Please show your work and use words to explain your steps where appropriate.

Problem 1 (9pts) Let R be a commutative ring and suppose $A \in R^{m \times n}$ and $B \in R^{n \times p}$. Prove $(AB)^T = B^T A^T$.

Problem 2 (10pts) Calculate the following matrix expressions if possible, if not possible then write n/a. Let u = [1, 2, 3] and $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -1 & 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$ (a.) $A^TA + u^Tu$

(b.)
$$Au^T + Bw$$

Problem 3 (10pts) Suppose x + y = -3 and (2 + i)x - y = 4. Find the solution in \mathbb{C} and leave your answer in Cartesian form. (use any solution techinque you see fit)

Problem 4 (15pts) Find
$$A^{-1}$$
 for $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Problem 5 (10pts) Solve the following system of equations for arbitrary $a, b, c \in \mathbb{R}$:

$$x + y + 2z = a$$
$$x + 2y + z = b$$
$$y + z = c$$

by multiplication by inverse matrix (no credit will be awarded for row reduction here)

Problem 6 (10pts) Consider the system x + y + z + w = 10 and y + 2z + w = 7 over a field \mathbb{F} . Find the augmented coefficient matrix [A|b] and calculate $\operatorname{rref}[A|b]$. Also, find the solution set.

Problem 7 (20pts) Let $V_1, V_2, V_3, V_4 \in \mathbb{R}^3$ and define $S = \{V_1, V_2, V_3, V_4\}$. Let W = (a, b, c) for some $a, b, c \in \mathbb{R}$. You are given

$$\operatorname{rref}[V_1|V_2|V_3|V_4|W] = \begin{bmatrix} 1 & 0 & -1 & 0 & 2a - b \\ 0 & 1 & 3 & 0 & b - a \\ 0 & 0 & 0 & 1 & c - b \end{bmatrix}$$

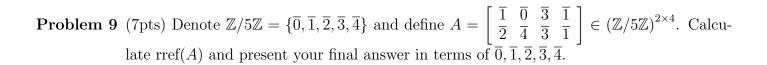
(a.) Write W as a linear combination of V_1, V_2 and V_4 .

(b.) Find the set of all $(a, b, c) \notin \text{span}\{V_1, V_2, V_3\}$

- (c.) is S linearly independent? Discuss.
- (d.) is $A = [V_1|V_2|V_3]$ invertible? Explain.

(e.) what are the solutions of $xV_1 + yV_2 + zV_4 = 0$? Discuss.

Problem 8 (5pts) Is $\mathbb{Z}/9\mathbb{Z}$ a field? Explain. You are given that $\mathbb{Z}/9\mathbb{Z}$ forms a commutative ring with respect to the usual operations.



Problem 10 (7pts) Let \mathbb{F} be a field and let $A_j \in \mathbb{F}^{n \times n}$ be invertible for $j \in \mathbb{N}$. Prove

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$$

for all $k \in \mathbb{N}$ by an inductive argument. You may use the fact for square matrices A, B, AB = I iff $A^{-1} = B$. Your proof should include a proof for the k = 2 case.

Problem 11 (7pts) If a 3×3 matrix A row-reduces to the identity matrix by the row-operation $r_1 \to r_1 + r_2$ followed by $r_2 \leftrightarrow r_3$ then find A and A^{-1} .