Please show your work and use words to explain your steps where appropriate.

Notational Conventions for Test: *V and W denote vector spaces over a field $\mathbb{F}$. Also, $L(V,W)$ is the set of linear transformations from V to W. $\mathbb{R}[x]$ is the set of real polynomials. $P_n(\mathbb{F}) = span_{\mathbb{F}}\{1, x, x^2, \ldots, x^n\}$. Also, $S \leq V$ means S is a subspace of V.* **LI** *means* **L***inearly* **I***ndependent*

Problem 1 (5pt) Define what is meant by the statement: *β is a basis for V* (assume $V \neq 0$)

β is a LI generating set for V; β LI & $span(\beta) = V$

Problem 2 (10pt) $P_2(\mathbb{R})$ has basis $\beta = \{(x+1)^2, x+1, 1\}$. If $f(x) = 8x^2 + 6x + 21$ then find $[f(x)]_\beta$.

If $c_1(x+1)^2 + c_2(x+1) + c_3(1) = 8x^2 + 6x + 21$

then $x^2(c_1) + x(2c_1 + c_2) + 1(c_1 + c_2 + c_3) = 8x^2 + 6x + 21$

Equating coeff of LI $\{x^2, x, 1\}$ yields,

$$c_1 = 8$$
$$2c_1 + c_2 = 6 \Rightarrow c_2 = 6 - 2c_1 = -10$$
$$c_1 + c_2 + c_3 = 21 \Rightarrow c_3 = 21 - 8 + 10 = 23$$

$\therefore \boxed{[f(x)]_\beta = (8, -10, 23)}$

Problem 3 (5pt) Is $\mathbb{Q}$ a subspace of $V = \mathbb{R}$ where V is a real vector space. ?

Let $q \in \mathbb{Q}$ note $\sqrt{2}\, q \notin \mathbb{Q}$ but $\sqrt{2} \in \mathbb{R}$ is an allowed scalar multiplication $\therefore \mathbb{Q} \not\leq \mathbb{R}$.

Problem 4 (10pt) Let $W = \{ax^2 + bx^2 + ax + b \mid a, b \in \mathbb{C}\}$. Prove $W \leq \mathbb{C}[x]$.

$w \in W \Rightarrow w = a(x^2 + x) + b(x^2 + 1)$

$\Rightarrow \underline{W = span_{\mathbb{C}}\{x^2 + x, x^2 + 1\}}$. thus $W \leq \mathbb{C}(x)$ by span is subspace Th$^{\underline{m}}$

Problem 5 (10pt) Let $T : V \to W$ be a linear transformation. Prove $\text{Ker}(T) \leq V$.

Note $T(0) = 0 \therefore Ker(T) \neq \emptyset$. Let $x, y \in Ker(T)$ and $\alpha \in \mathbb{F}$. Consider, $x, y \in Ker(T)$ means $T(x) = 0, T(y) = 0$

thus $T(\alpha x + y) = \alpha T(x) + T(y) = \alpha(0) + 0 = 0$

$\Rightarrow \alpha x + y \in Ker(T) \therefore \alpha x, x + y \in Ker(T)$. Hence $Ker(T) \leq V$ By Subspace Test Th$^{\underline{m}}$.

Problem 6 (10pt) Let B be a fixed square matrix in $\mathbb{R}^{n\times n}$ and define $T(A) = AB - BA$ for all $A \in \mathbb{R}^{n\times n}$. Show T is a linear transformation.

$$T(\alpha A + C) = (\alpha A + C)B - B(\alpha A + C)$$
$$= \alpha(AB - BA) + CB - BC$$
$$= \alpha T(A) + T(C) \quad \forall A, C \in \mathbb{R}^{n\times n} \text{ and } \alpha \in \mathbb{R}$$

Thus T is a linear transformation.

Problem 7 (10pt) Suppose $T(1,1) = (3,5)$ and $T(0,1) = (8,8)$ and extend T linearly to a transformation of $\mathbb{R}^2$. Find the standard matrix of T. Is T an isomorphism ?

$$T(1,1) = T(1,0) + \overbrace{T(0,1)}^{(8,8)} = (3,5)$$

$$\Rightarrow T(1,0) = (3,5) - (8,8) = (-5,-3)$$

$$\therefore \quad [T] = [T(e_1) | T(e_2)] = \boxed{\begin{bmatrix} -5 & 8 \\ -3 & 8 \end{bmatrix} = [T]} \hookrightarrow T \text{ \underline{linear}}$$

Observe, rank $[T] = 2 \therefore T$ is surjection from $\mathbb{R}^2 \to \mathbb{R}^2$ thus T is injection. In summary T is 1-1, onto, linear trans. on $\mathbb{R}^2$. T is an <u>isomorphism</u>.

