Please show your work and use words to explain your steps where appropriate. You can work together, but, the solution you turn in must be your own work. Copy ideas not steps. This quiz is worth at least 100pts (8pts per problem)

Problem 1 Curtis §22 exercise #5 on page 192-193.

Problem 2 Let $\beta = \{e^{2x}, xe^{2x}, e^x\}$ and define $V = \text{span}_{\mathbb{R}}(\beta)$. Let $T = D - 2$ where $D = d/dx$. Show that β is a Jordan basis for T.

Problem 3 Let $T = L_A$ where $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 7 \\ 0 & 7 & 4 \end{bmatrix}$ find an eigenbasis for T. Also, find transformations E_1, E_2, E_3 for which $E_j\mathbb{R}^3 = \text{Ker}(T - \alpha_j)$ and $\mathbb{R}^3 = E_1\mathbb{R}^3 \oplus E_2\mathbb{R}^3 \oplus E_3\mathbb{R}^3$ where E_1, E_2, E_3 are idempotent and pairwise commuting with $E_1E_2 = E_1E_3 = E_2E_3 = 0$

Problem 4 Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. I'll be nice and tell you the eigenvalues are $\lambda = 8$ and -1. Find an eigenbasis for A, diagonalize A and calculate A^n explicitly.

Problem 5 Continuing the previous problem, let $T = L_A$ and find the standard matrices of transformations E_j for which $E_j\mathbb{R}^3 = \text{Ker}(T - \alpha_j)$ and $\mathbb{R}^3$ is the direct sum of the ranges of $E_j\mathbb{R}^3$ where E_j are idempotent and pairwise commuting with products $E_iE_j = 0$ for $i \neq j$.

Problem 6 Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ find an basis for $\mathbb{R}^2$ consisting of eigenvectors for A. Find the **order** $m_v(x) \in \mathbb{R}[x]$ for each eigenvector v in your basis. Also, find the minimal and characteristic polynomials for A.

Problem 7 Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$ find an eigenvector v belonging to the characteristic root $\alpha = 5$. Calculuate the **order** $m_v(x) \in \mathbb{R}[x]$. Also, find the minimal and characteristic polynomials for A. (the other eigenvalues of A are not particularly nice numbers)

Problem 8 Let $T : V \to V$ where $\dim_{\mathbb{Q}}(V) = 4$. You are given that there exists a basis $\beta = \{a, b, c, v\}$ for V where

$$T(a) = 3a, \quad (T-3)(b) = a, \quad (T-3)(c) = b, \quad T(v) = 7v$$

Find $[T]_{\beta,\beta}$ and find the Jordan form for T. Also, find the minimal and characteristic polynomials for T. Find $f_1(x), f_2(x) \in \mathbb{Q}[x]$ for which $E_1 = f_1(T)$ and $E_2 = f_2(T)$ are operators such that $E_1V \oplus E_2V = V$ where E_1, E_2 are idempotent endomorphisms for which $E_1E_2 = E_2E_1 = 0$.

Problem 9 Let V be a real vector space. Furthermore, suppose $T : V \to V$ is a linear transformation such that there exist $v, w \in V_{\mathbb{C}}$ with

$$T_{\mathbb{C}}(v) = \lambda v, \qquad T_{\mathbb{C}}(w) = \lambda w + v$$

where $\lambda = 2+3i$. If $v = v_1+iv_2$ and $w = w_1+iw_2$ are both nonzero where $v_1, v_2, w_1, w_2 \in V$ then let $\beta = \{v_1, v_2, w_1, w_2\}$ and find $[T]_{\beta,\beta}$. Also, find $[T_{\mathbb{C}}]_{\gamma,\gamma} \in \mathbb{C}^{2\times 2}$ for $\gamma = \{v, w\}$. Neji Hoogerwerf wonders, can you see some simple correspondence between the real and complex matrices you found in this problem?

Problem 10 Find the companion matrix of $p(x) = (x^2+4x+13)^3$ (use the set-up as given in Chapter 7 of Curtis)

Problem 11 Consider the order 4 differential equation $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$ where $y' = \frac{dy}{dt}$ etc. You can convert this to a system of four first order linear differential equations by **reduction of order**. Let

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad x_4 = y'''$$

Find $A \in \mathbb{R}^{4\times 4}$ for which $\vec{x}' = (x_1', x_2', x_3', x_4') = A(x_1, x_2, x_3, x_4)$. Find a Jordan basis for A and calculate the general vector solution of $\vec{x}' = A\vec{x}$. Finally, extract the general solution to the differential equation in y from the first component of your general vector solution.

Problem 12 Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ where A is as was given in Problem 10 of this Quiz. Also, explicitly find the matrix exponential e^{tA}.

Problem 13 Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ where A is as was given in Problem 11 of this Quiz. Also, explicitly find the matrix exponential e^{tA}.

Bonus: (S-Rank) suppose S, T are linear transformations on a finite dimensional vector space V over a field $\mathbb{F}$. Furthermore, both S and T are diagonable and $S \circ T = T \circ S$. **Prove** there exists a basis β of V for which both $[S]_{\beta,\beta}$ and $[T]_{\beta,\beta}$ are diagonal matrices. This result is can be phrased: *commuting linear transformations are simultaneously diagonalizable.*

TAKEHOME QUIZ 2 SOLUTION:

PROBLEM 1 §22 #5 of p. 192–193 CURTIS

Prove: if $T \in L(V,V)$ then T is invertible iff the constant term of the minimal polynomial of T is different than zero.

Describe how to compute T^{-1} from the min. poly. Show T^{-1} can always be expressed as $f(T)$ for some poly. $f(x)$.

If we have Th $= 24.10$ then this is very easy over $\mathbb{C}$ as $\det(T) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n \neq 0$ iff $\lambda_1, \lambda_2, \dots, \lambda_n \neq 0$ iff zero not e-value of T.

