Please show your work and use words to explain your steps where appropriate.

Note: *V and W are finite dimensional vector space over a field $\mathbb{F}$ unless otherwise specified*

Problem 1 (10pts) If $T : V \to V$ is a linear transformation and $x \neq 0$ has $T(x) = 2x$ then show that $T^3 : V \to V$ has x as an eigenvector with eigenvalue 8.

$$T^3(x) = T(T(T(x)))$$
$$= T(T(2x)) = T(2(2x)) = 2(2(2x)) = 8x.$$

$\therefore$ x is e-vector with $\lambda = 8$ for T^3.

Problem 2 (10pts) Let $W = \text{span}\{(1,1,1), (0,1,0)\}$ find an orthonormal basis β for W.

Let $u_1 = (0,1,0)$

$$\tilde{u}_2 = (1,1,1) - [(0,1,0) \cdot (1,1,1)](0,1,0) = (1,0,1) \quad \therefore u_2 = \tfrac{1}{\sqrt{2}}(1,0,1)$$

$\therefore$ $\underline{\{(0,1,0), \tfrac{1}{\sqrt{2}}(1,0,1)\} \text{ serves as } \beta}$.

Problem 3 (10pts) Let $S = \{(1,0,0,0), (2,1,-2,-3)\}$. Find a basis for $S^\perp$ (w.r.t. dot-product on $\mathbb{R}^4$)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \end{bmatrix} \quad \therefore (x_1, x_2, x_3, x_4) \in \text{Null}[S]^T$$

has $x_1 = 0$ and $x_2 - 2x_3 - 3x_4 = 0$ hence,

$$(x_1, x_2, x_3, x_4) = (0, 2x_3 + 3x_4, x_3, x_4)$$
$$= x_3(0,2,1,0) + x_4(0,3,0,1)$$

$\therefore$ $\boxed{\gamma = \{(0,2,1,0), (0,3,0,1)\} \text{ is basis for } S^\perp}$

Problem 4 (10pts) Suppose $V = W_1 \oplus W_2 \oplus W_3$ where $\beta = \beta_1 \cup \beta_2 \cup \beta_3$ is a basis for V formed by concatenating bases $\beta_1, \beta_2, \beta_3$ for W_1, W_2, W_3 respective where $\dim(W_j) = d_j$ for $j = 1, 2, 3$. Suppose $[T]_{\beta,\beta} = \begin{bmatrix} A & 0 & M_1 \\ 0 & B & M_2 \\ 0 & 0 & C \end{bmatrix}$ where $A \in \mathbb{F}^{d_1 \times d_1}$, $B \in \mathbb{F}^{d_2 \times d_2}$ and $C \in \mathbb{F}^{d_3 \times d_3}$. Suppose the induced maps T_{V/W_3} and $T_{V/(W_1 \oplus W_2)}$ are invertible. Prove T is invertible.

$$[T_{V/W_3}] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad [T_{V/W_1 \oplus W_2}] = C$$

But, $\det [T]_{\beta\beta} = \det A \det B \det C = \underbrace{\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}}_{\neq 0} \underbrace{\det(C)}_{\neq 0} \neq 0$

by invertibility of induced maps.

$\therefore \det [T]_{\beta\beta} \neq 0 \Rightarrow \boxed{T \text{ invertible}}$

Problem 5 (10pts) Let (V, g) be a real geometry and suppose $S, T \in L(V)$ are g-orthonormal. Prove $S \circ T$ is also g-orthonormal.

Let $x, y \in V$,

$$g\big((S\circ T)(x), (S\circ T)(y)\big) = g\big(S(T(x)), S(T(y))\big) \quad g\text{-orthog. of } S$$
$$= g(T(x), T(y)) \quad g\text{-orthog. of } T$$
$$= g(x, y) \quad \forall x, y \in V$$

$\therefore$ $S\circ T$ is also g-orthogonal as we know $S\circ T \in L(V)$.

