We assume $\mathbb{F}$ is a field and V, W are vector spaces over $\mathbb{F}$.

Problem 1 Let A, B, N, S be invertible square matrices. Solve the following equation for X:

$$(BA)^{-1}XS^{-1} = (NA)^2$$

Problem 2 Let V_1, V_2 be subspaces of V over $\mathbb{F}$. Prove $V_1 \cap V_2$ is a subspace of V.

Problem 3 You are given $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $T(e_1) = (0, 1, 1)$ and $T(e_2) = (1, 2, 3)$ and $T(e_3) = (1, 1, 2)$. Find the bases for the kernel and image of T. Is T an surjection ? Is T an injection ? Is T an isomorphism ?

Problem 4 Friedberg, Insel and Spence 5th edition, §1.6#17, page 56.

Problem 5 Friedberg, Insel and Spence 5th edition, §2.1#5, page 74.

Problem 6 Friedberg, Insel and Spence 5th edition, §2.1#15, page 75.

Problem 7 Friedberg, Insel and Spence 5th edition, §2.1#17, page 76.

Problem 8 Let $T(f(x)) = \begin{bmatrix} f(0) & if'(0) \\ if(1) & f'(1) \end{bmatrix}$ define a linear transformation from $V = P_2(\mathbb{R})$ to $W = \mathbb{C}^{2\times 2}$. If we use basis $\beta = \{1, x, x^2\}$ for $V(\mathbb{R})$ and $\gamma = \{E_{11}, iE_{11}, E_{12}, iE_{12}, E_{21}, iE_{21}, E_{22}, iE_{22}\}$ for $W(\mathbb{R})$ then find $[T]_{\beta,\gamma}$. Is T an injective map ?

Problem 9 Define $T : P_3(\mathbb{R}) \to \mathbb{R}^{1\times 3}$ by $T(f(x)) = [f(1), f(2), f(1) + f(2)]$. Find a basis β for $P_3(\mathbb{R})$ and γ for $\mathbb{R}^{1\times 3}$ for which $[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right]$ where $p = rank(T)$.

Problem 10 Let $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & -1 & 3 \\ 1 & 3 & -2 & 4 \end{bmatrix}$. If $\beta = \{1, x, x^2, x^3\} \subseteq P_3(\mathbb{R})$ and $\gamma = \{E_{11}, E_{22}, E_{12} + E_{21}\} \subseteq \mathbb{R}^{2\times 2}$ and $[T]_{\beta,\gamma} = A$ then find the formula for $T(a + bx + cx^2 + dx^3)$ and find the basis for $Ker(T)$.

Problem 11 Suppose $L : P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ is a linear transformation for which

$$L(1) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \& \quad L(t) = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \& \quad L(t^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

. Let $\beta = \{t^2, t, 1\}$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ serve as bases for $P_2(\mathbb{R})$ and $\mathbb{R}^{2\times 2}$ respective.

(a.) find $[T]_{\beta,\gamma}$

(b.) calculate the rank and nullity of T

(c.) find a basis for $Ker(T)$

Problem 12 Suppose $W_1 \leq W_2 \leq V$ over $\mathbb{F}$. Prove $\operatorname{ann}(W_2) \leq \operatorname{ann}(W_1) \leq V^*$.

Problem 13 Friedberg, Insel and Spence 5th edition, §2.6#15, page 127.

Problem 14 Let $V = P_4(\mathbb{Q}) \times \mathbb{Q}^{2\times 2}$. Find an isomorphism of V and $\mathbb{Q}^{3\times 3}$.

Problem 15 Find an isomorphism of $V = \{A \in \mathbb{R}^{3\times 3} \mid A^T = A\}$ to $\mathbb{C}^n$ for an appropriate choice of n.

Problem 16 (Bonus) Friedberg, Insel and Spence 5th edition, §1.7#3, page 62.

MATH 321, QUIZ 2 SOLUTION

$\boxed{P1}$ Let A, B, N, S be invertible square matrices.

Suppose $(BA)^{-1} X S^{-1} = (NA)^2$. Solve for X,

$$(BA)(BA)^{-1} X S^{-1} S = (BA)(NA)^2 S$$

: multiplied by BA on left and S on the right.

Thus,

$$I X I = BA(NA)(NA)S$$

$$\therefore \boxed{X = BANANAS}$$

$\boxed{P2}$ Let V_1, V_2 be subspaces of V over $\mathbb{F}$.