//

I'll try to behave and argue from a §22 viewpoint,

$\Rightarrow$) If $T : V \to V$ is invertible and $m(x) = x^m + \cdots + a_2 x^2 + a_1 x + a_0$ then $m(T) = T^m + \cdots + a_2 T^2 + a_1 T + a_0 1$. Suppose $a_0 = 0$ towards a $\rightarrow\leftarrow$. Observe

$$\begin{aligned} m(T) &= T^m + \cdots + a_2 T^2 + a_1 T \\ &= T(T^{m-1} + \cdots + a_2 T + a_1 1) = 0 \end{aligned}$$

Hence, as T^{-1} exists we obtain $T^{m-1} + \cdots + a_2 T + a_1 1 = 0$ which gives $g(x) = x^{m-1} + \cdots + a_2 x + a_1$ a polynomial of lesser degree than $m(x)$ for which $g(T) = 0$. This $\rightarrow\leftarrow$ the minimality of $m(x)$. Thus $a_0 \neq 0$.

$\Leftarrow$) If $m(x) = x^m + \cdots + a_1 x + a_0$ with $a_0 \neq 0$ is the minimal polynomial for T then $m(T) = T^m + \cdots + a_1 T + a_0 1 = 0$ hence $1 = \frac{-1}{a_0}(T^m + \cdots + a_2 T^2 + a_1 T) = \frac{-1}{a_0}(T^{m-1} + \cdots + a_2 T + a_1 1)T$ and we find $\underbrace{T^{-1} = \frac{-1}{a_0}(T^{m-1} + \cdots + a_2 T + a_1 1)}_{*} \in \mathbb{F}[T]$

the method to find T^{-1} is apparent from $*$ for example, $m(T) = T^2 + 3T - 2 \Rightarrow \underline{T^{-1} = \frac{1}{2}(T+3)}$ etc.

PROBLEM 2 $\beta = \{ e^{2x}, xe^{2x}, e^{x} \}$ and $V = \text{span}_{\mathbb{R}} \{\beta\}$

Let $T = D - 2$ where $D = d/dx$. Show β is Jordan Basis

$$T(e^{2x}) = D(e^{2x}) - 2e^{2x} = 2e^{2x} - 2e^{2x} = 0.$$

$$T(xe^{2x}) = D(xe^{2x}) - 2xe^{2x} = e^{2x} + 2xe^{2x} - 2xe^{2x} = e^{2x}$$

$$T(e^{x}) = D(e^{x}) - 2e^{x} = e^{x} - 2e^{x} = -e^{x}$$

$$[T]_{\beta\beta} = \left[[0]_\beta \,\middle|\, [e^{2x}]_\beta \,\middle|\, [-e^{x}]_\beta \right] = \underbrace{\begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & 0 \\ & 0 \\ 0 \quad 0 & \boxed{-1} \end{bmatrix}}_{\text{Jordan form.}}$$

$\therefore$ $\underline{\beta \text{ is Jordan Basis.}}$

PROBLEM 3 Let $T = L_A$ for $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 7 \\ 0 & 7 & 4 \end{bmatrix}$ and find e-basis of A (aka for T)

$$\det(A - \lambda I) = \det \begin{bmatrix} \boxed{7-\lambda} & 0 & 0 \\ 0 & \boxed{\begin{matrix} 4-\lambda & 7 \\ 7 & 4-\lambda \end{matrix}} \\ 0 & \end{bmatrix} = (7-\lambda) \det \begin{bmatrix} 4-\lambda & 7 \\ 7 & 4-\lambda \end{bmatrix}$$

$$= (7-\lambda)\left((\lambda-4)^2 - 49 \right)$$

$$= -(\lambda - 7)(\lambda - 11)(\lambda + 3)$$

$\lambda = 7$

$$A - 7I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 7 \\ 0 & 7 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ observe $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has $\underline{Au_1 = 7u_1}$.

$\lambda = 11$

$$A - 11I = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -7 & 7 \\ 0 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ observe $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ has $\underline{Au_2 = 11u_2}$.

$\lambda = -3$

$$A + 3I = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 7 & 7 \\ 0 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ observe $u_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ has $\underline{Au_3 = -3u_3}$

Thus $\boxed{\beta = \{ (1,0,0), (0,1,1), (0,1,-1) \} \text{ is e-basis for } T}$

$\boxed{\text{PROBLEM 3}}$ find transformations $E_1, E_2, E_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ for which $E_j \mathbb{R}^3 = \text{Ker}(T - \alpha_j)$ and $\mathbb{R}^3 = E_1\mathbb{R}^3 \oplus E_2\mathbb{R}^3 \oplus E_3\mathbb{R}^3$ where E_1, E_2, E_3 are idempotent and pairwise commuting $E_1E_2 = E_1E_3 = E_2E_3 = 0$

$$m(x) = (x-7)(x-11)(x+3)$$

is minimal polynomial since T is diagonalizable $\{u_1, u_2, u_3\}$ is e-basis

following the proof of the primary decomposition th$^{\underline{m}}$,

$$q_1(x) = \frac{m(x)}{x-7} = (x-11)(x+3)$$

$$q_2(x) = \frac{m(x)}{x-11} = (x-7)(x+3)$$

$$q_3(x) = \frac{m(x)}{x+3} = (x-7)(x-11)$$

$\exists\, a_1, a_2, a_3$ for which $a_1q_1 + a_2q_2 + a_3q_3 = 1$. Easiest possibility $a_1, a_2, a_3 \in \mathbb{R}[x]$ and in fact $a_1, a_2, a_3 \in \mathbb{R}$ itself, Let's see if it's possible,

$$a_1(x-11)(x+3) + a_2(x-7)(x+3) + a_3(x-7)(x-11) = 1$$

Evaluate at $x = 7, 11, -3$ to obtain,

$$-40a_1 = 1, \quad 56a_2 = 1, \quad 140a_3 = 1$$

Thus, $\underbrace{\frac{-1}{40}(x-11)(x+3)}_{f_1(x)} - \underbrace{\frac{1}{56}(x-7)(x+3)}_{f_2(x)} + \underbrace{\frac{1}{140}(x-7)(x-11)}_{f_3(x)} = 1$.

We define $E_j = f_j(T)$.