Problem 8 (10pt) Assume $S = \{v, w\} \subset V$ is LI. **Prove or disprove:** $T = \{v + 2w, 3v + 4w\}$ is LI.

Assume $\{v, w\}$ is LI. Consider, if

$$c_1(v + 2w) + c_2(3v + 4w) = 0$$

then $(c_1 + 3c_2)v + (2c_1 + 4c_2)w = 0$.

Thus $\underbrace{c_1 + 3c_2 = 0}_{*}$ and $\underbrace{2c_1 + 4c_2 = 0}_{**}$ by LI of S.

Hence $c_1 = -3c_2 \Rightarrow 2(-3c_2) + 4c_2 = -2c_2 = 0 \therefore \underline{c_2 = 0}$ and $\underline{c_1 = 0}$. (you could solve $*$ and $**$ lots of ways)

Thus $T = \{v + 2w, 3v + 4w\}$ is LI.//

Problem 9 (20pt) If $T: V \to W$ and $S: W \to U$ are injective linear transformations then prove $S \circ T$ is an injective linear transformation.

Let T, S be injective then $Ker(T) = \{0\}$ and $Ker(S) = \{0\}$.

Suppose $x \in Ker(S \circ T)$ then $(S \circ T)(x) = 0$. Thus,

$S(T(x)) = 0 \Rightarrow T(x) \in Ker(S)$. Thus $T(x) = 0$.

But, $T(x) = 0 \Rightarrow x \in Ker(T) \therefore x = 0$. Since $0 \in Ker(S \circ T)$ we have shown $Ker(S \circ T) = \{0\} \therefore$ <u>$S \circ T$ is injective</u>.

It remains to show $S \circ T$ is a linear transformation.

$$\begin{aligned}(S \circ T)(\alpha x + y) &= S(T(\alpha x + y)) && \text{: def}^{\underline{n}} \text{ of composite} \\ &= S(\alpha T(x) + T(y)) && \text{: linearity of } T \\ &= \alpha S(T(x)) + S(T(y)) && \text{: linearity of } S \\ &= \alpha (S \circ T)(x) + (S \circ T)(y) && \text{: def}^{\underline{n}} \text{ of comp.}\end{aligned}$$

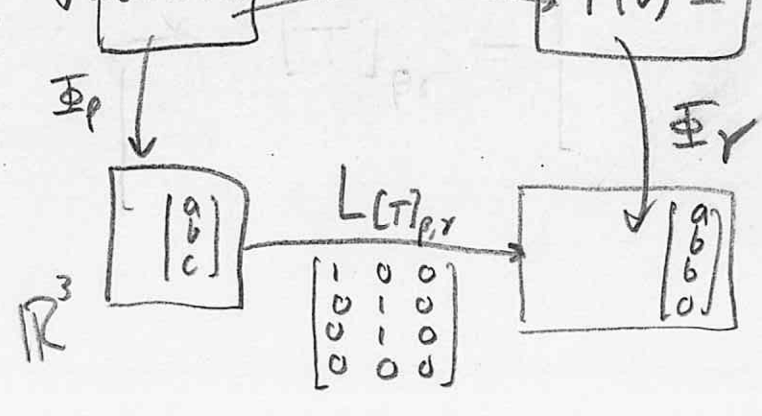
it follows $S \circ T \in L(V, U)$.

Problem 10 (5pt) A linear manifold has the same dimension as its directing space; $\mathcal{M} = p + \mathcal{S}$ then $\dim\mathcal{M} = \dim\mathcal{S}$. If $\mathcal{S} = \text{span}\{v_1, v_2, v_3, v_4\}$ where $\mathcal{S} \leq V$ and $\dim(V) = 3$ then what are the possible dimensions of $\mathcal{M}$?

