda_i$ and as $\lambda_i = i\alpha$ & $\lambda_j = i\beta$ for $\alpha \neq \beta$ real we find $\bar{\lambda}_j + \lambda_i = -i\beta + i\alpha = i(\alpha - \beta) \neq 0$.

Thus $\langle x, y \rangle = 0$ follows from $\star$ and we find $E_i \perp E_j$.

P161

Let $\beta = \{f_1, \dots, f_n\}$ be a basis for V over $\mathbb{R}$.
Let $\gamma = \{g_1, \dots, g_m\}$ be a basis for W over $\mathbb{R}$. As usual $V^* = \mathcal{L}(V, \mathbb{R})$ and $W^* = \mathcal{L}(W, \mathbb{R})$ with dual bases $\beta^* = \{f^1, \dots, f^n\}$ and $\gamma^* = \{g^1, \dots, g^m\}$ defined by $f^i(f_j) = \delta_{ij}$ and $g^k(g_l) = \delta_{kl}$ for all $1 \leq i,j \leq n$ and $1 \leq k, l \leq m$.
Suppose $T : V \to W$ is a linear transformation.
Define $S : W^* \to V^*$ by $(S(\alpha))(v) = \alpha(T(v))$ $\forall v \in V$
show $S \in \mathcal{L}(W^*, V^*)$ and find $[S]_{\gamma^*, \beta^*}$

$$\begin{aligned}
(S(c\alpha + \beta))(x) &= (c\alpha + \beta)(T(x)) \\
&= c\alpha(T(x)) + \beta(T(x)) \\
&= c(S(\alpha))(x) + (S(\beta))(x) \\
&= (c\,S(\alpha) + S(\beta))(x) \qquad \forall x \in V,\ \alpha, \beta \in W^*,\ c \in \mathbb{F}
\end{aligned}$$

Hence $S(c\alpha + \beta) = c\,S(\alpha) + S(\beta)$ and thus $S \in \mathcal{L}(W^*, V^*)$.

$$[S]_{\gamma^*, \beta^*} = \left[\, [S(g^1)]_{\beta^*} \,\middle|\, \cdots \,\middle|\, [S(g^m)]_{\beta^*} \,\right]$$

Notice $S(g^j) \in V^* \Rightarrow S(g^j) = \sum_{i=1}^{n} S(g^j)(f_i) f^i$

thus $([S(g^j)]_{\beta^*})_i = S(g^j)(f_i)$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.

That is,
$$\begin{aligned}
\left([S]_{\gamma^*, \beta^*}\right)_{ij} &= (S(g^j))(f_i) \\
&= g^j(T(f_i)) \\
&= ([T(f_i)]_\gamma)_j \\
&= ([T]_{\beta,\gamma})_{ji}
\end{aligned}$$

Thus, $\boxed{[S]_{\gamma^*, \beta^*} = ([T]_{\beta,\gamma})^T}$

(I give an expanded sol^n ...)

(Extra Sol'n, wasn't assigned, but... nice)

If $\alpha(v_1, v_2, v_3) = v_1 + 3v_3$ then find $[\alpha]_{\beta^*}$ in terms of $\beta^* = \{e^1, e^2, e^3\}$ as defined in P118

$$\alpha(v) = v^1 + 3v^3 \quad \text{for} \quad v = (v^1, v^2, v^3)$$
$$= e^1(v) + 3e^3(v)$$
$$= (1 \cdot e^1 + 0 \cdot e^2 + 3 \cdot e^3)(v) \hookrightarrow \alpha = e^1 + 0 \cdot e^2 + 3 \cdot e^3$$
$$\therefore \boxed{[\alpha]_{\beta^*} = (1, 0, 3)}$$

Given $V = \text{span}\,\beta$, $V^* = \text{span}\,\beta^*$ & $W = \text{span}\,\gamma$, $W^* = \text{span}\,\gamma^*$. If $T : V \longrightarrow W$ a linear tranformation and we define $S^t : W^* \to V^*$ by $(S(\alpha))(v) = \alpha(T(v))$ $\forall v \in V$ and $\alpha \in W^*$. Then show $S \in \mathcal{L}(W^*, V^*)$ and calculate $[S]_{\gamma^*, \beta^*}$