PROBLEM 3 continued

$$[E_1] = \frac{-1}{40}\underbrace{\begin{bmatrix}-4&0&0\\0&-7&7\\0&7&-7\end{bmatrix}}_{A-11I}\underbrace{\begin{bmatrix}10&0&0\\0&7&7\\0&7&7\end{bmatrix}}_{A+3I} = \frac{-1}{40}\begin{bmatrix}-40&0&0\\0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

$$[E_2] = \frac{-1}{56}\underbrace{\begin{bmatrix}0&0&0\\0&-3&7\\0&7&-3\end{bmatrix}}_{A-7I}\underbrace{\begin{bmatrix}10&0&0\\0&7&7\\0&7&7\end{bmatrix}}_{A+3I} = \frac{1}{56}\begin{bmatrix}0&0&0\\0&28&28\\0&28&28\end{bmatrix} = \begin{bmatrix}0&0&0\\0&1/2&1/2\\0&1/2&1/2\end{bmatrix}$$

$$[E_3] = \frac{1}{140}\underbrace{\begin{bmatrix}0&0&0\\0&-3&7\\0&7&-3\end{bmatrix}}_{A-7I}\underbrace{\begin{bmatrix}-4&0&0\\0&-7&7\\0&7&-7\end{bmatrix}}_{A-11I} = \frac{1}{140}\begin{bmatrix}0&0&0\\0&70&-70\\0&-70&70\end{bmatrix} = \begin{bmatrix}0&0&0\\0&1/2&-1/2\\0&-1/2&1/2\end{bmatrix}$$

In other words, we calculate

$$\boxed{E_1 = L_{\left[\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right]}, \quad E_2 = \frac{1}{2}L_{\left[\begin{smallmatrix}0&0&0\\0&1&1\\0&1&1\end{smallmatrix}\right]}, \quad E_3 = \frac{1}{2}L_{\left[\begin{smallmatrix}0&0&0\\0&1&-1\\0&-1&1\end{smallmatrix}\right]}}$$

It is simple to check $[E_i][E_j] = 0$ for $i \neq j$

hence $E_1E_2 = E_2E_3 = E_1E_3 = 0$ and $E_iE_j = E_jE_i$ for $i \neq j$.

Moreover, $[E_1]^2 = [E_1]$, $[E_2]^2 = [E_2]$ and $[E_3]^2 = [E_3]$

Hence $\underbrace{\underline{E_1^2 = E_1},\ \underline{E_2^2 = E_2} \text{ and } \underline{E_3^2 = E_3}}_{E_1,\, E_2,\, E_3 \text{ idempotent.}}$.

<u>Remark</u>: if we trust the general arguments in CUATIS then there is no need to check that E_i are idempotents, pairwise commuting to zero with $E_1 + E_2 + E_3 = \mathrm{Id}$.

That said, it is clear that

$$[E_1] + [E_2] + [E_3] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \therefore \quad \underline{E_1 + E_2 + E_3 = \mathrm{Id}}.$$

Problem 3 continued

$$\text{Col}[E_1] = \text{span}\{(1,0,0)\} = \text{Ker}(T-7)$$

$$\text{Col}[E_2] = \text{span}\{(0,1,1)\} = \text{Ker}(T-11)$$

$$\text{Col}[E_3] = \text{span}\{(0,1,-1)\} = \text{Ker}(T+3)$$

Hence, as $\text{Col}[E_j] = E_j(\mathbb{R}^3)$ for $E_j : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

we conclude $E_j(\mathbb{R}^3) = \text{Ker}(T-\alpha_j)$

for $\alpha_1 = 7$, $\alpha_2 = 11$, $\alpha_3 = -3$.

//

We can attack this problem a different way. My initial approach intends to illustrate the proof of the primary decomp. Th$^{\underline{m}}$ in CURTIS. Setting aside that goal, the spectral th$^{\underline{m}}$ helps! Notice $A^T = A$ and γ below is orthonormal e-basis!

$$\gamma = \{ \underbrace{(1,0,0)}_{v_1}, \underbrace{\tfrac{1}{\sqrt{2}}(0,1,1)}_{v_2}, \underbrace{\tfrac{1}{\sqrt{2}}(0,1,-1)}_{v_3} \}$$

following our homework (from after this was due!)

$$A = \underbrace{\lambda_1 v_1 v_1^T}_{\widetilde{E}_1} + \underbrace{\lambda_2 v_2 v_2^T}_{\widetilde{E}_2} + \underbrace{\lambda_3 v_3 v_3^T}_{\widetilde{E}_3}$$

$\lambda_1 = \alpha_1 = 7$
$\lambda_2 = \alpha_2 = 11$
$\lambda_3 = \alpha_3 = -3$

where $\widetilde{E}_1 = 7\begin{bmatrix}1\\0\\0\end{bmatrix}[1\ 0\ 0] = \begin{bmatrix}7&0&0\\0&0&0\\0&0&0\end{bmatrix} \neq E_1$

$\widetilde{E}_2 = \frac{11}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}[0\ 1\ 1] = \frac{11}{2}\begin{bmatrix}0&0&0\\0&1&1\\0&1&1\end{bmatrix} \neq E_2$

$\widetilde{E}_3 = \frac{-3}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix}[0\ 1\ -1] = -\frac{3}{2}\begin{bmatrix}0&0&0\\0&1&-1\\0&-1&1\end{bmatrix} \neq E_3$

oh, the spectral decomp. of A is related but not the same. $\widetilde{E}_j^{\,2} \neq \widetilde{E}_j$

We do have $\widetilde{E}_j^{\,2} = k_j \widetilde{E}_j$ for some scale factor k_j.

