Please show your work and use words to explain your steps where appropriate.

Note: *V and W are finite dimensional vector space over a field $\mathbb{F}$ unless otherwise specified*

Problem 1 (10pts) If $T : V \to V$ is a linear transformation and $x \neq 0$ has $T(x) = 2x$ then show that $T^3 : V \to V$ has x as an eigenvector with eigenvalue 8.

Problem 2 (10pts) Let $W = \text{span}\{(1, 1, 1), (0, 1, 0)\}$ find an orthonormal basis β for W.

Problem 3 (10pts) Let $S = \{(1, 0, 0, 0), (2, 1, -2, -3)\}$. Find a basis for $S^\perp$ (w.r.t. dot-product on $\mathbb{R}^4$)

Problem 4 (10pts) Suppose $V = W_1 \oplus W_2 \oplus W_3$ where $\beta = \beta_1 \cup \beta_2 \cup \beta_3$ is a basis for V formed by concatenating bases $\beta_1, \beta_2, \beta_3$ for W_1, W_2, W_3 respective where $\dim(W_j) = d_j$ for $j = 1, 2, 3$. Suppose $[T]_{\beta,\beta} = \begin{bmatrix} A & 0 & M_1 \\ 0 & B & M_2 \\ 0 & 0 & C \end{bmatrix}$ where $A \in \mathbb{F}^{d_1 \times d_1}, B \in \mathbb{F}^{d_2 \times d_2}$ and $C \in \mathbb{F}^{d_3 \times d_3}$. Suppose the induced maps T_{V/W_3} and $T_{V/(W_1 \oplus W_2)}$ are invertible. Prove T is invertible.

Problem 5 (10pts) Let (V, g) be a real geometry and suppose $S, T \in L(V)$ are g-orthonormal. Prove $S \circ T$ is also g-orthonormal.

Problem 6 (10pts) Fix $x \in V$. Let $h : V^* \times V^* \to \mathbb{F}$ be defined by

$$h(\alpha, \beta) = \alpha(x)\beta(x)$$

for all $\alpha, \beta \in V^*$. Show h is a symmetric bilinear mapping.

Problem 7 (15pts) Let $\beta = \{v_1, \dots, v_n\}$ and $\beta^* = \{v^1, \dots, v^n\}$ form bases for V and V^* where $v^i : V \to \mathbb{F}$ is the linear transformation for which $v^i(v_j) = \delta_{ij}$ for all $1 \leq i, j \leq n = \dim(V)$. Prove:

(a.) if $x = \sum_{i=1}^n x^i v_i$ then $x^i = v^i(x)$.

(b.) if $\alpha = \sum_{i=1}^n \alpha_i v^i$ then $\alpha_i = \alpha(v_i)$

Problem 8 (10pts) If $W \leq V$ then what condition is needed in order that $x + W = y + W$?

Problem 9 (10pts) Let $T : V \to V$ be a linear transformation and $V = W_1 \oplus W_2$ where W_1, W_2 are T-invariant subspaces. Let us propose a definition for $S : V/W_1 \to V/(W_1 \cap W_2)$ by the rule $S(x + W_1) = T(x) + W_1 \cap W_2$. What condition (if any) is needed for T to be a well-defined linear transformation ?

Problem 10 (10pts) Suppose $W \leq V$. Let $T : V \to V/W$ be defined by $T(x) = x + W$. Show how the first isomorphism theorem and the rank-nullity theorem for T can be used to prove $\dim(V/W) = \dim(V) - \dim(W)$.

Problem 11 (10pts) Apply the first isomorphism theorem to $T : M \times N \to M + N$ where $T(x, y) = x + y$ for each $(x, y) \in M \times N$ (yes, T is clearly linear). Then, explain why the dimension formula $\dim(M + N) = \dim(M) + \dim(N) - \dim(M \cap N)$ naturally follows.

Problem 12 (20pts) Let $A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Use inner product $\langle A, B \rangle = \text{trace}(AB^T)$ to answer the following:

(a.) Show $\{A_1, A_2, A_3\}$ is orthogonal

(b.) Let $W = \text{span}\{A_1, A_2, A_3\}$ and find an ortho**normal** basis for W.

(c.) Construct the formula for $\text{Proj}_W \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(d.) Find a basis for $\text{ann}(W)$.

Choose your own adventure: pick just one of these to work

Problem 13 (25pts) Prove that any real symmetric matrix A has a cube root. In other words, show there exists M for which $M^3 = A$.

Problem 14 (25pts) Let $g(a(x), b(x)) = \int_0^1 a(x)b(x)\,dx$ define an inner product on $P_1(\mathbb{R})$. Also, define the dual vector $\alpha : P_1(\mathbb{R}) \to \mathbb{R}$ by $\alpha(f(x)) = \int_0^1 xf(x)\,dx$ for each $f(x) \in P_1(\mathbb{R})$. Let $\beta = \{v_1, v_2\}$ form a basis for $P_1(\mathbb{R})$ where $v_1 = 1, v_2 = x$. Find:

(**a**.) g_{ij}, (**b**.) g^{ij}, (**c**.) $\sharp\alpha$ (**d**.)Riesz vector for α