Problem 6 (10pts) Fix $x \in V$. Let $h : V^* \times V^* \to \mathbb{F}$ be defined by

$$h(\alpha, \beta) = \alpha(x)\beta(x)$$

for all $\alpha, \beta \in V^*$. Show h is a symmetric bilinear mapping.

$$h(c\alpha_1 + \alpha_2, \beta) = (c\alpha_1 + \alpha_2)(x)\,\beta(x)$$
$$= (c\alpha_1(x) + \alpha_2(x))\beta(x)$$
$$= c\alpha_1\beta(x) + \alpha_2(x)\beta(x)$$
$$= c\,h(\alpha_1, \beta) + h(\alpha_2, \beta)$$

Also,

$$h(\alpha, \beta) = \alpha(x)\beta(x) = \beta(x)\alpha(x) = h(\beta, \alpha)$$

Thus $h(\beta, c\alpha_1 + \alpha_2) = h(c\alpha_1 + \alpha_2, \beta) = c\,h(\alpha_1, \beta) + h(\alpha_2, \beta) = c\,h(\beta, \alpha_1) + h(\beta, \alpha_2)$.

Problem 7 (15pts) Let $\beta = \{v_1, \ldots, v_n\}$ and $\beta^* = \{v^1, \ldots, v^n\}$ form bases for V and V^* where $v^i : V \to \mathbb{F}$ is the linear transformation for which $v^i(v_j) = \delta_{ij}$ for all $1 \le i, j \le n = \dim(V)$. Prove:

(a.) if $x = \sum_{i=1}^n x^i v_i$ then $x^i = v^i(x)$.

(b.) if $\alpha = \sum_{i=1}^n \alpha_i v^i$ then $\alpha_i = \alpha(v_i)$

(a.) Observe $V^j(x) = V^j\left(\sum_{i=1}^n x^i V_i\right) = \sum_{i=1}^n x^i V^j(V_i)$: by linearity of $V^j \in V^*$

$$= \sum_{i=1}^n x^i \delta_{ij} \quad : \text{def}^{\underline{n}} \text{ of dual basis.}$$

$$= x^j \quad \therefore \underline{V^j(x) = x^j} \;// \quad \forall j = 1, 2, \ldots, n.$$

(b.) Consider,

$$\alpha(V_j) = \left(\sum_{i=1}^n \alpha_i V^i\right)(V_j)$$
$$= \sum_{i=1}^n \alpha_i \underbrace{V^i(V_j)}_{\delta_{ij}} = \alpha_j \quad \therefore \underline{\alpha(V_i) = \alpha_i \quad \forall i = 1, 2, \ldots, n} \;//$$

Problem 8 (10pts) If $W \leq V$ then what condition is needed in order that $x + W = y + W$?

Best.

$\boxed{\text{We proved that } x + W = y + W \iff y - x \in W.}$

you could also say $\exists\, w_1, w_2 \in W$ s.t. $x + w_1 = y + w_2$ etc.

Problem 9 (10pts) Let $T : V \to V$ be a linear transformation and $V = W_1 \oplus W_2$ where W_1, W_2 are T-invariant subspaces. Let us propose a definition for $S : V/W_1 \to V/(W_1 \cap W_2)$ by the rule $S(x + W_1) = T(x) + W_1 \cap W_2$. What condition (if any) is needed for T to be a well-defined linear transformation ?

$T(W_1) \leq W_1$ and $T(W_2) \leq W_2$ is given by T-invariance.

Well-defined? $x + W_1 = y + W_1 \iff y - x \in W_1$

consider, $S(x+W_1) = T(x) + W_1 \cap W_2$ & $S(y+W_1) = T(y) + W_1 \cap W_2$

Notice $x \in W_1 \oplus W_2 \Rightarrow x = x_1 + x_2$ where $x_1 \in W_1$ & $x_2 \in W_2$

thus $T(x) = T(x_1) + T(x_2)$. We need $T(y) - T(x) \in W_1 \cap W_2$

$T(y-x) \in W_1 \cap W_2 = \{0\}$

$\therefore$ $y - x \in \text{Ker}(T)$

$\boxed{\text{Thus } W_1 \subset \text{Ker } T \text{ does nicely.}}$

Problem 10 (10pts) Suppose $W \leq V$. Let $T : V \to V/W$ be defined by $T(x) = x + W$. Show how the first isomorphism theorem and the rank-nullity theorem for T can be used to prove $\dim(V/W) = \dim(V) - \dim(W)$.