PROBLEM 4

$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ has $\lambda_1 = 8$, $\lambda_2 = -1$. Find e-basis of A and calc. A^n explicitly.

$$A - 8I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$u \in \mathrm{Null}(A - 8I)$ has
$u = (u_1, u_2, u_3) = (u_3, \frac{u_3}{2}, u_3)$
$\therefore$ $\underline{\mathrm{Null}(A-8I) = \mathrm{span}\{(2,1,2)\}}$.

$$A + I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$V = (V_1, V_2, V_3) \in \mathrm{Null}(A + I)$
$V = (-\frac{V_2}{2} - V_3, V_2, V_3)$
$\therefore$ $\underline{\mathrm{Null}(A+I) = \mathrm{span}\{(-1,2,0), (-1,0,1)\}}$

We find e-basis, after normalizing,

$$\underline{\beta = \left\{ \tfrac{1}{3}(2,1,2), \tfrac{1}{\sqrt{5}}(1,-2,0), \tfrac{1}{\sqrt{2}}(1,0,-1) \right\}}.$$

$$\begin{bmatrix} 8 & & \\ & -1 & \\ & & -1 \end{bmatrix} = D = [\beta]^{-1} A [\beta] = [\beta]^T A [\beta] \quad \text{or} \quad A = [\beta] D [\beta]^T$$

$-([\beta]^T [\beta] = I$ since β orthonormal$)-$

$$A^n = [\beta] D \underbrace{[\beta]^T [\beta]}_{I} D \underbrace{[\beta]^T}_{I} \cdots \underbrace{[\beta]}_{I} D [\beta]^T = [\beta] D^n [\beta]^T$$

(can prove by induction if you wish..)

$$A^n = [\beta] \begin{bmatrix} 8^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/\sqrt{5} & 1/\sqrt{2} \\ 1/3 & -2/\sqrt{5} & 0 \\ 2/3 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2(8)^n/3 & 8^n/3 & 2(8)^n/3 \\ (-1)^n/\sqrt{5} & -2(-1)^n/\sqrt{5} & 0 \\ (-1)^n/\sqrt{2} & 0 & (-1)^n/-\sqrt{2} \end{bmatrix}$$

$$\therefore \boxed{A^n = \begin{bmatrix} \frac{4}{9}8^n + \frac{(-1)^n}{5} + \frac{(-1)^n}{2} & \frac{2}{9}8^n - \frac{2}{5}(-1)^n & \frac{4}{9}8^n - \frac{(-1)^n}{2} \\ \frac{2}{9}8^n - \frac{2(-1)^n}{5} & \frac{1}{9}8^n + \frac{4}{5}(-1)^n & \frac{2}{9}8^n \\ \frac{4}{9}8^n - \frac{(-1)^n}{2} & \frac{2}{9}8^n & \frac{4}{9}8^n + \frac{(-1)^n}{2} \end{bmatrix}}$$

PROBLEM 5

Since A is diagonalizable we have $m(x) = (x-8)(x+1)$

Hence, $q_1(x) = x+1$ and $q_2(x) = x-8$

Find $a_1, a_2 \in \mathbb{R}$ for which $a_1 q_1(x) + a_2 q_2(x) = 1$

set $x = -1$, $q_1(-1) = 0 \therefore a_2 q_2(-1) = -9a_2 = 1 \therefore \underline{a_2 = -1/9}$

Likewise $x = 8$, $q_2(8) = 0 \therefore a_1 q_1(8) = 9a_1 = 1 \therefore a_1 = 1/9$

Set $f_1(x) = \frac{1}{9}(x+1)$ and $f_2(x) = \frac{-1}{9}(x-8)$

Hence $E_1 = f_1(A) = \frac{1}{9}(A+I)$ and $E_2 = f_2(A) = \frac{1}{9}(8I - A)$

Observe $E_1 + E_2 = \frac{1}{9}(A+I) + \frac{1}{9}(8I - A) \stackrel{\checkmark}{=} I$

and

$$\boxed{E_1 = \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}} \quad \text{has } E_1^2 = E_1$$

$$\boxed{E_2 = \frac{-1}{9}\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}} \quad \text{has } E_2^2 = E_2$$

Indeed $\text{Col}(E_1) = \text{span}\{(2,1,2)\} = \text{Ker}(T-8)$

and $\text{Col}(E_2) = \underbrace{\text{span}\{(-1,2,0), (-1,0,1)\}}_{\text{not entirely obvious by inspection}} = \text{Ker}(T+1)$

However, a short calculation reveals,

(see last pg. for Null(A+I) basis described as

$$\frac{1}{9}(-5,2,4) - \frac{2}{9}(2,-8,2) = (-1,2,0)$$

$$\frac{2}{9}(-5,2,4) + \frac{1}{18}(2,-8,2) = (-1,0,1)$$

Thus $\text{Ker}(T+1) = \text{Null}(A+I) = \text{span}\{(-1,2,0), (-1,0,1)\} = \text{span}\{(-5,2,4), (2,-8,2)$

PROBLEM 5 continued

$A^T = A$ so we can also investigate real spectral th$^{\underline{m}}$'s application. Note orthormal basis

$$\beta = \{ \underbrace{\tfrac{1}{3}(2,1,2)}_{u}, \underbrace{\tfrac{1}{\sqrt{5}}(-1,2,0)}_{v}, \underbrace{\tfrac{1}{\sqrt{2}}(-1,0,1)}_{w} \}$$

Allows us to create,

$$\tilde{E}_1 = 8uu^T = \frac{8}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} = 8E_1$$

$$\tilde{E}_2 = vv^T = \frac{1}{5}\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \tilde{\tilde{E}}_2 = ww^T = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\tilde{E}_2 \tilde{\tilde{E}}_2 = \frac{1}{10}\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Remark</u>: I can see connection between E_1 and $\tilde{E}_1$ easily. However for $\dim(\mathrm{Null}(A+I)) = 2$ it is not immediately apparent the connection between the outer products of the $\lambda = -1$ e-vectors v & w and E_2. <u>They</u> <u>are</u> <u>related</u> since $\mathrm{Range}(\tilde{E}_2) + \mathrm{Range}(\tilde{\tilde{E}}_2) = \underbrace{\mathrm{span}\{v, w\}}_{\mathrm{Null}(A+I)}$.

(explain the end of this unfinished story for bonus pts...)

PROBLEM 6

$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ find basis for $\mathbb{R}^2$ of e-vectors of A and find $m_v(x) \in \mathbb{R}[x]$ for each e-vector in basis. finally find $m(x)$ and $char_A(x)$

$$Char_A(x) = \det(xI - A) = \det\begin{bmatrix} x-2 & -3 \\ -3 & x-2 \end{bmatrix} = (x-2)^2 - 9$$

$\therefore$ $\underline{Char_A(x) = x^2 - 4x - 5}$.