Since $\dim V = 3$ we can have 0, 1, 2 or <u>at most</u> 3 LI vectors amongst $\{v_1, v_2, v_3, v_4\}$ thus <u>$\dim(\mathcal{M}) = 0, 1, 2, 3$ (possibilities)</u>.

Problem 11 (15pt) Let $T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f'(0) & 0 \end{bmatrix}$ for each $f(x) \in P_2(\mathbb{R})$. Note $T: P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$. Let $P_2(\mathbb{R})$ have basis $\beta = \{1, x, x^2\}$ and $\mathbb{R}^{2\times 2}$ have basis $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Find $[T]_{\beta\gamma}$

$$T(a + bx + cx^2) = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$$



$$\boxed{[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Problem 12 (20pt) If $\dim(V) = n$ and $T \in L(V,V)$ then prove $n = \text{rank}(T) + \text{nullity}(T)$. (your argument should involve the construction and extension of an appropriate basis)

Let $\tilde{\beta} = \{V_1, V_2, \dots, V_k\}$ be basis for $Ker(T) \leq V$.

extend $\tilde{\beta}$ to $\beta = \{V_1, \dots, V_k, W_1, \dots, W_{n-k}\}$ where $W_j \notin Ker(T)$

for $j = 1, 2, \dots, n-k$. Consider, $x \in V$ then

$$T(x) = T\left(\sum_{i=1}^{k} x_i V_i + \sum_{j=1}^{n-k} y_j W_j\right) \leftarrow \left(\text{notation invented here}\right)$$

$$= \sum_{i=1}^{k} x_i \underbrace{T(V_i)}_{0} + \sum_{j=1}^{n-k} y_j T(W_j) \in \text{span}\{T(W_j)\}_{j=1}^{n-k}$$

Thus $\gamma = \{T(W_1), \dots, T(W_{n-k})\}$ is a generating set for $T(V)$.

We argue γ is LI. Consider,

$$C_1 T(W_1) + \cdots + C_{n-k} T(W_{n-k}) = 0$$

$$\Rightarrow T(C_1 W_1 + \cdots + C_{n-k} W_{n-k}) = 0$$

Thus $C_1 W_1 + \cdots + C_{n-k} W_{n-k} \in Ker(T)$ hence $C_1 = 0, \dots, C_{n-k} = 0$ (otherwise we obtain a $\rightarrow\leftarrow$ since nonzero $C_j \Rightarrow W_j \in Ker T$)

Thus γ is basis for $T(V)$ and as $\#\gamma = n-k$ and $\#\tilde{\beta} = k$ it follows $n = \dim(Ker T) + \dim(T(V))$. //

Problem 13 (10pt) Suppose $T : V \to V$ is an injective linear transformation. Prove T is surjective. *Hint: use result of previous problem!*

T injective linear trans. $\Rightarrow Ker(T) = \{0\} \Rightarrow \nu(T) = 0$

Thus $n = \cancel{\nu(T)}^{0} + rk(T) = rk(T) \therefore \dim(T(V)) = n$
and as $T(V) \leq V$ it follows $\underline{T(V) = V}$.

Problem 14 (10pt) Suppose V, W are finite dimensional vector spaces over a field $\mathbb{F}$. Show $L(V, W)$ is isomorphic to $L(W, V)$.

Let β be basis for V $(\#\beta = n)$
Let γ be basis for W $(\#\gamma = m)$
then $L(V, W) \approx \mathbb{F}^{m \times n}$ by $\psi_1(S) = [S]_{\beta,\gamma}$
But, $\mathbb{F}^{m \times n} \approx \mathbb{F}^{n \times m}$ by $\psi_2(A) = A^T$. Consider
$\mathbb{F}^{n \times m} \approx L(W, V)$ by $\psi_3(S) = [S]_{\gamma,\beta}$ thus $L(V, W) \approx L(W, V)$.