First Isomorphism Thm for $T : V \longrightarrow V/W$

gives $\underline{V/\text{Ker } T \approx \text{Range}(T)}^{*}$. But,

$\text{Ker } T = \{x \in V \mid x + W = W\} = W$

and T is a surjection as $x + W = T(x)$ for each $x + W \in V/W$.

But, $\dim V = \underbrace{\dim(\text{Ker } T)}_{\dim W} + \underbrace{\dim(\text{Range } T)}_{\dim(V/W) \text{ by } *}$ $\therefore$ $\dim(V/W) = \dim(\text{Range}(T)) = \dim V - \dim W.$

Problem 11 (10pts) Apply the first isomorphism theorem to $T : M \times N \to M + N$ where $T(x, y) = x + y$ for each $(x, y) \in M \times N$ (yes, T is clearly linear). Then, explain why the dimension formula $\dim(M + N) = \dim(M) + \dim(N) - \dim(M \cap N)$ naturally follows.

$$\text{Ker } T = \{(x,y) \in M \times N \mid T(x,y) = x + y = 0\} \to \underbrace{y = -x}_{\text{both } x \,\&\, y \text{ are in } M \,\&\, N \;\therefore\; x \in M \cap N.}$$
$$= \{(x, -x) \in M \times N \mid x \in M \cap N\}$$

Also, if $x + y \in M + N$ then $T(x,y) = x + y$

thus T is a surjection. Observe,

$\dfrac{M \times N}{\text{Ker } T} \approx M + N$ $\therefore$ $\underbrace{\dim(M \times N)}_{} - \underbrace{\dim(\text{Ker } T)}_{} = \dim(M+N)$

$\underline{\dim M + \dim N - \dim(M \cap N) = \dim(M+N).}$

Note, $\psi(x) = (x, -x)$ is isomorphism from $M \cap N$ to $\text{Ker}(T)$. (thus $\dim M \cap N = \dim(\text{Ker } T)$)

Problem 12 (20pts) Let $A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Use inner product $\langle A, B\rangle = \text{trace}(AB^T)$ to answer the following:

(a.) Show $\{A_1, A_2, A_3\}$ is orthogonal

(b.) Let $W = \text{span}\{A_1, A_2, A_3\}$ and find an ortho**normal** basis for W.

(c.) Construct the formula for $\text{Proj}_W \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(d.) Find a basis for $\text{ann}(W)$.

(a.) $A_1A_2^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A_1A_3^T = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, $A_2A_3^T = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore \langle A_1, A_2\rangle = 0$, $\langle A_1, A_3\rangle = 0$, $\langle A_2, A_3\rangle = 0$.

Thus $\{A_1, A_2, A_3\}$ forms orthogonal set.

(b.) $A_1A_1^T = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\therefore \langle A_1, A_1\rangle = 4 \Rightarrow \underline{\|A_1\| = 4}$

$\therefore \underline{\|A_1\| = 2}$.

Likewise,

$\|A_2\| = 2$ and $\|A_3\| = \sqrt{2}$ so $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}$.

$\therefore$ $\boxed{\beta = \left\{ \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \text{ is orthonormal basis for } W.}$

(c.) $\text{Proj}_W(A) = \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\rangle \frac{A_1}{4} + \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle \frac{A_2}{4} + \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \frac{A_3}{2}$

$= (-a+b+c-d)\frac{A_1}{4} + (a+b+c+d)\frac{A_2}{4} + (b-c)\frac{A_3}{2}$

$= \frac{-a+b+c-d}{4}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{a+b+c+d}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{b-c}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} a/2 + b/2 - \cancel{c/4} + \cancel{c/4} & b/2 + b/2 + c/2 - c/2 \\ b/2 + c/2 - b/2 + c/2 & b/2 + c/2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} a+b & b \\ c & b+c \end{bmatrix}$

(d.) $\text{Ann}(W) = \left\{ \alpha : \mathbb{R}^{2\times 2} \to \mathbb{R} \mid \alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \text{ for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W \right\}$

$4 = \dim(\text{ann } W) + \dim W$ an. by inspection $\boxed{\text{Ann}(W) = \text{span}\{E^{11} - E^{22}\}}$

Choose your own adventure: pick just one of these to work

Problem 13 (25pts) Prove that any real symmetric matrix A has a cube root. In other words, show there exists M for which $M^3 = A$.