Since $x^2 - 4x - 5 = (x-5)(x+1)$ and $m(x) \mid Char_A(x)$ we deduce (as $m(x)$ & $char_A(x)$ share same factors) $\underline{m(x) = x^2 - 4x - 5}$. Now, for the e-basis,

$\underline{\alpha = -1}$ $A + I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ $\therefore$ $v_1 = (1, -1)$ will do.

$\underline{\alpha = 5}$ $A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$ $\therefore$ $v_2 = (1, 1)$ will do.

We find $\boxed{\beta = \{(1,-1), (1,1)\} \text{ is e-basis.}}$

Moreover, as $m_v(x) \mid m(x)$ we have either $m_v(x) = x - 5$ or $m_v(x) = x + 1$ as $m_v(x)$ is <u>monic</u>

More to the point,

$$(A + I)(v_1) = 0 \Rightarrow \boxed{m_{v_1}(x) = x + 1}$$

$$(A - 5I)(v_2) = 0 \Rightarrow \boxed{m_{v_2}(x) = x - 5}$$

} cannot find smaller polynomials on which span v_1, span v_2 respectively vanish.

$\boxed{\text{PROBLEM 7}}$ $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}$ find v belonging to e-value $\alpha = 5$
calculate $m_v(x)$ & find $m(x)$
$char_A(x)$

$$A - 5I = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -5 & 4 \\ 2 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -13/10 \\ 0 & 1 & -4/5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= \tfrac{13}{10} x_3 \\ x_2 &= \tfrac{4}{5} x_3 \end{aligned}$$

If $(x_1, x_2, x_3) \in Null(A - 5I)$ then set $x_3 = 10$ to obtain $\boxed{(13, 8, 10) = v}$

$$\det(xI - A) = \det \begin{bmatrix} x-3 & -2 & -1 \\ 0 & x & -4 \\ -2 & -3 & x \end{bmatrix}$$

$$= (x-3)(x^2-12) + 2(0-8) - 1(2x)$$

$$= (x-3)x^2 - 12(x-3) - 16 - 2x$$

$$= \boxed{x^3 - 3x^2 - 14x + 20 = char_A(x)}$$

Observe, $x^3 - 3x^2 - 14x + 20$ has $(x-5)$ as factor

$$\begin{array}{r|l} & x^2 + 2x - 4 \\ \hline x - 5 & x^3 - 3x^2 - 14x + 20 \\ & x^3 - 5x^2 \\ \hline & \quad 2x^2 - 14x + 20 \\ & \quad 2x^2 - 10x \\ \hline & \qquad -4x + 20 \\ & \qquad -4x + 20 \\ \hline & \qquad\qquad \boxed{0} \end{array}$$

Thus $char_A(x) = (x-5)(x^2 + 2x - 4)$
$= (x-5)((x+1)^2 - 5)$
$= (x-5)(x+1+\sqrt{5})(x+1-\sqrt{5})$

Hence $\boxed{m(x) = x^3 - 3x^2 - 14x + 20}$ as $char_A(x)$ & $m(x)$ have same zeros.

$\boxed{\text{PROBLEM 8}}$ $T: V \to V$, $\dim_{\mathbb{Q}}(TV) = 4$

$V = \operatorname{span}\beta$ where $\beta = \{a, b, c, v\}$ and

$T(a) = 3a$, $(T-3)(b) = a$, $(T-3)(c) = b$, $T(v) = 7v$

$$[T]_{\beta\beta} = \Big[\, [3a]_\beta \,\Big|\, [T(b)]_\beta \,\Big|\, [T(c)]_\beta \,\Big|\, [T(v)]_\beta \,\Big] \quad - \begin{cases} T(b) = a + 3b \\ T(c) = 3c + b \end{cases}$$

$$= \left[\begin{array}{c|c|c|c} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Thus $[T]_{\beta\beta} = \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & \boxed{7} \end{array}\right]$ ← Jordan form. (found it.)

It follows that $\boxed{\mathrm{char}_T(x) = m(x) = (x-3)^3(x-7)}$

Let $q_1(x) = \dfrac{m(x)}{(x-3)^3} = x - 7$ & $q_2(x) = \dfrac{m(x)}{x-7} = (x-3)^3$

want to find $a_1(x), a_2(x)$ s.t.

$$a_1(x)\, q_1(x) + a_2(x)\, q_2(x) = 1$$

$$a_1(x)(x-7) + a_2(x)(x-3)^3 = 1 \quad (*)$$

Best case, $a_1(x)$ is 2nd order, and $a_2(x)$ is constant

$\underline{x=7}$ $\quad a_2(7-3)^3 = 1 \quad \therefore \underline{a_2 = 1/64}$.

$\underline{x=3}$ $\quad a_1(3)(3-7) = 1 \quad \therefore \underline{a_1(3) = -1/4}$.

Differentiate $(*)$!

$$a_1'(x)(x-7) + a_1(x) + \cancelto{0}{a_2'(x)}(x-3)^3 + 3a_2(x)(x-3)^2 = 0$$

$\underline{x=3}$ $\quad a_1'(3)(-4) + a_1(3) = 0 \Rightarrow a_1'(3) = \left(\frac{1}{4}\right)\left(\frac{1}{-4}\right) = \frac{-1}{16}$

Diff. again & evaluate at $x=3$, $\quad a_1''(3)(-4) + a_1'(3) + a_1'(3) = 0$

Continuing PROBLEM 8

$a_1(3) = \frac{-1}{4}$, $a_1'(3) = \frac{-1}{16}$, $a_1''(3) = \frac{-2a_1'(3)}{-4} = \frac{-1}{32}$

$$a_1(x) = a_1(3) + a_1'(3)(x-3) + \tfrac{1}{2}a_1''(3)(x-3)^2 \quad (\text{Taylor's Th}^{m})$$

$$= \frac{-1}{4} - \frac{1}{16}(x-3) - \frac{1}{64}(x^2 - 6x + 9)$$

$$= \underline{\frac{-13}{64} + \frac{1}{32}x - \frac{1}{64}x^2}.$$

Hence, following the primary decomp. Th^{m}, we define,

$E_1 = f_1(T)$ where $f_1(x) = \left(\frac{-13}{64} + \frac{1}{32}x - \frac{1}{64}x^2\right)(x-7)$

So, $\boxed{E_1 = \left(\frac{-13}{64}\mathrm{Id} + \frac{1}{32}T - \frac{1}{64}T^2\right)(T-7)}$

Likewise,

$E_2 = f_2(T)$ where $f_2(x) = \frac{1}{64}(x-3)^3$

So, $\boxed{E_2 = \frac{1}{64}(T - 3\,\mathrm{Id})^3}$

and, by primary decomp. Th^{m}, $E_1^2 = E_1$, $E_2^2 = E_2$

and $E_1E_2 = E_2E_1 = 0$ where $E_1 + E_2 = \mathrm{Id}_V$.