Consider, for $\alpha, \beta \in W^*$ and $c \in \mathbb{R}$,
$$(S(c\alpha + \beta))(v) = (c\alpha + \beta)(T(v))$$
$$= c\,\alpha(T(v)) + \beta(T(v))$$
$$= (c\,S(\alpha) + S(\beta))(v) \quad \forall v \in V \Rightarrow S \in \mathcal{L}(W^*, V^*).$$

Next, consider, $\gamma^* = \{g^1, g^2, \ldots, g^m\}$ so,
$$[S]_{\gamma^*, \beta^*} = \Big[[S(g^1)]_{\beta^*} \Big| [S(g^2)]_{\beta^*} \Big| \cdots \Big| [S(g^m)]_{\beta^*} \Big] \quad *$$
$$= \left[\begin{bmatrix} S(g^1)(f_1) \\ S(g^1)(f_2) \\ \vdots \\ S(g^1)(f_n) \end{bmatrix} \middle| \begin{bmatrix} S(g^2)(f_1) \\ S(g^2)(f_2) \\ \vdots \\ S(g^2)(f_n) \end{bmatrix} \middle| \cdots \middle| \begin{bmatrix} S(g^m)(f_1) \\ S(g^m)(f_2) \\ \vdots \\ S(g^m)(f_n) \end{bmatrix} \right]$$
$$= \begin{bmatrix} g^1(T(f_1)) & g^2(T(f_1)) & \cdots & g^m(T(f_1)) \\ \vdots & & & \\ g^1(T(f_n)) & g^2(T(f_n)) & \cdots & g^m(T(f_n)) \end{bmatrix} \quad **$$
$$= \begin{bmatrix} ([T(f_1)]_\gamma)^T \\ \vdots \\ ([T(f_n)]_\gamma)^T \end{bmatrix}$$
$$= \Big[[T(f_1)]_\gamma \Big| \cdots \Big| [T(f_n)]_\gamma \Big]^T$$
$$= ([T]_{\beta, \gamma})^T$$

I'll expand on * & ** next

$*$: $[S(g^1)]_{\beta^*} = (c_1, c_2, \dots, c_n)$ means that

$S(g^1) = c_1 f^1 + c_2 f^2 + \dots + c_n f^n$ and we can

select the values of $c_1, c_2, \dots, c_n$ by using $f^i(f_j) = \delta_{ij}$

$$(S(g^1))(f_i) = c_1 \underbrace{f^1(f_i)}_{0} + \dots + c_i \underbrace{f^i(f_i)}_{1} + \dots + c_n \underbrace{f^n(f_i)}_{0} = c_i$$

Therefore,

$$[S(g^1)]_{\beta^*} = \big((S(g^1))(f_1),\ (S(g^1))(f_2),\ \dots,\ (S(g^1))(f_n)\big)$$

Of course this calculation holds for $\alpha \in V^*$ just the same

$$[\alpha]_{\beta^*} = (\alpha(f_1), \alpha(f_2), \dots, \alpha(f_n)).$$

$**$: $[T(f_1)]_\gamma^T = [g^1(T(f_1)), g^2(T(f_1)), \dots, g^m(T(f_1))]$

Observe $[T(f_1)]_\gamma = (c_1, c_2, \dots, c_m)$ indicates that

$T(f_1) = c_1 g_1 + c_2 g_2 + \dots + c_m g_m$. However, as

$g^j(g_i) = \delta_{ij}$ we deduce $g^j(T(f_1)) = c_j$

hence $[T(f_1)]_\gamma^T = [g^1(T(f_1)), \dots, g^m(T(f_1))]$.

Likewise, for any $w \in W$ we have

$[w]_\gamma = (g^1(w), g^2(w), \dots, g^m(w))$ where $\gamma^* = \{g^1, \dots, g^m\}$ is the dual-basis to $\gamma = \{g_1, \dots, g_m\}$.

__Remark:__ you'll notice the notation for $[T]_{\beta,\gamma}$ is replaced with $[T]_\beta^\gamma$ in other texts such as Damiano & Little (see Prop. 2.2.15, page 78, $\underline{[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha}$ translate $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$)

continued ↙

continued

It is instructive to explore how the matrix of $T: V \to W$ depends explicitly on the basis β for V and γ^* for W^*.