Since $0 \in V_1$ and $0 \in V_2$ we find $0 \in V_1 \cap V_2$ thus $V_1 \cap V_2 \neq \emptyset$. Let $c \in \mathbb{F}$ and $x, y \in V_1 \cap V_2$ then $x, y \in V_1$ and $x, y \in V_2$. Thus $cx + y \in V_1$ and $cx + y \in V_2$ since $V_1 \leq V$ and $V_2 \leq V$.

Consequently, $cx + y \in V_1 \cap V_2$ and so $cx \in V_1 \cap V_2$ and $x + y \in V_1 \cap V_2$ and we conclude $V_1 \cap V_2 \leq V$ by the subspace test.

P3 $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ has $T(e_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T(e_3) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ (and we assume T is linear). Observe the standard matrix

$$[T] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

thus $\underline{\text{rref}\,[T] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}\,*$ we find $col_3 = col_2 - col_1$ for $[T]$ by CCP, indeed $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$$\text{Col}\,[T] = \boxed{\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} = \text{Im}\,(T)}.$$

The basis for $\text{Ker}\,(T) = \text{Null}\,[T]$ is found from studying $[T]x = 0 \Rightarrow x_1 = x_3$ and $x_2 = -x_3$ from $*$ thus $x = (x_3, -x_3, x_3) = x_3(1, -1, 1)$ hence

$$\boxed{\text{Null}\,[T] = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \text{Ker}\,(T)}$$

Since $\text{Im}\,(T) \neq \mathbb{R}^3$ we find T is not onto.
Since $\text{Ker}\,(T) \neq \{0\}$ we find T is not one-to-one.
Since T is not both a surjection and injection we find T is not a bijection, thus T is not an isomorphism.

<u>Remark:</u> we only get $\text{Ker}(T) = \text{Null}\,[T]$ and $\text{Im}\,(T) = \text{Col}\,[T]$ in the context $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. In other contexts we can't use rref and CCP directly like we did here.

P4) text, §1.6 #17, p. 56

$W = \{ A \in \mathbb{F}^{n\times n} / A^T = -A \}$

Find a basis for W and calculate $\dim(W)$.

If $A \in W$ then $A^T = -A \Rightarrow A_{ji} = -A_{ij} \; \forall i,j$.

If $i = j$ then $A_{ii} = -A_{ii} \Rightarrow 2A_{ii} = 0 \Rightarrow \underline{A_{ii} = 0}$.

(I assume $2 \neq 0$ in the field $\mathbb{F}$)

Thus, if $A \in W$ then

$$\begin{aligned}
A &= \sum_{i \neq j} A_{ij} E_{ij} \\
&= \sum_{i<j} A_{ij} E_{ij} + \sum_{i>j} A_{ij} E_{ij} \\
&= \sum_{i<j} A_{ij} E_{ij} + \sum_{k<l} A_{lk} E_{lk} \\
&= \sum_{i<j} A_{ij} E_{ij} + \sum_{k<l} -A_{kl} E_{lk} \\
&= \sum_{i<j} A_{ij} E_{ij} - \sum_{i<j} A_{ij} E_{ji} \\
&= \sum_{i<j} A_{ij} (E_{ij} - E_{ji})
\end{aligned}$$

relabel sum $i \mapsto l$, $j \mapsto k$

~~switch back~~ $A_{lk} = -A_{kl}$

$k \to i$, $l \to j$

Thus $\beta = \{ E_{ij} - E_{ji} \mid 1 \leq i < j \leq n \}$ serves as a basis since $W = \operatorname{span} \beta$ and β is LI by the above calculation.

$\boxed{P4}$ continued

$$\beta = \{ E_{ij} - E_{ji} \mid 1 \le i < j \le n \}$$

we can count,

$i = 1$ then $2 \le j \le n \Rightarrow n-1$ choices

$i = 2$ then $3 \le j \le n \Rightarrow n-2$ choices

$\vdots \qquad\qquad \vdots$

$i = n-1$ then $j = n \Rightarrow n-(n-1) = 1$ choice

thus

$$\dim(W) = \underbrace{(n-1) + (n-2) + \cdots + (n-(n-1))}_{(n-1)\text{-fold summands}}$$

$$= n(n-1) - (1 + 2 + \cdots + (n-1))$$

$$= n(n-1) - \frac{1}{2}((n-1))(n-1+1)$$

Gauss' Formula.

$$= n^2 - n - \frac{1}{2}(n^2 - n)$$

$$= \frac{1}{2}n^2 - \frac{1}{2}n$$

$$= \boxed{\frac{1}{2}n(n-1)}$$

Let's check my counting.