PROBLEM 9 Suppose $T: V \longrightarrow V$ and $\exists\, v, w \in V_{\mathbb{C}}$ with $T_{\mathbb{C}}(v) = \lambda v$ and $T_{\mathbb{C}}(w) = \lambda w + v$ where $\lambda = 2+3i$ and $v = v_1 + i v_2$ and $w = w_1 + i w_2$ and $\beta = \{v_1, v_2, w_1, w_2\}$ is basis for V find $[T]_{\beta\beta}$ and relate it to $[T_{\mathbb{C}}]_{\gamma\gamma} \in \mathbb{C}^{2\times 2}$ for $\gamma = \{v, w\}$

$$T_{\mathbb{C}}(w) = (2+3i)w + v \longrightarrow [T_{\mathbb{C}}(w)]_{\gamma} = \begin{bmatrix} 1 \\ 2+3i \end{bmatrix}$$

$$T_{\mathbb{C}}(v) = (2+3i)v \longrightarrow [T_{\mathbb{C}}(v)]_{\gamma} = \begin{bmatrix} 2+3i \\ 0 \end{bmatrix}$$

Thus $\boxed{[T_{\mathbb{C}}]_{\gamma\gamma} = \begin{bmatrix} 2+3i & 1 \\ 0 & 2+3i \end{bmatrix}}$

Notice, $T_{\mathbb{C}}(w_1 + i w_2) = T(w_1) + i\, T(w_2)$ by def$^{\underline{n}}$ of complexification.

$$(2+3i)(w_1 + i w_2) + v_1 + i v_2 = T(w_1) + i\, T(w_2)$$

$$2w_1 - 3w_2 + v_1 + i(v_2 + 3w_1 + 2w_2) = T(w_1) + i\, T(w_2)$$

$$\Rightarrow \underline{T(w_1) = v_1 + 2w_1 - 3w_2}. \quad \text{I}$$

$$\underline{T(w_2) = v_2 + 3w_1 + 2w_2}. \quad \text{II}$$

Likewise, $T_{\mathbb{C}}(v_1 + i v_2) = T(v_1) + i\, T(v_2)$ compare

$$(2+3i)(v_1 + i v_2) = 2v_1 - 3v_2 + i(3v_1 + 2v_2)$$

Thus, $\underline{T(v_1) = 2v_1 - 3v_2}$ III and $\underline{T(v_2) = 3v_1 + 2v_2}$ IV

By I, II, III and IV since $[T]_{\beta\beta} = \big[\, [T(v_1)]_{\beta} \,\big|\, [T(v_2)]_{\beta} \,\big|\, [T(w_1)]_{\beta} \,\big|\, [T(w_2)]_{\beta} \,\big]$

we find,

$$[T_{\mathbb{C}}]_{\gamma\gamma} = (2+3i)I + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{[T]_{\beta\beta} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}} = \underbrace{\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}}\otimes\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$M(2+3i) = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, $M: \mathbb{C} \longrightarrow \mathbb{R}^{2\times 2}$
a matrix rep. of $2+3i$.

PROBLEM 10 Consider $P(x) = (x^2+4x+13)^3$ in view of Lemma 25.12 p. 221

$$A_{P(x)} = \begin{bmatrix} C_{q(x)} & B & 0 \\ 0 & C_{q(x)} & B \\ 0 & 0 & C_{q(x)} \end{bmatrix} \qquad q(x) = x^2+4x+13$$

$$\hookrightarrow A_{q(x)} = \begin{bmatrix} 0 & -13 \\ 1 & -4 \end{bmatrix}$$

$$\boxed{A_{P(x)} = \left[\begin{array}{cc|cc|cc} 0 & -13 & 0 & 1 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -13 & 0 & 1 \\ 0 & 0 & 1 & -4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -13 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right]} \qquad (*)$$

Some of you, well, many, expanded $P(x) = x^6+12x^5+87x^4+376x^3+1131x^2+2028x+2197$

Gives companion matrix $\begin{bmatrix} 0&0&0&0&0&-2197 \\ 1&0&0&0&0&-2028 \\ 0&1&0&0&0&-1131 \\ 0&0&1&0&0&-376 \\ 0&0&0&1&0&-87 \\ 0&0&0&0&1&-12 \end{bmatrix}$ but, I wanted $(*)$

PROBLEM 11 $y^{(4)}+4y'''+6y''+4y'+y=0$

Set $x_1 = y$, $x_2 = y'$, $x_3 = y''$, $x_4 = y'''$

$x_1' = x_2$
$x_2' = x_3$
$x_3' = x_4$
$x_4' = y^{(4)} = -x_1 - 4x_2 - 6x_3 - 4x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ -1&-4&-6&-4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Notice $A^T = \begin{bmatrix} 0&0&0&-1 \\ 1&0&0&-4 \\ 0&1&0&-6 \\ 0&0&1&-4 \end{bmatrix}$ is the companion matrix of $x^4+4x^3+6x^2+4x+1$. It follows $\underline{\mathrm{char}_A(x) = (x+1)^4}$.

PROBLEM 11 continued

$$A+I = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -4 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \overset{\cdots}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(x_1, x_2, x_3, x_4) \in \text{Null}(A+I)$ has $x_1 = -x_4$, $x_2 = x_4$, $x_3 = -x_4$

Set $x_4 = 1$ to obtain e-vector $\underline{V_1 = (-1, 1, -1, 1)}$.