Assume $\dim V = n$ and $\dim W = m$,

$$T(v) = \sum_{i=1}^{m} g^i [T(v)]\, g_i \quad (\text{by } **)$$

$$= \sum_{i=1}^{m} g^i \Big[T\Big(\sum_{j=1}^{n} f^j(v) f_j\Big)\Big] g_i \quad (\text{by } ** \text{ for } V)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f^j(v)\, g^i [T(f_j)]\, g_i$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \Big(\underbrace{g^i [T(f_j)]}\, \underbrace{f^j(v)}\Big)\, g_i$$

$$\left([T(v)]_\gamma\right)^i = \sum_{i=1}^{m} \sum_{j=1}^{n} A^i_{\ j}\, V^j$$

Our notation has been $\left([T]_{\beta,\gamma}\right)_{ij} = g^i(T(f_j))$

But, you can see that $\left([T]_\beta^\gamma\right)^i_j = g^i(T(f_j))$

is more reflective of the role β and γ^* play in defining the matrix of T. In particular, to use other bases $[T]_{\bar\beta}^{\bar\gamma} = \bar g^i(T(\bar f_j))$

we see immediately the transformed matrix transforms the same way as $\bar\beta$ but, instead of γ, as $\bar\gamma^*$ relates to γ^*. It turns out $\bar\beta$ & $\bar\beta^*$ are inversely related to β and β^* respective. This must occur as $f^i(f_j) = \delta_{ij}$ and $\bar f^i(\bar f_j) = \delta_{ij}$ as well...

P162

Let $T: V \longrightarrow W$ and $S: W^* \longrightarrow V^*$ be defined, where as in P144, $(S(\alpha))(v) = \alpha(T(v))$ $\forall v \in V, \alpha \in W^*$

(a.) if T is surjective then $T(V) = W$. Let $\dim(V) = n$, $\dim(W) = m$ thus $[T]_{\beta\gamma}$ is $m \times n$ and $T(V) = W \Rightarrow$ Col $[T]_{\beta\gamma}$ has dimension m or, if you prefer, $\exists$ m-LI columns of $[T]_{\beta\gamma}$.

We found $[S]_{\gamma^*\beta^*} = \left([T]_{\beta,\gamma}\right)^T \leftarrow$ has m-LI rows

Thus $[S]_{\gamma^*,\beta^*}$ is an $n \times m$ matrix with m-LI rows which indicates each column in $[S]_{\gamma^*\beta^*}$ is a pivot column $\therefore$ Null $[S]_{\gamma^*\beta^*} = \{0\} \Rightarrow \operatorname{Ker}(S) = \{0\}$

$\therefore$ <u>S is injective.</u>

(b.) if S is injective then $[S]_{\gamma^*\beta^*}$ is an $n \times m$ matrix with m-LI columns since Null $[S]_{\gamma^*\beta^*} = \{0\}$.

But, $[T]_{\beta,\gamma} = \left([S]_{\gamma^*\beta^*}\right)^T$ is an $m \times n$ matrix with m-LI rows and hence m-LI columns

$\Rightarrow \dim\left(\text{Col } [T]_{\beta,\gamma}\right) = m \Rightarrow T(V) = W$

$\therefore$ <u>T is surjective.</u>

(c.) If T is an isomorphism then T is injective and surjective $\therefore$ S is injective by (a.). Since $\dim V = \underbrace{\dim(\cancel{\operatorname{Ker}} T)}_{0} + \dim(T(V)) \Rightarrow \dim V = \dim W$.

Then $\dim W^* = \underbrace{\dim(\cancel{\operatorname{Ker}} S)}_{0} + \dim(S(W^*)) \Rightarrow \underline{\dim(S(W^*)) = \dim V^*}$ as $\dim V = \dim V^*$ and $\dim W = \dim W^*$. Hence $\underline{S(W^*) = V^*}$.

Thus S is an isomorphism. (I leave converse to reader)

$\boxed{P155}$ Suppose $A^T = A$ has orthonormal e-basis $\beta = \{u, v, w\}$ with distinct, nonzero e-values $\lambda_1, \lambda_2, \lambda_3$. Define $E_1 = \lambda_1 uu^T$, $E_2 = \lambda_2 vv^T$, $E_3 = \lambda_3 ww^T$

Observe $\quad Au = \lambda_1 u \quad$ and $\quad E_1 u = \lambda_1 u \underbrace{u^T u}_{1} = \lambda_1 u.$

$Av = \lambda_2 v \quad$ and $\quad E_2 v = \lambda_2 v \underbrace{v^T v}_{1} = \lambda_2 v$

$Aw = \lambda_3 w \quad$ and $\quad E_3 w = \lambda_3 w \underbrace{w^T w}_{1} = \lambda_3 w$

Moreover, $E_1 v = 0$ and $E_1 w = 0$ as $u^T v = u^T w = 0$.
Likewise $E_2 u = 0$ and $E_2 w = 0$ as $v^T u = v^T w = 0$.
and $E_3 u = 0$ and $E_3 v = 0$ since $w^T u = 0$ & $w^T v = 0$.

