Please show your work and use words to explain your steps where appropriate. You can work together, but, the solution you turn in must be your own work. Copy ideas not steps. This quiz is worth at least 100pts (8pts per problem)

Problem 1 Curtis §22 exercise #5 on page 192-193.

Problem 2 Let $\beta = \{e^{2x}, xe^{2x}, e^x\}$ and define $V = \text{span}_{\mathbb{R}}(\beta)$. Let $T = D - 2$ where $D = d/dx$. Show that β is a Jordan basis for T.

Problem 3 Let $T = L_A$ where $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 7 \\ 0 & 7 & 4 \end{bmatrix}$ find an eigenbasis for T. Also, find transformations E_1, E_2, E_3 for which $E_j\mathbb{R}^3 = \text{Ker}(T - \alpha_j)$ and $\mathbb{R}^3 = E_1\mathbb{R}^3 \oplus E_2\mathbb{R}^3 \oplus E_3\mathbb{R}^3$ where E_1, E_2, E_3 are idempotent and pairwise commuting with $E_1E_2 = E_1E_3 = E_2E_3 = 0$

Problem 4 Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. I'll be nice and tell you the eigenvalues are $\lambda = 8$ and -1. Find an eigenbasis for A, diagonalize A and calculate A^n explicitly.

Problem 5 Continuing the previous problem, let $T = L_A$ and find the standard matrices of transformations E_j for which $E_j\mathbb{R}^3 = \text{Ker}(T - \alpha_j)$ and $\mathbb{R}^3$ is the direct sum of the ranges of $E_j\mathbb{R}^3$ where E_j are idempotent and pairwise commuting with products $E_iE_j = 0$ for $i \neq j$.

Problem 6 Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ find an basis for $\mathbb{R}^2$ consisting of eigenvectors for A. Find the **order** $m_v(x) \in \mathbb{R}[x]$ for each eigenvector v in your basis. Also, find the minimal and characteristic polynomials for A.

Problem 7 Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$ find an eigenvector v belonging to the characteristic root $\alpha = 5$. Calculuate the **order** $m_v(x) \in \mathbb{R}[x]$. Also, find the minimal and characteristic polynomials for A. (the other eigenvalues of A are not particularly nice numbers)

Problem 8 Let $T : V \to V$ where $\dim_{\mathbb{Q}}(V) = 4$. You are given that there exists a basis $\beta = \{a, b, c, v\}$ for V where

$$T(a) = 3a, \quad (T - 3)(b) = a, \quad (T - 3)(c) = b, \quad T(v) = 7v$$

Find $[T]_{\beta,\beta}$ and find the Jordan form for T. Also, find the minimal and characteristic polynomials for T. Find $f_1(x), f_2(x) \in \mathbb{Q}[x]$ for which $E_1 = f_1(T)$ and $E_2 = f_2(T)$ are operators such that $E_1V \oplus E_2V = V$ where E_1, E_2 are idempotent endomorphisms for which $E_1E_2 = E_2E_1 = 0$.

Problem 9 Let V be a real vector space. Furthermore, suppose $T : V \to V$ is a linear transformation such that there exist $v, w \in V_{\mathbb{C}}$ with

$$T_{\mathbb{C}}(v) = \lambda v, \qquad T_{\mathbb{C}}(w) = \lambda w + v$$

where $\lambda = 2+3i$. If $v = v_1+iv_2$ and $w = w_1+iw_2$ are both nonzero where $v_1, v_2, w_1, w_2 \in V$ then let $\beta = \{v_1, v_2, w_1, w_2\}$ and find $[T]_{\beta,\beta}$. Also, find $[T_{\mathbb{C}}]_{\gamma,\gamma} \in \mathbb{C}^{2\times 2}$ for $\gamma = \{v, w\}$. Neji Hoogerwerf wonders, can you see some simple correspondence between the real and complex matrices you found in this problem?

Problem 10 Find the companion matrix of $p(x) = (x^2 + 4x + 13)^3$ (use the set-up as given in Chapter 7 of Curtis)

Problem 11 Consider the order 4 differential equation $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$ where $y' = \frac{dy}{dt}$ etc. You can convert this to a system of four first order linear differential equations by **reduction of order**. Let

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y'''$$

Find $A \in \mathbb{R}^{4\times 4}$ for which $\vec{x}\,' = (x_1', x_2', x_3', x_4') = A(x_1, x_2, x_3, x_4)$. Find a Jordan basis for A and calculate the general vector solution of $\vec{x}' = A\vec{x}$. Finally, extract the general solution to the differential equation in y from the first component of your general vector solution.

Problem 12 Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ where A is the companion matrix you found for Problem 10 of this Quiz. This will involve extracting 6 real solutions from a 3-chain of generalized complex e-vectors.

Problem 13 Let A be the matrix studied in Problem 11 of this Quiz. Calculate the matrix exponential e^{tA}.

Bonus: (S-Rank) suppose S, T are linear transformations on a finite dimensional vector space V over a field $\mathbb{F}$. Furthermore, both S and T are diagonable and $S \circ T = T \circ S$. **Prove** there exists a basis β of V for which both $[S]_{\beta,\beta}$ and $[T]_{\beta,\beta}$ are diagonal matrices. This result is can be phrased: *commuting linear transformations are simultaneously diagonalizable.*