Please show your work and use words to explain your steps where appropriate.

Notational Conventions for Test: *V and W denote vector spaces over a field $\mathbb{F}$. Also, $L(V,W)$ is the set of linear transformations from V to W. $\mathbb{R}[x]$ is the set of real polynomials. $P_n(\mathbb{F}) = span_{\mathbb{F}}\{1, x, x^2, \ldots, x^n\}$. Also, $S \leq V$ means S is a subspace of V.* **LI** *means* **L***inearly* **I***ndependent*

Problem 1 (5pt) Define what is meant by the statement: *β is a basis for V* (assume $V \neq 0$)

Problem 2 (10pt) $P_2(\mathbb{R})$ has basis $\beta = \{(x+1)^2, x+1, 1\}$. If $f(x) = 8x^2 + 6x + 21$ then find $[f(x)]_\beta$.

Problem 3 (5pt) Is $\mathbb{Q}$ a subspace of $V = \mathbb{R}$ where V is a real vector space. ?

Problem 4 (10pt) Let $W = \{ax^2 + bx^2 + ax + b \mid a, b \in \mathbb{C}\}$. Prove $W \leq \mathbb{C}[x]$.

Problem 5 (10pt) Let $T : V \to W$ be a linear transformation. Prove $\mathrm{Ker}(T) \leq V$.

Problem 6 (10pt) Let B be a fixed square matrix in $\mathbb{R}^{n\times n}$ and define $T(A) = AB - BA$ for all $A \in \mathbb{R}^{n\times n}$. Show T is a linear transformation.

Problem 7 (10pt) Suppose $T(1,1) = (3,5)$ and $T(0,1) = (8,8)$ and extend T linearly to a transformation of $\mathbb{R}^2$. Find the standard matrix of T. Is T an isomorphism ?

Problem 8 (10pt) Assume $S = \{v, w\} \subset V$ is LI. **Prove or disprove:** $T = \{v + 2w, 3v + 4w\}$ is LI.

Problem 9 (20pt) If $T : V \to W$ and $S : W \to U$ are injective linear transformations then prove $S \circ T$ is an injective linear transformation.

Problem 10 (5pt) A linear manifold has the same dimension as its directing space; $\mathcal{M} = p + \mathcal{S}$ then $\dim\mathcal{M} = \dim\mathcal{S}$. If $\mathcal{S} = \text{span}\{v_1, v_2, v_3, v_4\}$ where $\mathcal{S} \leq V$ and $\dim(V) = 3$ then what are the possible dimensions of $\mathcal{M}$?

Problem 11 (15pt) Let $T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f'(0) & 0 \end{bmatrix}$ for each $f(x) \in P_2(\mathbb{R})$. Note $T : P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$. Let $P_2(\mathbb{R})$ have basis $\beta = \{1, x, x^2\}$ and $\mathbb{R}^{2\times 2}$ have basis $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Find $[T]_{\beta\gamma}$

Problem 12 (20pt) If $\dim(V) = n$ and $T \in L(V, V)$ then prove $n = \text{rank}(T) + \text{nullity}(T)$. (your argument should involve the construction and extension of an appropriate basis)

Problem 13 (10pt) Suppose $T : V \to V$ is an injective linear transformation. **Prove** T is surjective. *Hint: use result of previous problem!*

Problem 14 (10pt) Suppose V, W are finite dimensional vector spaces over a field $\mathbb{F}$. Show $L(V, W)$ is isomorphic to $L(W, V)$.

Problem 15 (10pt) Let $T(A) = BAB$ for each $A \in \mathbb{R}^{n\times n}$ where $B = E_{ij}$ for a particular choice of i, j. Find a basis for $\mathrm{Ker}(T)$.

Problem 16 (10pt) Let $T \in L(V)$. Show that the matrix of $T^2 : V \to V$ with respect to basis β is given by $[T]_{\beta,\beta}[T]_{\beta,\beta}$.