**Please show your work and use words to explain your steps where appropriate.**

**Problem 1** (9pts) Let $R$ be a commutative ring and suppose $A \in R^{m\times n}$ and $B \in R^{n\times p}$. Prove $(AB)^T = B^T A^T$.

Let $A, B$ be as above and suppose $(i,j) \in \mathbb{N}_m \times \mathbb{N}_p$,

$$((AB)^T)_{ij} = (AB)_{ji} \quad \text{: def}^{\underline{n}} \text{ of transpose.}$$
$$= \sum_{k=1}^{n} A_{jk} B_{ki} \quad \text{: def}^{\underline{n}} \text{ of matrix multiplication}$$
$$= \sum_{k=1}^{n} (A^T)_{kj} (B^T)_{ik} \quad \text{: def}^{\underline{n}} \text{ of transpose.}$$
$$= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} \quad \text{: multiplication in } R \text{ commutes.}$$
$$= (B^T A^T)_{ij} \quad \text{: def}^{\underline{n}} \text{ of matrix multiplication.}$$

Thus $(AB)^T = B^T A^T$ as the above holds $\forall (i,j) \in \mathbb{N}_m \times \mathbb{N}_p$ //

**Problem 2** (10pts) Calculate the following matrix expressions if possible, if not possible then write n/a. Let $u = [1,2,3]$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -1 & 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

(a.) $A^T A + u^T u$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 8 \\ 3 & 8 & 25 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 2 & 6 \\ 2 & 8 & 14 \\ 6 & 14 & 34 \end{bmatrix}}$$

(b.) $Au^T + Bw$ $= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix} + \begin{bmatrix} 65 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 75 \\ 15 \end{bmatrix}}$

**Problem 3** (10pts) Suppose $x + y = -3$ and $(2+i)x - y = 4$. Find the solution in $\mathbb{C}$ and leave your answer in Cartesian form. (use any solution techinque you see fit)

$$x + y = -3$$
$$(2+i)x - y = 4$$
$$(3+i)x = 1 \quad \therefore \quad x = \frac{1}{3+i} = \frac{3-i}{10}$$

$$y = -3 - x = -3 - \left(\frac{3-i}{10}\right) = \frac{-30-3+i}{10} = \frac{-33+i}{10}$$

$$\therefore \boxed{x = \frac{3}{10} - \frac{i}{10} \ \ \& \ \ y = \frac{-33}{10} + \frac{i}{10} \quad \text{aka } \frac{1}{10}(3-i, -33+i)}$$

**Problem 4** (15pts) Find $A^{-1}$ for $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_2]{r_1 - r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{r_3/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right] \xrightarrow[r_2 + r_3]{r_1 - 3r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & +1/2 & -3/2 \\ 0 & 1 & 0 & -1/2 & +1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

(right block: $A^{-1}$)

$$\therefore \boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & +1 & -3 \\ -1 & +1 & 1 \\ 1 & -1 & 1 \end{bmatrix}} \qquad \text{Check, } AA^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \overset{\checkmark}{=} I$$

**Problem 5** (10pts) Solve the following system of equations for arbitrary $a, b, c \in \mathbb{R}$:

$$x + y + 2z = a$$
$$x + 2y + z = b$$
$$y + z = c$$

by multiplication by inverse matrix (no credit will be awarded for row reduction here)

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a + b - 3c \\ -a + b + c \\ a - b + c \end{bmatrix}$$

$$\boxed{x = \tfrac{1}{2}(a + b - 3c)}$$
$$\boxed{y = \tfrac{1}{2}(-a + b + c)}$$
$$\boxed{z = \tfrac{1}{2}(a - b + c)}$$

**Problem 6** (10pts) Find the solution set of the system $x + y + z + w = 10$ and $y + 2z + w = 7$.

(added instructions here)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 2 & 1 & 7 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 & 7 \end{array}\right]$$

Thus $x = z + 3$, $y = -2z - w + 7$ and $z, w$ are free

$$\boxed{\text{Sol}^{\text{n}}\text{ Set} = \{ (z+3, -2z - w + 7, z, w) \mid z, w \in \mathbb{F} \}}$$

**Problem 7** (20pts) Let $V_1, V_2, V_3, V_4 \in \mathbb{R}^3$ and define $S = \{V_1, V_2, V_3, V_4\}$. Let $W = (a, b, c)$ for some $a, b, c \in \mathbb{R}$. You are given

$$\text{rref}[V_1|V_2|V_3|V_4|W] = \left[\begin{array}{cccc|c} V_1 & V_2 & V_3 & V_4 & W \\ 1 & 0 & -1 & 0 & 2a-b \\ 0 & 1 & 3 & 0 & b-a \\ 0 & 0 & 0 & 1 & c-b \end{array}\right]$$

**(a.)** Write $W$ as a linear combination of $V_1, V_2$ and $V_4$.

CCP $\Rightarrow$ $W = (2a-b)V_1 + (b-a)V_2 + (c-b)V_4$.