Thus,
$$\left.\begin{aligned} Au &= (E_1 + E_2 + E_3)u \\ Av &= (E_1 + E_2 + E_3)v \\ Aw &= (E_1 + E_2 + E_3)w \end{aligned}\right\} *$$

But, as $L_A(x) = Ax$ if we set $\beta = \{u, v, w\}$ then $*$ shows $[L_A]_{\beta\beta} = [E_1 + E_2 + E_3]_{\beta\beta}$

Consequently, $P^{-1}AP = P^{-1}(E_1 + E_2 + E_3)P$

thus $\underline{A = E_1 + E_2 + E_3}$.

To see $\text{Col}(E_1) = \text{Null}(A - \lambda_1 I)$ suppose $x \in \text{Col}(E_1)$ then as $\text{Col}(E_1) = \text{span}\{u\}$ since $E_1 = uu^T$ we find $x = cu$. Note $(A - \lambda_1 I)cu = c(Au - \lambda_1 u) = c(\lambda u - \lambda u) = 0$ So $x \in \text{Null}(A - \lambda I)$ $\therefore$ $\text{Col}(E_1) \subseteq \text{Null}(A - \lambda_1 I)$. Conversely, $\text{Null}(A - \lambda_1 I) = \text{span}\{u\}$; oh well, in short $\text{Col}(E_1) = \text{span}\{u\} = \text{Null}(A - \lambda_1$
Likewise $\text{Null}(A - \lambda_2 I) = \text{span}\{v\} = \text{Col}(E_2)$, $\text{Null}(A - \lambda_3 I) = \text{span}\{w\} = \text{Col}(E_3)$. //

P156

$$u = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} \quad \& \quad v = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix} \quad \& \quad w = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Let $A = 12uu^T + 2vv^T + 18ww^T$

$$= \frac{12}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}[1,-1,1] + \frac{2}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}[0,1,1] + \frac{18}{6}\begin{bmatrix}2\\1\\-1\end{bmatrix}[2,1,-1]$$

$$= 4\begin{bmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} + 3\begin{bmatrix}4 & 2 & -2\\2 & 1 & -1\\-2 & -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4 & -4 & 4\\-4 & 4 & -4\\4 & -4 & 4\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} + \begin{bmatrix}12 & 6 & -6\\6 & 3 & -3\\-6 & -3 & 3\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}16 & 2 & -2\\2 & 8 & -6\\-2 & -6 & 8\end{bmatrix}}$$

Remark: there are at least ~~two~~ other natural answers here,

$$A = 2uu^T + 12vv^T + 18ww^T$$

OR

$$A = 2uu^T + 18vv^T + 12vv^T$$

etc...

$$\begin{bmatrix}12 & 2 & 18\\2 & 12 & 18\\2 & 18 & 12\\12 & 18 & 2\\18 & 2 & 12\\18 & 12 & 2\end{bmatrix}$$ 6 patterns.

$\boxed{P157}$ $e^M = I + M + \cdots + \frac{1}{n!}M^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$

(a.) If $D = \sum_{i=1}^{n} \lambda_i E_{ii}$ then $D^k = \sum_{i=1}^{n} \lambda_i^k E_{ii}$

(we've proved $(\mathrm{diag}(a_1, a_2, \ldots, a_n))^k = \mathrm{diag}(a_1^k, a_2^k, \ldots, a_n^k)$.)

thus

$$e^D = \sum_{k=0}^{\infty} \frac{1}{k!} D^k$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k & & & \\ & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_2^k & & \\ & & \ddots & \\ & & & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k \end{bmatrix}$$
$$= \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} . // \overset{*}{=} e^{\lambda} I \text{ when } \lambda_1 = \lambda_2 = \cdots = \lambda_n .$$