Problem 14 (25pts) Let $g(a(x), b(x)) = \int_0^1 a(x)b(x)\,dx$ define an ~~inner product~~ metric on $P_1(\mathbb{R})$. Also, define the dual vector $\alpha : P_1(\mathbb{R}) \to \mathbb{R}$ by $\alpha(f(x)) = \int_0^1 xf(x)\,dx$ for each $f(x) \in P_1(\mathbb{R})$. Let $\beta = \{v_1, v_2\}$ form a basis for $P_1(\mathbb{R})$ where $v_1 = 1, v_2 = x$. Find:

(a.) g_{ij}, (b.) g^{ij}, (c.) $\sharp\alpha$ (d.)Riesz vector for α

PROBLEM 13

$A^T = A \Rightarrow \exists P \in GL(n)$ s.t. $P^{-1}AP = D \in \mathbb{R}^{n\times n}$

Now $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and we can take a cube-root of any real # thus $\bar{M} = \text{diag}(\sqrt[3]{\lambda_1}, \dots, \sqrt[3]{\lambda_n}) \in \mathbb{R}^{n\times n}$ and $\bar{M}^3 = D$. Let $M = P\bar{M}P^{-1}$ and observe,

$$\begin{aligned} M^3 &= (P\bar{M}P^{-1})(P\bar{M}P^{-1})(P\bar{M}P^{-1}) \\ &= P\bar{M}\bar{M}\bar{M}P^{-1} \\ &= PDP^{-1} \quad : (P^{-1}AP = D \Rightarrow A = PDP^{-1}) \\ &= A. \end{aligned}$$

PROBLEM 14 Let $V_1 = 1$, $V_2 = x$

(a.)
$$\left.\begin{aligned} g_{11} &= g(1,1) = \int_0^1 dx = 1 \\ g_{12} &= g(1,x) = \int_0^1 x\,dx = \tfrac{1}{2} \\ g_{22} &= g(x,x) = \int_0^1 x^2\,dx = \tfrac{1}{3} \end{aligned}\right\} \quad \underline{G = (g_{ij}) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}}.$$

(b.) $$G^{-1} = \frac{1}{\frac{1}{3} - \frac{1}{4}}\begin{bmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{bmatrix} = 12\begin{bmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} = (g^{ij})$$

(c.) $\alpha(1) = \int_0^1 x\,dx = \frac{1}{2}$ and $\alpha(x) = \int_0^1 x^2\,dx = \frac{1}{3}$

thus $\alpha = \frac{1}{2}V^1 + \frac{1}{3}V^2$ where $\{V^1, V^2\}$ is dual to $\{V_1, V_2\}$

We have $\alpha_1 = 1/2$ and $\alpha_2 = 1/3$. Recall

$$(\#\alpha)^i = \sum_{j=1}^{2} g^{ij}\alpha_j$$

$$(\#\alpha)^1 = g^{11}\alpha_1 + g^{12}\alpha_2 = 4/2 - 6/3$$

$$(\#\alpha)^2 = g^{21}\alpha_1 + g^{22}\alpha_2 = -6/2 + 12/3 = 1$$

Thus $\alpha^1 = 0$ and $\alpha^2 = 1$

In conclusion, $\boxed{\#\alpha = x}$

(d.) the Riesz vector is precisely $\#\alpha$! Note $\alpha(y) = \langle x, y\rangle$

Please show your work and use words to explain your steps where appropriate.