Next, find V_2 such that $(A+I)V_2 = V_1$

$$\text{rref}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & -4 & -6 & -3 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= -3 - x_4 \\ x_2 &= 2 + x_4 \\ x_3 &= -1 - x_4 \end{aligned} \quad \text{set } \underline{x_4 = 0}$$

Obtain $\underline{V_2 = (-3, 2, -1, 0)}$. with $(A+I)V_2 = V_1$

Next, seek V_3 s.t. $(A+I)V_3 = V_2$

$$\text{rref}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & -4 & -6 & -3 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -6 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

set $x_4 = 0$ $\rightarrow$ $\underline{V_3 = (-6, 3, -1, 0)}$.

Next, seek V_4 s.t. $(A+I)V_4 = V_3$

$$\text{rref}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -6 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & -4 & -6 & -3 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \underline{V_4 = (-10, 4, -1, 0)}$$

(set $x_4 = 0$)

From the magic formula

$$e^{tA} = e^{-t}\left(I + t(A+I) + \frac{t^2}{2}(A+I)^2 + \frac{t^3}{6}(A+I)^3 + \cdots\right)$$

And the chain conditions

$(A+I)V_1 = 0$, $(A+I)V_2 = V_1$, $(A+I)^2V_2 = 0$, $(A+I)V_3 = V_2$, $(A+I)^2V_3 = V_1$, $(A+I)^3V_3 = 0$, $(A+I)V_4 = V_3$, $(A+I)^2V_4 = V_2$, $(A+I)^3V_4 = V_1$, $(A+I)^4V_4 = 0$

From which we may simplify the general sol$^{\text{n}}$

$$\vec{x} = c_1 e^{tA}V_1 + c_2 e^{tA}V_2 + c_3 e^{tA}V_3 + c_4 e^{tA}V_4$$ as follows

PROBLEM 11 continued

$$\boxed{\vec{X}(t) = c_1 e^{-t}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix}-3\\2\\-1\\0\end{bmatrix} + t\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right) + c_3 e^{-t}\left(\begin{bmatrix}-6\\3\\-1\\0\end{bmatrix} + t\begin{bmatrix}-3\\2\\-1\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right) + c_4 e^{-t}\left(\begin{bmatrix}-10\\4\\-1\\0\end{bmatrix} + t\begin{bmatrix}-6\\3\\-1\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}-3\\2\\-1\\0\end{bmatrix} + \frac{t^3}{6}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right)}$$

The 1^{st} component of the sol^{n} above gives $X_1 = y$ hence,

$$\boxed{y = c_1 e^{-t} + c_2 e^{-t}(-3-t) + c_3 e^{-t}\left(-6-3t-\tfrac{1}{2}t^2\right) + c_4 e^{-t}\left(-10-6t-\tfrac{3}{2}t^2-\tfrac{1}{6}t^3\right)}$$

$$y = (c_1 - 3c_2 - 6c_3 - 10c_4)e^{-t} + (-c_2 - 3c_3 - 6c_4)te^{-t} + \left(-c_3/2 - \tfrac{3c_4}{2}\right)t^2e^{-t} - \frac{c_4}{6}t^3e^{-t}$$

$$\underline{y = b_1 e^{-t} + b_2 te^{-t} + b_3 t^2 e^{-t} + b_4 t^3 e^{-t}}$$

(formulation usual for Math 334, don't be fooled, our sol^{n} is the same general sol^{n}, just with ugly constants)

$\boxed{\text{PROBLEM 12}}$ Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ for A of <u>PROBLEM 10</u>

$B^2 = 0$, $N^3 = 0$

$$A = \begin{bmatrix} 0 & -13 \\ 1 & -4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = C \otimes I + B \otimes N$$

$$\begin{aligned} A^2 = (C\otimes I + B \otimes N)^2 &= (C\otimes I)^2 + (C\otimes I)(B\otimes N) + (B\otimes N)(C\otimes I) + (B\otimes N)^2 \\ &= C^2 \otimes I + CB \otimes N + BC \otimes N + \underbrace{B^2}_{0} \otimes N^2 \\ &= \begin{bmatrix} -13 & 52 \\ -4 & 3 \end{bmatrix} \otimes I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes N + \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \otimes N \end{aligned}$$

$$C^2 = \begin{bmatrix} 0 & -13 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 0 & -13 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -13 & 52 \\ -4 & 3 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & -13 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad BC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -13 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -13 & 52 \\ -4 & 3 \end{bmatrix} \otimes I + \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \otimes N$$

$$\begin{aligned} A^2 + 4A + 13I &= \begin{bmatrix} -13 & 52 \\ -4 & 3 \end{bmatrix} \otimes I + \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \otimes N \\ &+ \begin{bmatrix} 0 & -52 \\ 4 & -16 \end{bmatrix} \otimes I + \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \otimes N \\ &+ \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} \otimes I \\ &= I \otimes N \quad (\text{cool.}) \end{aligned}$$

Hence $(A^2 + 4A + 13I)^2 = (I\otimes N)^2 = I^2 \otimes N^2 = I \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $(A^2 + 4A + 13I)^3 = (I \otimes N)^3 = I^3 \otimes N^3 = I \otimes (0) = 0.$

PROBLEM 12 continued

$$A^2+4A+13I = (A+2I)^2+9I = (A+2I+3iI)(A+2I-3iI)$$

Thus $A^2+4A+13I = \left(A+(2+3i)I\right)\left(A+(2-3i)I\right)$

and $(A^2+4A+13I)^2 = \left(A+(2+3i)I\right)^2\left(A+(2-3i)I\right)^2$

After some calculation,

$$\left(A+(2+3i)I\right)^2 = 6\left[\begin{array}{c|c} -3+2i & -13i \\ \hline i & -3-2i \end{array}\right]\otimes I + \left[\begin{array}{c|c} 1 & 6i \\ \hline 0 & 1 \end{array}\right]\otimes N$$