(b.) $J_k(\lambda) = \lambda I + N_k$ where $N_k^{k-1} \neq 0$ yet $N_k = 0$

$$e^{J_k(\lambda)} = e^{\lambda I + N_k}$$
$$= e^{\lambda I} e^{N_k}$$
$$= \boxed{e^{\lambda} I \left(I + N_k + \frac{1}{2} N_k^2 + \cdots + \frac{1}{(k-1)!} N_k^{k-1} \right)}$$

Thus,

$$e^{tJ_k(\lambda)} = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2 & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & & \vdots \\ & & 1 & & \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix} \leftarrow \text{need later.}$$

P157 continued

(c.) $A \oplus B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$

$$(A \oplus B)^2 = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] = \left[\begin{array}{c|c} A^2 & 0 \\ \hline 0 & B^2 \end{array}\right] = A^2 \oplus B^2$$

etc... continuing, $(A \oplus B)^k = A^k \oplus B^k$ for $k \in \mathbb{N}$.

and by def^n $(A \oplus B)^0 = I = I_a \oplus I_b = A^0 \oplus B^0$

for A an $a \times a$ and B a $b \times b$. Also,

$c(A \oplus B) = (cA) \oplus (cB)$

$$e^{A \oplus B} = \sum_{k=0}^{\infty} \frac{(A \oplus B)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{A^k}{k!} \oplus \frac{B^k}{k!}$$

$$= \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right) \oplus \left(\sum_{k=0}^{\infty} \frac{B^k}{k!}\right)$$

$$= \underline{e^A \oplus e^B}.$$

(d.) $$P^{-1}e^M P = P^{-1}\left(\sum_{k=0}^{\infty} \frac{M^k}{k!}\right) P$$

$P^{-1}M^k P = (P^{-1}MP)^k \ \forall k.$
$P^{-1}MMP = P^{-1}MPP^{-1}MP$
$= (P^{-1}MP)^2$ etc.

$$= \sum_{k=0}^{\infty} P^{-1} \frac{M^k}{k!} P$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (P^{-1}MP)^k$$

$$= e^{P^{-1}MP}.$$

P157 continued

(e.) For $M \in \mathbb{C}^{n \times n}$ $\exists P$ such that

$$P^{-1}MP = J_{r_1}(\lambda_1) \oplus J_{r_2}(\lambda_2) \oplus \cdots \oplus J_{r_k}(\lambda_k) = J$$

by the existence of Jordan Form over $\mathbb{C}$.

Then note $M = PJP^{-1}$ hence

$$
\begin{aligned}
e^M &= e^{PJP^{-1}} \\
&= P e^J P^{-1} \quad : \text{ by (d.)} \\
&= P \exp\left[J_{r_1}(\lambda_1) \oplus \cdots \oplus J_{r_k}(\lambda_k)\right] P^{-1} \quad \text{extending (c.)} \\
&= \underbrace{P\left[\exp(J_{r_1}(\lambda_1)) \oplus \cdots \oplus \exp(J_{r_k}(\lambda_k))\right]P^{-1}}_{\text{nice formula for } e^M.}
\end{aligned}
$$

Notice we can calculate $\exp(J_{r_i}(\lambda_i))$ for $i = 1, 2, \dots, k$ via formula found in (b.)

Example:

$$\exp\left[t\begin{bmatrix} \boxed{2} & & & & \\ & \boxed{3} & & & \\ & & -1 & 1 & 0 \\ & & 0 & -1 & 1 \\ & & 0 & 0 & -1 \end{bmatrix}\right] = e^{2t} \oplus e^{3t} \oplus \left[e^{-t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\exp\left(t\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right) = e^{3t}\begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{etc.}$$

$\boxed{P159}$

(a.) Solve $\frac{dx}{dt} = Ax$ given $A = J_3(7)$

$$x = e^{tA}c = e^{7t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

equally fine.

multiplying.

$$\boxed{x = \left((c_1 + c_2 t + c_3 t^2/2)e^{7t},\ (c_2 + t c_3)e^{7t},\ c_3 e^{7t}\right)}$$

(b.) $A = J_2(3) \oplus J_1(1)$ solve $\frac{dx}{dt} = Ax$

$$e^{tA} = \exp(tJ_2(3)) \oplus \exp(tJ_1(1))$$

$$= e^{3t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \oplus e^{t}[1]$$

$$= \begin{bmatrix} e^{3t} & te^{3t} & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

Thus $\boxed{x = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

$\boxed{P160}$ Suppose $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$, not both zero,