$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \hookrightarrow \dim(W) = 1$ for $n = 2$

$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \hookrightarrow \dim(W) = 3$ for $n = 3$

$\frac{1}{2}3(3-1) = \frac{1}{2}(6) = 3.$

P5 text §2.1 #5 p. 74

Let $T: P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(f) = xf + f'$

Calculate Ker (T), Im (T), nullity (T), rank (T) and verify the dimension Th$^{\underline{m}}$ and decide 1-1 or onto

Show T is linear. Let $c \in \mathbb{R}$ and $f(x), g(x) \in P_2(\mathbb{R})$

$$\begin{aligned} T(cf(x)+g(x)) &= x(cf(x)+g(x)) + (cf(x)+g(x))' \\ &= c(xf(x) + f'(x)) + xg(x) + g'(x) \\ &= c\,T(f(x)) + T(g(x)) \end{aligned}$$

Thus $T \in \mathcal{L}(P_2(\mathbb{R}), P_3(\mathbb{R}))$. Consider $f(x) = ax^2 + bx + c \in P_2(\mathbb{R})$ and suppose $f(x) \in \text{Ker}(T)$

then $\underline{x(ax^2+bx+c) + 2ax + b = 0}$ $*$

$$\Rightarrow a(x^3) + bx^2 + (c+2a)x + b = 0$$

$$\Rightarrow a = 0, \quad b = 0, \quad c+2a = 0, \quad b = 0$$

Hence, $a = 0$, $b = 0$, $c = 0$ $\therefore f(x) = 0$ and we find $\boxed{\text{Ker}(T) = 0}$. Observe, by $*$

$$\begin{aligned} T(ax^2+bx+c) &= ax^3 + bx^2 + (c+2a)x + b \\ &= a(x^3+2x) + b(x^2+1) + cx \end{aligned}$$

Thus $\boxed{\text{Im}(T) = \text{span}\{x^3+2x,\ x^2+1,\ x\}}$. We've found

$$\text{rank}(T) = \dim(\text{Im}(T)) = 3$$
$$\text{nullity}(T) = \dim(\text{Ker}(T)) = 0$$

This verifies $\dim(P_2(\mathbb{R})) = 3 = \text{rank}(T) + \text{nullity}(T)$. We have T is injective but not surjective since $\text{Im}(T) \neq P_3(\mathbb{R})$.

P6 text §2.1 # 15 p. 75

Recall $P(\mathbb{R}) = \mathbb{R}[x]$ (my notation)

Define $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by $T(f(x)) = \int_0^x f(t)\,dt$.

Prove that T is linear and one-to-one, but not onto.

Let $c \in \mathbb{R}$ and $f(x), g(x) \in \mathbb{R}[x]$ then

$$\begin{aligned} T(cf(x)+g(x)) &= \int_0^x (cf(t)+g(t))\,dt \\ &= c\int_0^x f(t)\,dt + \int_0^x g(t)\,dt \\ &= c\,T(f(x)) + T(g(x)) \end{aligned}$$

Therefore, $T \in \mathcal{L}(\mathbb{R}[x], \mathbb{R}[x])$. Let $f(x) \in \mathrm{Ker}(T)$ then if $f(x) = c_0 + c_1 x + \cdots + c_n x^n$ then $T(f(x)) = 0$ yields

$$\int_0^x (c_0 + c_1 t + \cdots + c_n t^n)\,dt = 0$$

$$c_0 x + \tfrac{1}{2}c_1 x^2 + \cdots + \tfrac{1}{n+1} c_n x^{n+1} = 0 \quad *$$

Consequently, $c_0 = 0$, $c_1 = 0, \ldots, c_n = 0$ and we find $f(x) = 0$ thus $\mathrm{Ker}(T) = 0$ and it follows that T is injective. From $*$ we find $T(f(x)) = 1 = c_0 x + \cdots + \frac{1}{n+1} c_n x^{n+1}$ has no solution thus $\mathrm{Im}(T) \neq \mathbb{R}[x]$ since $1 \notin \mathrm{Im}(T)$. T is not onto since it fails to map to the constant polynomials.