Or, to be explicit, $I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ hence

$$\left(A+(2+3i)I\right)^2 = \left[\begin{array}{cc|cc|cc} -18+12i & -78i & 1 & 6i & 0 & 0 \\ 6i & -18-12i & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -18+12i & -78i & 1 & 6i \\ 0 & 0 & 6i & -18-12i & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & -18+12i & -78i \\ 0 & 0 & 0 & 0 & 6i & -18-12i \end{array}\right]$$

Consider $M = \left[\begin{array}{c|c} -18+12i & -78i \\ \hline 6i & -18-12i \end{array}\right] \sim \left[\begin{array}{c|c} -18+12i & -78i \\ \hline -18+12i & (-18-12i)(2+3i) \end{array}\right]$

$6i(2+3i) = 12i-18.$

$$\underbrace{-6(3+2i)(2+3i)}_{-6(6+4i+9i-6)}$$

$$-6(13i) = -78i$$

Hence M has rank 1 and

$Mz = 0$ for $(z_1, z_2) = z$ must solve $6iz_1 - (18+12i)z_2 = 0$ or $iz_1 = (3+2i)z_2 \Rightarrow z_1 = (-3i+2)z_2$ let $z_2 = 1$ then $z_1 = -3i+2 = 2-3i$. Notice then

$$z = \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} \longrightarrow v_3 = \begin{bmatrix} 2-3i \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \text{Null}\left(A+(2+3i)I\right)^2$$

PROBLEM 12 continued

$$(A + (2+3i)I)V_3 = \begin{bmatrix} 2+3i & -13 & 0 & 1 & 0 & 0 \\ 1 & -2+3i & 0 & 0 & 0 & 0 \\ 0 & 0 & 2+3i & -13 & 0 & 1 \\ 0 & 0 & 1 & -2+3i & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+3i & -13 \\ 0 & 0 & 0 & 0 & 1 & -2+3i \end{bmatrix} \begin{bmatrix} 2-3i \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= (0,\ 2-3i-2+3i,\ 0,\ 0,\ 0,\ 0)$$

$$= (0,0,0,0,0,0).$$

ok, so V_3 is e-vector with $\lambda = 2+3i$ for A.

I was hoping for V_3 a generalized e-vector of order 3, but, I chose poorly.

Let's try again. Set $\underline{V_3 = u_1}$.

$(A + (2+3i)I)u_2 = u_1$ has sol$^{=}$ $u_2 = (1, 0, -18-12i, -6i, 0, 0)$

$$\underline{u_2 = (1, 0, -18-12i, -6i, 0, 0)}.$$

and again,

$(A + (2+3i)I)u_3 = u_2$ has sol$^{=}$ $\underline{u_3 = (0, 0, 2-9i, 1, -72+108i, -36)}$.

It follows $\{u_1, u_2, u_3\}$ is complex 3-chain. This gives us basis $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ for $\mathbb{R}^6$ where

$$\left.\begin{aligned} a_1 &= (2, 1, 0, 0, 0, 0) \\ b_1 &= (-3, 0, 0, 0, 0, 0) \end{aligned}\right\} \quad u_1 = a_1 + ib_1$$

$$\left.\begin{aligned} a_2 &= (1, 0, -18, 0, 0, 0) \\ b_2 &= (0, 0, -12, -6, 0, 0) \end{aligned}\right\} \quad u_2 = a_2 + ib_2$$

$$\left.\begin{aligned} a_3 &= (0, 0, 2, 1, -72, -36) \\ b_3 &= (0, 0, -9, 0, 108, 0) \end{aligned}\right\} \quad u_3 = a_3 + ib_3$$

PROBLEM 12 continued

The sol$^{\underline{n}}$ of $\frac{d\vec{x}}{dt} = A\vec{x}$ follows from extracting $Re(\vec{z})$, $Im(\vec{z})$ for sol$^{'s}$ $\vec{z} = e^{tA}\vec{u}_j$ for $j = 1, 2, 3$

For example, $\vec{z}_1 = e^{(2+3i)t}\left[I + t(A + (2+3i)I) + \cdots\right]\vec{u}_1 =$

$$\vec{z}_1 = e^{2t}(\cos 3t + i \sin 3t)(a_1 + ib_1)$$

$$\vec{z}_1 = \underbrace{e^{2t}\cos 3t\, a_1 - e^{2t}\sin 3t\, b_1}_{Re(\vec{z}_1)} + i(\underbrace{e^{2t}\sin 3t\, a_1 + e^{2t}\cos 3t\, b_1}_{Im(\vec{z}_1)})$$

Likewise $\vec{z}_2 = e^{tA}u_2 = e^{(2+3i)t}\left[I + t(A + (2+3i)I) + \cdots\right]u_2$

$$\vec{z}_2 = e^{2t}(\cos 3t + i \sin 3t)(u_2 + t u_1)$$

$$= (e^{2t}\cos 3t)(a_2 + t a_1) - (e^{2t}\sin 3t)(b_2 + t b_1) + \cdots$$

$$+ i\Big[\underbrace{(e^{2t}\sin 3t)(a_2 + t a_1) + (e^{2t}\cos 3t)(b_2 + t b_1)}_{Im(\vec{z}_2)}\Big]$$

Likewise for $\vec{z}_3 = e^{tA}u_3$,

$$\boxed{\begin{aligned}\vec{X} = {} & c_1(e^{2t}\cos 3t\, a_1 - e^{2t}\sin 3t\, b_1) + c_2 e^{2t}(\sin 3t\, a_1 + \cos 3t\, b_1) + \\ & + c_3 e^{2t}(\cos 3t\,(a_2 + t a_1) - \sin 3t\,(b_2 + t b_1)) + c_4 e^{2t}(\sin 3t\,(a_2 + t a_1) + \cos 3t\,(b_2 + t b_1)) \\ & + c_5 e^{2t}(\cos 3t\,(a_3 + t a_2 + \tfrac{t^2}{2}a_1) - \sin 3t\,(b_3 + t b_2 + \tfrac{1}{2}t^2 b_1)) + c_6 e^{2t}[\sin 3t\,(a_3 + t a_2 + \tfrac{t^2}{2}a_1) + \cos 3t\,(b_3 + t b_2 + \tfrac{t^2}{2}b\end{aligned}}$$

where $a_1, b, a_2, b_2, a_3, b_3$ we're given on last page.