(a.) find complex P s.t. $P^{-1}AP = \begin{bmatrix} a+ib & \\ & a-ib \end{bmatrix} = D$

$$\det(A - \lambda I) = \det\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix} = (\lambda - a)^2 + b^2$$

Thus A has $\lambda = a \pm ib$ as complex e-values.

$$A - (a+ib)I = \begin{bmatrix} -ib & -b \\ b & -ib \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$ assume $b \neq 0$

Thus $(u,v) \in \mathcal{E}_{\lambda = a+ib} \Rightarrow u = iv$

hence $\mathcal{E}_{a+ib} = \text{span}\{(i,1)\}$ and by complex conjugation $\mathcal{E}_{a-ib} = \text{span}\{(-i,1)\}$.

Hence we set

$$\underline{P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}}. \Rightarrow \underline{P^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}}.$$

Calculate,

$$P^{-1}AP = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} a+ib & -b+ia \\ -a+ib & b+ia \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} ia-b-b+ia & -ia+b-b+ia \\ -ia-b+b+ia & ia+b+b+ia \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} 2ia-2b & 0 \\ 0 & 2b+2ia \end{bmatrix}$$

$$= \underline{\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}}.$$

(6.) $$e^{tA} = e^{t(PDP^{-1})}$$
$$= P e^{tD} P^{-1}$$
$$= P \begin{bmatrix} e^{t(a+ib)} & \\ & e^{t(a-ib)} \end{bmatrix} P^{-1}$$
$$= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\bar{\lambda} t} \end{bmatrix} \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$$
$$= \begin{bmatrix} ie^{\lambda t} & -ie^{\bar{\lambda} t} \\ e^{\lambda t} & e^{\bar{\lambda} t} \end{bmatrix} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \frac{1}{2i}$$
$$= \frac{1}{2i} \left[\begin{array}{c|c} i(e^{\lambda t} + e^{\bar{\lambda} t}) & -e^{\lambda t} + e^{\bar{\lambda} t} \\ \hline e^{\lambda t} - e^{\bar{\lambda} t} & i(e^{\lambda t} + e^{\bar{\lambda} t}) \end{array} \right]$$
$$= \left[\begin{array}{c|c} \frac{1}{2}(e^{\lambda t} + e^{\bar{\lambda} t}) & \frac{-1}{2i}(e^{\lambda t} - e^{\bar{\lambda} t}) \\ \hline \frac{1}{2i}(e^{\lambda t} - e^{\bar{\lambda} t}) & \frac{1}{2}(e^{\lambda t} + e^{\bar{\lambda} t}) \end{array} \right]$$
$$= e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$ ← Real Functions

Since $e^{\lambda t} = e^{at+ibt} = e^{at} e^{ibt} = e^{at}(\cos bt + i \sin bt)$
and $e^{\bar{\lambda} t} = e^{at-ibt} = e^{at} e^{-ibt} = e^{at}(\cos bt - i \sin bt)$
Hence $\frac{1}{2}(e^{\lambda t} + e^{\bar{\lambda} t}) = e^{at} \cos(bt)$ whereas
$\frac{1}{2i}(e^{\lambda t} - e^{\bar{\lambda} t}) = e^{at} \sin(bt)$.

P160 continued

(c.) Let $A = aI + bJ$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

so $J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

and $J^3 = -J$ and $J^4 = I$ where

we should note $(taI)(tbJ) = (tbJ)(taI)$ thus,

$$e^{tA} = e^{taI + tbJ}$$

$$= e^{taI} e^{tbJ}$$

$$= e^{at} I \left(I + tbJ + \tfrac{1}{2} t^2 b^2 J^2 + \tfrac{1}{3!} t^3 b^3 J^3 + \cdots \right)$$

$$= e^{at} \Big(I \left(1 - \tfrac{1}{2} t^2 b^2 + \tfrac{1}{4!} t^4 b^4 + \cdots \right)$$

$$+ J \left(tb - \tfrac{1}{3!} t^3 b^3 + \cdots \right) \Big)$$

$$= e^{at} \left(\cos(bt) I + \sin(bt) J \right)$$

$$= e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

(d.) to solve $\begin{matrix} x' = ax - by \\ y' = bx + ay \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

thus e^{tA} is solⁿ matrix and gen. solⁿ is:

$$\boxed{X = c_1 e^{at} \begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix} + c_2 e^{at} \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix}.}$$