P7 text, §2.1 # 17, p. 76

Let V, W be finite dim'l vector spaces over $\mathbb{F}$ and let $T: V \longrightarrow W$ be linear.

(a.) Prove that if $\dim V < \dim W$ then T is not onto.

(b.) Prove that if $\dim V > \dim W$ then T is not one-to-one.

The key to proving both (a) and (b.) is the dimension Th^{m} a.k.a the rank/nullity Th$^{\underline{m}}$,

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

where $\operatorname{rank}(T) = \dim(\operatorname{Im}(T))$ and $\operatorname{nullity}(T) = \dim(\operatorname{Ker}(T))$.

(a.) Suppose $\dim V < \dim W$ then if $\operatorname{rank}(T) = \dim(W)$ then $\dim V = \dim W + \operatorname{nullity}(T) \geq \dim(W)$ which is a $\rightarrow\leftarrow$ thus $\operatorname{rank}(T) \neq \dim(W)$ and we conclude T is not onto.

(b.) Suppose $\dim V > \dim W$. If $\operatorname{nullity}(T) = 0$ then $\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T) = \operatorname{rank}(T)$ thus $\underline{\operatorname{rank}(T) > \dim W}_{*}$. However, we know

$\operatorname{Im}(T) \leq W \Rightarrow \cancel{\dim(T) \leq \dim(W)}$

$\Rightarrow \dim(\operatorname{Im}(T)) \leq \dim(W)$ that is $\underline{\operatorname{rank}(T) \leq \dim(W)}_{**}$

Clearly $*$ and $**$ contradict. Thus $\operatorname{nullity}(T) \neq 0$ and so $\operatorname{Ker}(T) \neq \{0\}$ $\therefore$ T is not injective.

$\boxed{P8}$ Let $T(f(x)) = \left[\begin{array}{c|c} f(0) & i f'(0) \\ \hline i f(1) & f'(1) \end{array}\right]$ define a linear transformation $T : \underbrace{P_2(\mathbb{R})}_{V(\mathbb{R})} \longrightarrow \underbrace{\mathbb{C}^{2\times 2}}_{W(\mathbb{R})}$

Use $\beta = \{1, x, x^2\}$

$\gamma = \{E_{11}, iE_{11}, E_{12}, iE_{12}, E_{21}, iE_{21}, E_{22}, iE_{22}\}$

$$[T]_{\beta,\gamma} = \left[\, [T(1)]_\gamma \,\middle|\, [T(x)]_\gamma \,\middle|\, [T(x^2)]_\gamma \,\right]$$

$$= \left[\, \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}_\gamma \,\middle|\, \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}_\gamma \,\middle|\, \begin{bmatrix} 0 & 0 \\ i & 2 \end{bmatrix}_\gamma \,\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\text{rank}([T]_{\beta,\gamma}) = 3$ and as

$\dim(P_2(\mathbb{R})) = rk(T) + \nu(T) = 3 + \nu(T) = 3$

we find $\nu(T) = \text{nullity}(T) = 0 \therefore \text{Ker}(T) = 0$

hence T is one-to-one.

That is, T is <u>injective</u>.

(you could also just solve $T(f(x)) = 0$ and deduce $f(x) = 0$ by direct calculation)

$\boxed{P10}$ $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & -1 & 3 \\ 1 & 3 & -2 & 4 \end{bmatrix} = [T]_{\beta,\gamma}$

where $\beta = \{1, x, x^2, x^3\} \subseteq P_3(\mathbb{R})$ and $\gamma = \{E_{11}, E_{22}, E_{12}+E_{21}\} \subseteq \mathbb{R}^{2\times 2}$

$$[T(1)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \hookrightarrow T(1) = E_{11} + E_{22} + E_{12} + E_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$[T(x)]_\gamma = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow T(x) = E_{11} + 2E_{22} + 3(E_{12}+E_{21}) = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$[T(x^2)]_\gamma = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \Rightarrow T(x^2) = -E_{22} - 2(E_{12}+E_{21}) = \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix}$$

$$[T(x^3)]_\gamma = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow T(x^3) = 2E_{11} + 3E_{22} + 4(E_{12}+E_{21}) = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$$