Note: *V and W are finite dimensional vector space over a field $\mathbb{F}$ unless otherwise specified*

Problem 1 Let $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ find the eigenvalues and eigenvectors of A over $\mathbb{C}$.

$$\det\begin{bmatrix} -\lambda & 3 \\ -3 & -\lambda \end{bmatrix} = \lambda^2 + 9 = 0 \Rightarrow \underline{\lambda = \pm 3i \text{ e-values}}.$$

$$A - 3iI = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \sim \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow (x_1, x_2) \in \text{Null}(A-3iI) \text{ has } -ix_1 + x_2 = 0 \quad \therefore x_2 = ix_1$$

Set $x_1 = 0$ obtain $v_1 = (1, i)$

then $Av_1 = 3iv_1 \Rightarrow Av_1^* = -3iv_1^* \therefore \underline{v_1^* = (1, -i)}$ is e-vector with e-value $\lambda = -3i$

Thus $\{(1, i), (1, -i)\}$ forms eigenbasis for A. Moreover, as we'd expect,

$$P^{-1}AP = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \frac{1}{-2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} \begin{bmatrix} 3i & -3i \\ -3 & -3 \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 3i & 0 \\ 0 & -3i \end{bmatrix} = D$$

$A = PDP^{-1}$

Problem 2 Calculate e^A where A is the matrix given in first problem.

$$e^A = e^{PDP^{-1}} = Pe^DP^{-1}$$

$$= P\begin{bmatrix} e^{3i} & 0 \\ 0 & e^{-3i} \end{bmatrix}P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{3i} & 0 \\ 0 & e^{-3i} \end{bmatrix} \begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} \frac{1}{2i}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} i\exp(3i) & \exp(3i) \\ i\exp(-3i) & -\exp(-3i) \end{bmatrix} \frac{1}{2i}$$

$$= \frac{1}{2i}\left[\begin{array}{c|c} ie^{3i} + ie^{-3i} & e^{3i} - e^{-3i} \\ \hline -e^{3i} + e^{-3i} & ie^{3i} + ie^{-3i} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \frac{1}{2}(e^{3i} + e^{-3i}) & \frac{1}{2i}(e^{3i} - e^{-3i}) \\ \hline \frac{1}{2i}(e^{-3i} - e^{3i}) & \frac{1}{2}(e^{3i} + e^{-3i}) \end{array}\right] = \boxed{\begin{bmatrix} \cos 3 & \sin 3 \\ -\sin 3 & \cos 3 \end{bmatrix}}$$

But, you probably did this the easier way. Calculate directly A^n then split into odd/even terms.

Problem 3 State and prove either the first, second or third isomorphism theorems.

Consider $T: V \to W$ linear

Let $\psi: V/\mathrm{Ker}\,T \to W$ be defined by

$\psi(x + \mathrm{Ker}\,T) = T(x)$. Is ψ well-defined?

Suppose $x + \mathrm{Ker}\,T = y + \mathrm{Ker}\,T$ then $y - x \in \mathrm{Ker}\,T$

thus $T(y-x) = 0$ or $T(x) = T(y)$ hence

$\psi(x + \mathrm{Ker}\,T) = \psi(y + \mathrm{Ker}\,T)$ hence ψ well-defined.

Furthermore, $\mathrm{Ker}\,\psi = \{x + \mathrm{Ker}\,T \mid T(x) = 0\} = \{\mathrm{Ker}\,T\}$

thus ψ is an injection. Consider $\tilde{\psi}: V/\mathrm{Ker}\,T \to T(V)$

note $w \in T(V) \Rightarrow w = T(x)$ for some $x \in V$ hence

$\tilde{\psi}(x + \mathrm{Ker}\,T) = T(x) = w \therefore \tilde{\psi}$ is a surjection and

by our previous comments concerning ψ, $\mathrm{Ker}\,\tilde{\psi} = \{\mathrm{Ker}\,T\}$

hence $\tilde{\psi}: V/\mathrm{Ker} \to T(V)$ is an isomorphism of $\underline{V/\mathrm{Ker}\,T \approx T(V)}$

and $\tilde{\psi}(x + \mathrm{Ker}\,T) = T(x)$.

