Please show your work and use words to explain your steps where appropriate.

Problem 1 (9pts) Let R be a commutative ring and suppose $A \in R^{m\times n}$ and $B \in R^{n\times p}$. Prove $(AB)^T = B^T A^T$.

Problem 2 (10pts) Calculate the following matrix expressions if possible, if not possible then write n/a. Let $u = [1, 2, 3]$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -1 & 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

(a.) $A^T A + u^T u$

(b.) $Au^T + Bw$

Problem 3 (10pts) Suppose $x + y = -3$ and $(2 + i)x - y = 4$. Find the solution in $\mathbb{C}$ and leave your answer in Cartesian form. (use any solution techinque you see fit)

Problem 4 (15pts) Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Problem 5 (10pts) Solve the following system of equations for arbitrary $a, b, c \in \mathbb{R}$:

$$\begin{aligned} x + y + 2z &= a \\ x + 2y + z &= b \\ y + z &= c \end{aligned}$$

by multiplication by inverse matrix (no credit will be awarded for row reduction here)

Problem 6 (10pts) Consider the system $x + y + z + w = 10$ and $y + 2z + w = 7$ over a field $\mathbb{F}$. Find the augmented coefficient matrix $[A|b]$ and calculate rref$[A|b]$. Also, find the solution **set**.

Problem 7 (20pts) Let $V_1, V_2, V_3, V_4 \in \mathbb{R}^3$ and define $S = \{V_1, V_2, V_3, V_4\}$. Let $W = (a, b, c)$ for some $a, b, c \in \mathbb{R}$. You are given

$$\text{rref}[V_1|V_2|V_3|V_4|W] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 2a-b \\ 0 & 1 & 3 & 0 & b-a \\ 0 & 0 & 0 & 1 & c-b \end{array}\right]$$

(a.) Write W as a linear combination of V_1, V_2 and V_4.

(b.) Find the set of all $(a, b, c) \notin \text{span}\{V_1, V_2, V_3\}$

(c.) is S linearly independent ? Discuss.

(d.) is $A = [V_1|V_2|V_3]$ invertible ? Explain.

(e.) what are the solutions of $xV_1 + yV_2 + zV_4 = 0$? Discuss.

Problem 8 (5pts) Is $\mathbb{Z}/9\mathbb{Z}$ a field ? Explain. You are given that $\mathbb{Z}/9\mathbb{Z}$ forms a commutative ring with respect to the usual operations.

Problem 9 (7pts) Denote $\mathbb{Z}/5\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ and define $A = \begin{bmatrix} \overline{1} & \overline{0} & \overline{3} & \overline{1} \\ \overline{2} & \overline{4} & \overline{3} & \overline{1} \end{bmatrix} \in (\mathbb{Z}/5\mathbb{Z})^{2\times 4}$. Calculate $\mathrm{rref}(A)$ and present your final answer in terms of $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}$.

Problem 10 (7pts) Let $\mathbb{F}$ be a field and let $A_j \in \mathbb{F}^{n\times n}$ be invertible for $j \in \mathbb{N}$. Prove

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$$

for all $k \in \mathbb{N}$ by an inductive argument. You may use the fact for square matrices A, B, $AB = I$ iff $A^{-1} = B$. Your proof should include a proof for the $k = 2$ case.

Problem 11 (7pts) If a 3×3 matrix A row-reduces to the identity matrix by the row-operation $r_1 \to r_1 + r_2$ followed by $r_2 \leftrightarrow r_3$ then find A and A^{-1}.