Consequently,

$$\begin{aligned} T(a+bx+cx^2+dx^3) &= aT(1) + bT(x) + cT(x^2) + dT(x^3) \\ &= a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} + c\begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix} + d\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix} \\ &= \boxed{\left[\begin{array}{c|c} a+b+2d & a+3b-2c+4d \\ \hline a+3b-2c+4d & a+2b-c+3d \end{array}\right]} \end{aligned}$$

Notice $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ thus $x \in \text{Null}(A)$

has $x_1 = -x_3 - x_4$ and $x_2 = x_3 - x_4 \Rightarrow x = (-x_3 - x_4, x_3 - x_4, x_3, x_4)$

thus $\text{Null}(A)$ has basis $\{(-1,1,1,0), (-1,-1,0,1)\}$.

$$\begin{aligned} \text{Ker}(T) &= \Phi_\beta^{-1}(\text{Null}[T]_{\beta,\gamma}) \quad \text{as } \Phi_\beta^{-1} \text{ is isomorphism} \\ &= \Phi_\beta^{-1}(\text{span}\{(-1,1,1,0), (-1,-1,0,1)\}) \\ &= \text{span}\{\Phi_\beta^{-1}(-1,1,1,0), \Phi_\beta^{-1}(-1,-1,0,1)\} \end{aligned}$$

P 10 continued

We find Ker (T) has basis given by

$$\{ \Phi_\beta^{-1}(-1,1,1,0),\ \Phi_\beta^{-1}(-1,-1,0,1) \}$$

Since $\beta = \{1, x, x^2, x^3\}$ we deduce

$$\boxed{\{-1+x+x^2,\ -1-x+x^3\}}$$

is basis for Ker (T).

Alternatively: set $T(a+bx+cx^2+dx^3) = 0$ and work out the 4 eq^{ns} in a, b, c, d to describe Ker (T). Should yield same outcome.

$\boxed{P11}$ $L : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^{2\times 2}$ is linear transformation for which $L(1) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ \$ $L(t) = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ \$ $L(t^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Let $\beta = \{t^2, t, 1\}$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ serve as bases for $P_2(\mathbb{R})$ and $\mathbb{R}^{2\times 2}$ respective

(a.) $$[L]_{\beta,\gamma} = \left[[L(t^2)]_\gamma \,\middle|\, [L(t)]_\gamma \,\middle|\, [L(1)]_\gamma \right]$$

$$= \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_\gamma \,\middle|\, \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}_\gamma \,\middle|\, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}_\gamma \right]$$

$$= \underline{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}}.$$

(b.) $$[L]_{\beta,\gamma} \sim \begin{bmatrix} 1 & 3 & 2 \\ 1 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

thus $\text{Null}\,[L]_{\beta,\gamma} = \{0\} \Rightarrow \text{Ker}(L) = \{0\}$

$\boxed{\text{nullity}(L) = 0}$

Since $\text{Im}(L) = \Phi_\gamma^{-1}(\text{Col}\,[L]_{\beta,\gamma})$ and $\text{rank}\,[L]_{\beta,\gamma} = 3$ we find $\dim(\text{Im}(L)) = 3$ thus $\boxed{\text{rank}(L) = 3}$

(c.) $\text{Ker}(L) = \{0\} = \text{span}(\emptyset)$

Basis for $\text{Ker}(L)$ is $\emptyset$.

$\boxed{P12}$ Suppose $W_1 \leq W_2 \leq V$ over $\mathbb{F}$

Let $\text{Ann}(W) = \{\alpha \in V^* \mid \alpha(x) = 0 \quad \forall x \in W\}$

Notice $0 \in V^*$ has $0(x) = 0 \quad \forall x \in V$ thus $0 \in \text{Ann}(W)$

and we see $\text{Ann}(W) \neq \emptyset$. Let $\alpha, \beta \in \text{ann}(W)$ and suppose $c \in \mathbb{F}$. Let $x \in W$,

$$
\begin{aligned}
(c\alpha + \beta)(x) &= c\alpha(x) + \beta(x) && : \text{def}^{\underline{n}} \text{ of } + \text{ in } V^* \\
&= c(0) + 0 && : \text{since } \alpha, \beta \in \text{ann}(W) \\
&= 0
\end{aligned}
$$

Thus $c\alpha$, $\alpha + \beta \in \text{ann}(W)$ and the subspace test gives that $\text{ann}(W) \leq V^*$. Since $W_1, W_2 \leq V$ we find $\text{ann}(W_1), \text{ann}(W_2) \leq V^*$.