Problem 4 Find the matrix of

$$Q(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 6xy + 6xz + 6yz.$$

Find the eigenvalues of A and write the formula for Q in terms of eigencoordinates. *hint: the vectors* $(-1, 1, 0), (-1, 0, 1)$ *and* $(1, 1, 1)$ *are all very interesting*

$$Q(v) = v^T \underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}}_{A} v$$

$$\underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\lambda = -1}, \quad \underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\lambda = -1}, \quad \underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}}_{\lambda = 8}$$

$$\boxed{Q(\bar{x} v_1 + \bar{y} v_2 + \bar{z} v_3) = -\bar{x}^2 - \bar{y}^2 + 8\bar{z}^2}$$

where $v_1 = \frac{1}{\sqrt{2}}(-1, 1, 0)$, $v_2 = \frac{1}{\sqrt{2}}(-1, 0, 1)$, $v_3 = \frac{1}{\sqrt{3}}(1, 1, 1)$

Problem 5 Let $Q(x,y) = 2x^2 + 6xy + 2y^2$ for all $(x,y) \in \mathbb{R}^2$.

- **(a.)** Find the matrix A for Q such that $Q(v) = v^T A v$,
- **(b.)** find eigenvalues of A,
- **(c.)** find an orthonormal eigenbasis $\beta = \{u_1, u_2\}$ for A
- **(d.)** if $v = \bar{x}u_1 + \bar{y}u_2$ then explicitly relate the eigencoordinates $\bar{x}, \bar{y}$ to the usual x, y coordinates
- **(e.)** write the formula for $Q(\bar{x}u_1 + \bar{y}u_2)$

(a.) $Q(x,y) = [x, y]\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \therefore \quad \underline{A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}}.$

(b.) $\det\begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} = (\lambda-2)^2 - 9 = (\lambda-5)(\lambda+1) \quad \therefore \quad \underline{\lambda_1 = 5,\ \lambda_2 = -1}.$

(c.) $A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \therefore \quad \underline{u_1 = \frac{1}{\sqrt{2}}(1,1)}.$ $\left(\begin{array}{l} x_1 - x_2 = 0 \\ \text{set } x_2 = 1 \\ \text{then normalize.} \end{array} \right)$

$A + I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \therefore \quad \underline{u_2 = \frac{1}{\sqrt{2}}(1,-1)}.$

$\underline{\beta = \left\{ \frac{1}{\sqrt{2}}(1,1),\ \frac{1}{\sqrt{2}}(1,-1) \right\} \text{ is orthonormal e-basis}}$

(d.) If $(x,y) = \bar{x}u_1 + \bar{y}u_2 = \underbrace{[u_1 | u_2]}_{[\beta]}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$ & $[\beta]^T[\beta] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

thus $\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = [\beta]^T \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(x+y) \\ \frac{1}{\sqrt{2}}(x-y) \end{bmatrix}$

that is, $\boxed{\bar{x} = \frac{1}{\sqrt{2}}(x+y) \text{ and } \bar{y} = \frac{1}{\sqrt{2}}(x-y)}$

(e.) $Q(\bar{x}u_1 + \bar{y}u_2) = \underline{5\bar{x}^2 - \bar{y}^2}.$

Problem 6 Let V be a real vector space and $V_\mathbb{C}$ its complexification. Suppose $T : V_\mathbb{C} \to V_\mathbb{C}$ is defined by $T(x+iy) = 3x + y + i(x-y)$ for each $x + iy \in V_\mathbb{C}$. Is T the complexification of a map $\tau : V \to V$? That is, does there exist $\tau \in L(V)$ such that $\tau_\mathbb{C} = T$?

$\tau_\mathbb{C}(x+iy) = \tau(x) + i\,\tau(y) = T(x+iy) = 3x+y + i(x-y)$

Can we have $\tau(x) = 3x+y$ and $\tau(y) = x-y$?

More to the point $\tau_\mathbb{C}(iy) = i\,\tau(y) = T(0+iy) = y - iy$

thus $\tau(y) = -y$ and $\tau(0) = y$ which is impossible for $y \neq 0$.

$\therefore$ no such $\tau : V \to V$ exists.