Problem 15 (10pt) Let $T(A) = BAB$ for each $A \in \mathbb{R}^{n\times n}$ where $B = E_{ij}$ for a particular choice of i, j. Find a basis for $\mathrm{Ker}(T)$.

$$A \in \mathrm{Ker}(T) \Rightarrow T(A) = BAB = 0$$

$$\Rightarrow E_{ij} A E_{ij} = 0 \qquad \boxed{E_{ij}E_{kl} = \delta_{jk} E_{il}}$$

$$\Rightarrow E_{ij} \sum_{k,l} A_{kl} E_{kl} E_{ij} = 0$$

$$\Rightarrow \sum_{k,l} A_{kl}\, \delta_{jk} E_{il} E_{ij} = 0$$

$$\Rightarrow \sum_{k,l} A_{kl}\, \delta_{jk} \delta_{li} E_{ij} = 0$$

$$\Rightarrow \underset{\text{scalar}}{A_{ji}}\ \underset{\text{nonzero matrix}}{E_{ij}} = 0 \quad \therefore A_{ji} = 0 \quad \text{// } A_{kl} \text{ free! for } k \neq j,\ i \neq l$$

BASIS FOR Ker(T) $\longrightarrow$ $\boxed{\beta = \{ E_{kl} \mid k \neq j,\ i \neq l \}}$

~~**Problem 16** (10pt) Let $T \in L(V)$. Show that the matrix of $T^2 : V \to V$ with respect to basis β is given by $[T]_{\beta,\beta}[T]_{\beta,\beta}$.~~

Alternative notation

$$BAB = E_{ij} A E_{ij} \quad , \quad E_{ij} = e_i^T e_j^T$$

$$= e_i e_j^T A e_i e_j^T$$

$$= e_i A_{ji} e_j^T$$

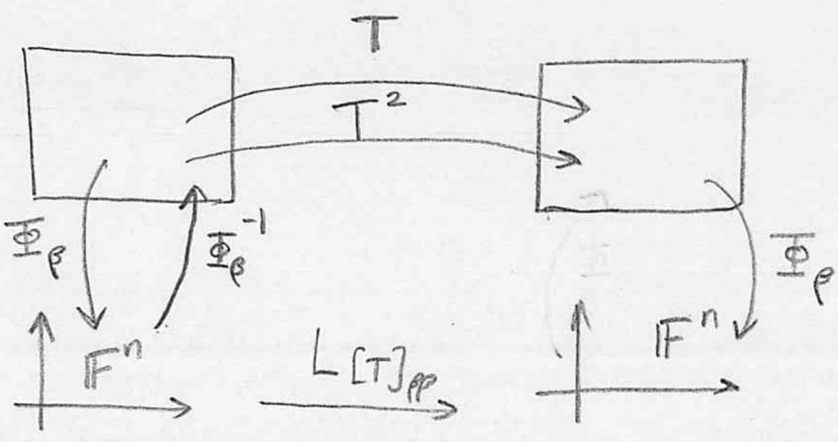
$$= A_{ji} E_{ij}$$

same conclusion, $A_{ji} = 0$
all other components free.
$\dim(\mathrm{Ker}(T)) = n^2 - 1$

$$\beta = \{ E_{11}, E_{12}, \ldots, \widehat{E_{ji}}, \ldots, E_{nn} \}$$

Problem 15 (10pt) Let $T(A) = BAB$ for each $A \in \mathbb{R}^{n\times n}$ where $B = E_{ij}$ for a particular choice of i, j. Find a basis for $\mathrm{Ker}(T)$.

Problem 16 (10pt) Let $T \in L(V)$. Show that the matrix of $T^2 : V \to V$ with respect to basis β is given by $[T]_{\beta,\beta}[T]_{\beta,\beta}$.



Def^n/ $[T]_{\beta\beta} = [L_{[T]_{\beta\beta}}] = [\Phi_\beta \circ T \circ \Phi_\beta^{-1}]$

Likewise $[T^2]_{\beta\beta} = [\Phi_\beta \circ T^2 \circ \Phi_\beta^{-1}]$. Consider,

$$\begin{aligned}
[T^2]_{\beta\beta} &= [\Phi_\beta \circ T \circ T \circ \Phi_\beta^{-1}] \\
&= [\Phi_\beta \circ T \circ \Phi_\beta^{-1} \circ \Phi_\beta \circ T \circ \Phi_\beta^{-1}] \\
&= [\Phi_\beta \circ T \circ \Phi_\beta^{-1}][\Phi_\beta \circ T \circ \Phi_\beta^{-1}] \\
&= [T]_{\beta\beta}[T]_{\beta\beta} . //
\end{aligned}$$

$S = \Phi_\beta \circ T \circ \Phi_\beta^{-1}$ is $\mathbb{F}^n \to \mathbb{F}^n$
$[S \circ S] = [S][S]$
we could prove with more time ☺