Suppose $\alpha \in \text{ann}(W_2)$ and let $x \in W_1$. Then as $W_1 \leq W_2$ we find $x \in W_2$ and so $\alpha(x) = 0$. Thus $\alpha \in \text{ann}(W_1)$ and it follows $\text{ann}(W_2) \subseteq \text{ann}(W_1)$. Hence,

$$\text{ann}(W_2) \leq \text{ann}(W_1) \leq V^*.$$

P13 text, §2.6 #15, p. 127

Suppose W is finite dim'l vector space and that $T: V \to W$ is linear. Prove that

$$\underbrace{\mathrm{Ker}\,(T^*) = \mathrm{ann}\,(\mathrm{Im}\,(T))}_{N(T^*) = (R(T))^0 \text{ in book's notation}}$$

Recall $T^*: W^* \to V^*$ is defined by
$(T^*(\alpha))(x) = \alpha(T(x)) \quad \forall x \in V,\ \alpha \in W^*.$

① Let $\alpha \in \mathrm{Ker}\,(T^*)$ then $T^*(\alpha) = 0$
Hence $(T^*(\alpha))(x) = \alpha(T(x)) = 0 \quad \forall x \in V.$
Thus $\alpha(y) = 0 \quad \forall y = T(x) \in \mathrm{Im}(T)$ (where $x \in V$.)
That means $\alpha \in \mathrm{ann}\,(\mathrm{Im}\,(T))$ and
so $\mathrm{Ker}\,(T^*) \subseteq \mathrm{ann}\,(\mathrm{Im}(T))$.

② Let $\beta \in \mathrm{ann}(\mathrm{Im}(T))$ then $\beta(y) = 0$
$\forall y \in \mathrm{Im}(T)$. Hence $\beta(T(x)) = 0 \quad \forall x \in V.$
Thus $(T^*(\beta))(x) = 0 \quad \forall x \in V.$
which shows $T^*(\beta) = 0 \therefore \beta \in \mathrm{Ker}(T^*)$
and so $\mathrm{ann}\,(\mathrm{Im}\,(T)) \subseteq \mathrm{Ker}\,(T^*)$.

$\boxed{P14}$ $V = P_4(\mathbb{Q}) \times \mathbb{Q}^{2\times 2}$

Find isomorphism of V and $\mathbb{Q}^{3\times 3}$

$$\Psi\left((a+bx+cx^2+dx^3+ex^4, \begin{bmatrix} f & g \\ h & i \end{bmatrix}\right) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Remark: many other correct answers exist.

$\boxed{P15}$ Find isomorphism of $V = \{A \in \mathbb{R}^{3\times 3} \mid A^T = A\}$ to $\mathbb{C}^n$ for appropriate choice of n.

$$V = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$$

$$\psi(A) = (A_{11} + iA_{12},\ A_{13} + iA_{22},\ A_{23} + iA_{33})$$

$\psi : V \longrightarrow \mathbb{C}^3$ is an isomorphism.

Remark: it's good if you checked linearity onto and 1-1 (or just onto or just 1-1) But, I've been lazy here ☺.

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$V = \mathbb{R}(\mathbb{Q})$

Prove $\dim(V) = \infty$. Hint use that π is transcendental

$\nexists$ $P(t) \in \mathbb{Q}[t]$ for which $P(\pi) = 0$.

Consider the set

$$\beta = \{\pi, \pi^2, \pi^3, \pi^4, \dots\}$$

If β is linearly dependent then $\exists$ rational $c_1, c_2, \dots, c_n \in \mathbb{Q}$, not all zero, for which

$$c_1 \pi + c_2 \pi^2 + \cdots + c_n \pi^n = 0$$

Then $P(t) = c_1 t + c_2 t^2 + \cdots + c_n t^n \in \mathbb{Q}[t]$ and $P(\pi) = c_1 \pi + \cdots + c_n \pi^n = 0 \Rightarrow \pi$ is not transcendental $\rightarrow\leftarrow$ given fact. Thus β is LI and so $\beta \subseteq \mathbb{R}(\mathbb{Q})$ and $|\beta| = \infty$

$\Rightarrow \dim(V) = \infty$.

(If $\dim_{\mathbb{Q}}(\mathbb{R}) < \infty$ then we cannot fit a LI set with ∞ly many vectors inside $\mathbb{R}$)