**(b.)** Find the set of all $(a, b, c) \notin \text{span}\{V_1, V_2, V_3\}$

CCP $\Rightarrow$ $\{(a,b,c) \mid c \neq b\} \subset \mathbb{R}^3$

**(c.)** is $S$ linearly independent ? Discuss.

No, $S$ has 4 vectors and at most 3 vectors can form LI subset of $\mathbb{R}^3$. Also, $V_3 = -V_1 + 3V_2$ by CCP.

**(d.)** is $A = [V_1|V_2|V_3]$ invertible ? Explain.

We see $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \neq I$

$\therefore A^{-1}$ d.n.e. (can also argue $A\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ thus $Av=0$ has $v \neq 0$ sol$^{\underline{n}}$. etc..

**(e.)** what are the solutions of $xV_1 + yV_2 + zV_4 = 0$ ? Discuss.

$$\underbrace{\left[V_1 | V_2 | V_4\right]}_{B}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

only has $(x,y,z) = (0,0,0)$ sol$^{\underline{n}}$ since $\text{rref}(B) = I$ from given data hence $B^{-1}$ exists and $Bv = 0$ iff $v = 0$.

**Problem 8** (5pts) Is $\mathbb{Z}/9\mathbb{Z}$ a field ? Explain. You are given that $\mathbb{Z}/9\mathbb{Z}$ forms a commutative ring with respect to the usual operations.

$(\bar{3})(\bar{3}) = \bar{9} = \bar{0}$ But, $\bar{3} \neq 0$ $\therefore$ $\bar{3}$ is a zero-divisor and hence $\frac{1}{\bar{3}}$ d.n.e. It follows $\mathbb{Z}/9\mathbb{Z}$ is not a field since $\exists$ a nonzero element which is not a unit.//

**Problem 9** (7pts) Denote $\mathbb{Z}/5\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ and define $A = \begin{bmatrix} \overline{1} & \overline{0} & \overline{3} & \overline{1} \\ \overline{2} & \overline{4} & \overline{3} & \overline{1} \end{bmatrix} \in (\mathbb{Z}/5\mathbb{Z})^{2\times 4}$. Calculate rref$(A)$ and present your final answer in terms of $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}$.

$$A = \begin{bmatrix} \overline{1} & \overline{0} & \overline{3} & \overline{1} \\ \overline{2} & \overline{4} & \overline{3} & \overline{1} \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} \overline{1} & \overline{0} & \overline{3} & \overline{1} \\ \overline{0} & \overline{4} & -\overline{3} & -\overline{1} \end{bmatrix} \xrightarrow{\overline{4}r_2} \begin{bmatrix} \overline{1} & \overline{0} & \overline{3} & \overline{1} \\ \overline{0} & \overline{16} & -\overline{12} & -\overline{4} \end{bmatrix} = \boxed{\begin{bmatrix} \overline{1} & \overline{0} & \overline{3} & \overline{1} \\ \overline{0} & \overline{1} & \overline{3} & \overline{1} \end{bmatrix} = \text{rref}(A)}$$

**Problem 10** (7pts) Let $\mathbb{F}$ be a field and let $A_j \in \mathbb{F}^{n\times n}$ be invertible for $j \in \mathbb{N}$. Prove

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$$

for all $k \in \mathbb{N}$ by an inductive argument. You may use the fact for square matrices $A, B$, $AB = I$ iff $A^{-1} = B$. Your proof should include a proof for the $k = 2$ case.

Observe if $A^{-1}, B^{-1}$ exist for square $A, B$ over a field $\mathbb{F}$ then $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I \therefore (AB)^{-1} = B^{-1}A^{-1}$.

Consider, $(A_1)^{-1} = A_1^{-1}$ hence $\underbrace{(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}}_{*}$ for $k=1$.

Suppose $*$ is true for $k \in \mathbb{N}$ (towards an induction argument)
Consider, for invertible $A_1, \dots, A_k$ over $\mathbb{F}$, (square matrices)

$$(A_1A_2\cdots A_kA_{k+1})^{-1} = [\underbrace{(A_1\cdots A_k)}_{A}\underbrace{(A_{k+1})}_{B}]^{-1}$$
$$= B^{-1}A^{-1} \quad \text{by initial sentences in this sol}^{\text{t}}.$$
$$= A_{k+1}^{-1}A_k^{-1}\cdots A_2^{-1}A_1^{-1} \quad \text{by induct. hypo. } (*).$$

Thus $(*)$ holds for $k+1$ and we conclude $(*)$ is true $\forall k \in \mathbb{N}$ by induction on $k$. //

**Problem 11** (7pts) If a $3 \times 3$ matrix $A$ row-reduces to the identity matrix by the row-operation $r_1 \to r_1 + r_2$ followed by $r_2 \leftrightarrow r_3$ then find $A$ and $A^{-1}$.

$$\text{rref}(A) = E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{E_2:\ r_2\leftrightarrow r_3}{\xrightarrow{r_2\leftrightarrow r_3}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E_1A} \underset{E_1:\ r_1+r_2}{\xrightarrow{r_1 - r_2}} \boxed{\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A} = A}$$

$$A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = A^{-1}}$$