Problem 7 (10pts) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = \frac{d^2 f}{dx^2}$ for each $f(x) = ax^2 + bx + c \in P_2(\mathbb{R})$. Find the eigenvalue of T and determine the geometric and algebraic multiplicity of the eigenvalue.

$$T(ax^2 + bx + c) = 2a \Rightarrow [T]_{\beta\beta} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[T]_{\beta\beta}} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ 0 \end{bmatrix} \quad \beta = \{1, x, x^2\} \qquad \therefore \boxed{\lambda = 0 \text{ with algebraic mult. } 3}$$

$$\ker T = \{ax^2 + bx + c \mid T(ax^2+bx+c) = 2a = 0\} = \mathrm{span}\{1, x\}$$

thus the $\boxed{\text{geometric multiplicity of } \lambda = 0 \text{ is } 2}$

Incidentally, to find e-vector of order 2 we can see

$$T(\tfrac{1}{2}x^2) = 1 \quad \text{thus}$$

$$\gamma = \{1, \tfrac{1}{2}x^2, x\} \hookrightarrow [T]_{\gamma\gamma} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 8 Define $\alpha(x, y, z) = x + y$, $\beta(x, y, z) = x + z$ and $\gamma(x, y, z) = y - z$ for all $(x, y, z) \in V = \mathbb{R}^3$. In fact, $\Upsilon^* = \{\alpha, \beta, \gamma\}$ forms a basis for V^*. Find the basis Υ for $\mathbb{R}^3$ for which Υ^* is the dual basis. Let $W = \ker(\alpha)$ and calculate $W^\perp$ relate it to the annihilator of $\{c\alpha \mid c \in \mathbb{R}\}$. If we want to identify orthogonal complements and annihilators then which isomorphism do we need to use?

$\alpha(v) = 1 \Rightarrow [1, 1, 0]v = 1$	$\alpha(w) = 0$	$\alpha(u) = 0$
$\beta(v) = 0 \Rightarrow [1, 0, 1]v = 0$	$\beta(w) = 1$	$\beta(u) = 0$
$\gamma(v) = 0 \Rightarrow [0, 1, -1]v = 0$	$\gamma(w) = 0$	$\gamma(u) = 1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Well folks, it seems the Prophecy of Laurenzo has come to fruitition. Υ = error.

In fact, Υ^* is not a basis!

$$\underline{\alpha - \beta = \gamma}$$

Sorry. I think this could be an interesting problem...

Problem 9 Let $A, B \in \mathbb{C}^{n\times n}$ and $[A,B] = AB - BA$ (the *commutator* of A and B). Consider the function $f(t) = e^{tA}Be^{-tA}$. **Calculate** $f'(t)$ **and** $f''(t)$. Assume Taylor's Theorem is known:

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \cdots$$

Set $t = 1$ to **derive the identity** $e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$. Finally, if $\mathrm{ad}_A(B) = [A,B]$ defines the *adjoint* map $\mathrm{ad}_A : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ we can define

$$e^{\mathrm{ad}_A} = I + \mathrm{ad}_A + \frac{1}{2}\mathrm{ad}_A^2 + \cdots \quad \text{(yes, an infinite series of operators!)}$$

Rewrite the identity in terms of the exponential of this *adjoint* map.

$$\frac{df}{dt} = \frac{d}{dt}\left(e^{tA}Be^{-tA}\right) = Ae^{tA}Be^{-tA} + e^{tA}B\left(-Ae^{-tA}\right)$$

$$f'(0) = AB - BA = [A,B]$$

$$\frac{d^2f}{dt^2} = A^2e^{tA}Be^{-tA} - Ae^{tA}BAe^{-tA} - Ae^{tA}BAe^{-tA} + e^{tA}BA^2e^{-tA}$$

$$\begin{aligned} f''(0) &= A^2B - ABA - ABA + BA^2 \\ &= A(AB - BA) - (AB - BA)A \\ &= A[A,B] - [A,B]A \\ &= [A,[A,B]] \end{aligned}$$

Thus, $f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \cdots$

$$f(t) = B + t[A,B] + \frac{t^2}{2}[A,[A,B]] + \cdots$$

$$\Rightarrow f(1) = \underline{e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots}.$$

Finally,

$$\begin{aligned} e^{\mathrm{ad}_A}(B) &= I(B) + \mathrm{ad}_A(B) + \frac{1}{2}\mathrm{ad}_A^2(B) + \cdots \\ &= B + [A,B] + \frac{1}{2}\mathrm{ad}_A([A,B]) + \cdots \\ &= B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots \\ &= \underline{e^A B e^{-A}}. \end{aligned}$$